On Gabber's uniformization theorems : outline and applications to étale cohomology [G]

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Of ramification and vanishing cycles, Tokyo, Sep. 11, 2007

preliminary notes (Sep. 19, 2007)

1. Finiteness and uniformization

Th. 1 (Gabber) : $f : X \to Y$ of f. t., Y noetherian, qe, $\Lambda = \mathbb{Z}/n\mathbb{Z}$, n invertible on Y, then Rf_* sends $D^b_c(X, \Lambda)$ to $D^b_c(Y, \Lambda)$.

quasi-excellent (qe) : noetherian, formal fibers geom. regular ; openness of Reg

history : (i) f proper : [SGA 4 XIV] : no qe necessary, n not needed to be invertible (needed in the non proper case)

(ii) $Y/\mathbb{Q}$: [SGA 4 XIX]

(iii) $f : X \to Y$, Y ft/ regular of dim ≤ 1 : Deligne [SGA 4 1/2]

Basic reduction : by (i) enough to show $Rj_*F \in D^b_c$ for any open dense immersion $j : U \to X$ with X of ft $/Y$, F constructible on U. If X regular, $Z = X - U =$ support of a sncd (strict normal crossings divisor), $F = \Lambda_U$, OK by *absolute purity* :

Th. 0 (Gabber) : In this case, with $Z = \sum D_i$ (D_i regular),

$$R^q j_* \Lambda = \Lambda^q(\oplus \Lambda_{D_i}(-1)).$$

Basic strategy : reduce to th. 0 by using *resolution*. Here's a typical example. Consider :

(dJ) *weak de Jong* : for any open dense immersion $j : U \to X$ with X of ft $/Y$, there exists a proper surjective $p : X' \to X$ with X' regular and $X' - U'$ (where $U' = p^{-1}(U)$) the complement of a sncd.

(Note : (i) "weak" because p not assumed generically finite ; (ii) by de Jong (dJ) is available for Y of finite type over a regular scheme of dimension ≤ 1.)

Then (dJ) + absolute purity + proper cohomological descent implies that Rf_* sends D^b_c to D^+_c.

Leaves the *issue* : to D^b_c ?

It Y of *finite (Krull) dimension*, then Rf_* is of finite cohomological dimension, hence D^b_c goes to D^b_c. The finiteness of the cohomological dimension follows, by reduction to the case of an open immersion, from :

Prop. 1 (Gabber). X noetherian, strictly local henselian, of dimension $d > 0$, ℓ invertible on X, then for any open U in X, one has $\mathrm{cd}_\ell(U) \leq 2d - 1$.

Gabber shows it's equivalent to proving $\mathrm{cd}_\ell(\eta) \leq 2d - 1$ for X integral with generic point η, then does it by using the Zariski-Riemann space Z of X (= proj. lim. of proper birational $X' \to X$) and the Leray spectral sequence of $\eta \to Z$. (In the qe case, could also use Gabber's affine Lefschetz th. (proved independently of th. 1 !), see §3.)

However, there are *bad* Y's which are of *infinite* dimension (Nagata's excellent pathologies ...). For such Y's, (dJ) is insufficient. Can show that Hironaka would suffice (i. e. variant of (dJ) with p *birational* (for X integral of f. t. $/Y$)). Gabber shows that in fact a weaker, *local* variant of Hironaka suffices. This is his *refined uniformization theorem* :

Th. 2 (Gabber) : Let X be noetherian, qe, with the prime ℓ invertible on X, and let $j : U \to X$ be a dense open immersion. Then there exists a finite family of maps of finite type $(p_i : X'_i \to X)$ with the following properties :

(a) X'_i is regular connected and the inverse image U'_i of U in X'_i is the complement of the support of a strict dnc (convention : the empty space = the sum of an empty family of divisors is considered as a sncd)

(b) the image by p_i of the maximal point of X'_i is a maximal point of X, $p = \coprod p_i : \coprod X'_i \to X$ is generically finite, and, up to thickenings, is dominated by a composition of maps of the following types : modification, finite flat surjective of degree prime to ℓ, Nisnevich covers (i. e. étale covers $V_i \to V$ with the property that for each point x in V there exists a point y in a V_j above x with $[k(y) : k(x)] = 1$).

Topological reformulation : S noetherian, (gf/S) = category of S-schemes X of finite type with the property that each maximal point x of X maps to a maximal point s of S with finite fiber at s (gf : "generically finite), define the *ℓ'-topology* on (gf/S) as the topology generated by thickenings, Nisnevich covers and *ℓ'-alterations* (i. e. proper, surjective, generically finite maps sending each maximal point to a maximal point with generic degree prime to ℓ). Then the factorization assertion in (b) is equivalent to saying that p is surjective for the ℓ'-topology (on (gf/X)) (use Gruson-Raynaud's flattening theorem).

Sketch of proof of Th. 2 $\Rightarrow$ Th. 1. WMA $n = \ell^r$. For each $x \in X$, $c(x) = \dim \mathcal{O}_{X,x}$ is finite. It may be the case that the function $x \mapsto c(x)$ is unbounded, we don't care. Indeed, Gabber observes that, because $c(x)$ is finite, by quasi-compactness it suffices to show that for any X of finite type over Y, any dense open $j : U \to X$, any constructible Λ-module F on U and any integer $c \geq 0$, the following property holds :

$P(j, F, c)$: there exists a closed subset T of X of codimension $> c$ such that $(Rj_*F)|X - T$ is in D^b_c.

This can be proven by induction on c. The property is trivial for $c = 0$. Assuming $P(-, -, c-1)$, to prove $P(-, -, c)$ a little dévissage using ZMT shows that it is even enough to prove $P(-, \Lambda, c)$. Take $p : X' \to X$ as in th. 2 and a factorization

$$\begin{tikzcd} & X' \arrow[d, "p"] \\ X_1 \arrow[ur, "h"] \arrow[r, "g"] & X \end{tikzcd} ,$$

where g is a composition of thickenings, modifications, finite flat maps of degree prime to ℓ and Nisnevich covers. Denote by $j' : U' \to X'$, $j_1 : U_1 \to X_1$ the corresponding pull-backs of j. By the induction assumption choose T in X of codimension c such that $Rj_*\Lambda|X - T$

is in D^b_c. Then try to *shrink* T, i. e. remove a dense open of it such that the remaining part, of codimension $> c$, does the job. May assume T irreducible, work *generically on* T and *modulo* D^b_c. By the induction assumption again, plus shrinking, may choose T_1 in X_1 of codimension c, lying above T, such that $Rj_{1*}\Lambda|X_1 - T_1$ is in D^b_c. Reduce to the case where T_1 is *quasi-finite* over T. Consider the natural map

$$a : Rj_*\Lambda|T \to R(g|T_1)_*(Rj_{1*}\Lambda|T_1).$$

One shows that *generically on* T *and modulo* D^b_c

(a) a has a left inverse

(b) $a = 0$.

This will formally imply that $Rj_*\Lambda|T$ is generically in D^b_c. It is enough to show (a) and (b) for g of one of the types : (i) finite flat surjective of degree prime to ℓ, (ii) modification, (iii) Nisnevich cover.

For (b), use the factorization of a through X' :

$$Rj_*\Lambda|T \to R(g|T_1)_*((Rj'_*\Lambda)|T_1) \to R(g|T_1)_*((Rj_{1*}\Lambda)|T_1).$$

By absolute purity (th. 0), $Rj'_*\Lambda|T_1$ is in D^b_c. This implies the result for types (i) and (ii). For type (iii), g is not proper, but there is a component V of X_1, whose restriction over T is generically sent isomorphically onto its image by $g|T_1$, and one can replace X_1 by V.

For (a) : in case (i), use trace map ; in case (iii), argue as above ; in case (iii), a is in fact an isomorphism (generically on T and mod D^b_c) (apply induction assumption to the cone of $\Lambda \to Rg_*\Lambda$, which is concentrated on a closed subset Σ of codimension ≥ 1, and the inclusion $\Sigma \cap U \subset \Sigma$).

Gabber also proved a variant of th. 2, where no prime ℓ is assumed to be invertible on X :

Th. 3 (Gabber) (*weak uniformization*). Same data as in th. 2, except that ℓ is not assumed to be invertible on X. Then there exists a family (p_i) as in th. 2, satisfying (a) and the variant (b') of (b) with the condition of degree prime to ℓ removed, and Nisnevich cover replaced by Zariski cover.

Reformulation : *h-topology* on (schemes of finite type / noetherian S) (Voevodsky) = generated by Zariski covers and proper surjective maps. Finer than étale. Any surjective family for the h-topology is dominated by a composition as in th. 3 (b'). Hence : (b') is equivalent to saying that p is surjective for the h-topology.

Th. 3 + "oriented" cohomological descent (Gabber) $\Rightarrow$ Rf_* sends D^b_c to D^+_c (hence Th. 3 $\Rightarrow$ Th. 1 if Y of finite dimension). Won't be discussed here.

2. Glimpses on the proof of the uniformization theorems

The proofs of th. 2 and th. 3 are similar. There are 3 main steps :

(1) Use of Artin-Popescu's approximation th. and new techniques of approximatton due to Gabber to reduce to the complete local case.

(2) In the local complete case, a partial algebraization (or fibration) th. enabling a proof by induction on the local dimension.

(3) Arguments of log geometry starting from de Jong's th. on nodal curves.

The 3 steps contain intermediate results which are of independent interest. For lack of time, I'll skip step (1), be brief on step (2) and concentrate on step (3) in the refined uniformization case (th. 2).

Main result in (2) :

Th. 4 (Gabber) (*partial algebraization*). (a) X complete noetherian local, *reduced*, dim $d > 0$, *equichar.* p, $Z \subset X$ closed subscheme. Then there exists Y regular complete noetherian local, equichar. p, $\dim(Y) < d$, $X' = Y$-scheme of f. t., $Z' \subset X'$ closed subscheme, $x \in X'$ a closed point above the closed point of Y, such that (X, Z) is the completion of (X', Z') at x :

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ X \arrow[r] & X' \arrow[d] \\ & Y \end{tikzcd}$$

(b) X complete noetherian local, *normal*, *mixed char.* $(0, p)$, dim $d \geq 2$, ℓ invertible on X, $Z \subset X$ a closed subscheme. Then there exists X_1 complete noetherian local, normal, finite over X, with *generic degree prime to* ℓ, Y regular complete noetherian local, mixed char. $(0, p)$, $\dim(Y) < d$, $X' = Y$-scheme of f. t., $Z' \subset X'$ closed subscheme, $x \in X'$ a closed point above the closed point of Y, such that (X_1, Z_1) (where $Z_1 = X_1 \times_X Z$) is the completion of (X', Z') at x :

$$\begin{tikzcd} Z \arrow[d] & Z_1 \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[d] \\ X & X_1 \arrow[l] \arrow[r] & X' \arrow[d] \\ & & Y \end{tikzcd}$$

Outline of proof given in Orgogozo's talk (note : only weaker form (no ℓ) needed for Gabber-Orgogozo's th. on p-dimension of fields). Relies on Elkik's algebraization th. for finite gen. étale maps, Epp's potential reducedness th., using the following refinement of Cohen structure th.

Lemma 1 (Cohen-Gabber). A complete noetherian local, *reduced*, equichar., residue field $k = A/m$, G a finite group acting on A, with $|G|$ invertible in A. Then there exists a G-equivariant map $h : k[[t_1, ..., t_d]] \to A$, with $k \to A$ lifting Id_k, G acting trivially on the t_i's, and h finite and generically étale.

Refines [EGA 0_{IV} 19.8.8 (ii)] in several respects : action of G, "reduced" instead of "integral" in the hypotheses, and "generically étale" in the conclusion.

(3) *Outline of proof of th. 2 (using (1) and th. 4)*

Using induction on dimension, we assume uniformization OK in dim $< d$. By step (1) and th. 4, as $X_1 \to X$ in th. 4 is covering for the ℓ'-topology, WMA the pair (X, Z) with X normal integral (and $Z \subset X$ closed, nowhere dense), equipped with a morphism $f : X \to Y$, with Y qe normal integral of dim. $< d$, and the generic fiber of f a curve. We have a closed point x in Z and we want to show that locally around x for the ℓ'-topology we can uniformize (X, Z). WMA X and Y affine, so that compactifying f (and normalizing and changing notations) WMA Y affine and f projective. From now on we forget about x and try to find (globally) a covering family $(X_i \to X)$ for the ℓ'-topology which uniformizes (X, Z). Replacing X by some blow-up (and Z by its inverse image), WMA Z is a *divisor*.

Starting point : Apply de Jong's th. on nodal curves [dJ, 2.4] to $(f : X \to Y, Z)$: get

• a finite group G and a *commutative* square (non necessarily cartesian) of G-schemes, with projective map :

$$(*) \qquad \begin{tikzcd} X' \arrow[r, "a"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"] & Y \end{tikzcd} ,$$

together with

• a divisor D in X'

• a G-stable closed and nowhere dense subset $T' \subset Y'$

satisfying the following properties :

- G acts trivially on X, Y, faithfully on X', Y', and *freely* on $Y' - T'$
- a, b are alterations and $X'/G \to X$, $Y'/G \to Y$ generically radicial
- f' is a nodal curve, smooth outside T'
- D is étale over Y and contained in the smooth locus of f
- $Z' := a^{-1}(Z)$ is contained in $D \cup f'^{-1}(T')$

Goal : (i) using the induction assumption on a suitable quotient of Y', make the pair (X', Z') log regular

(ii) get rid of the ℓ-part of a, i. e. ensure that the quotient (X_1, Z_1) of (X', Z') by an ℓ-Sylow of G is still log regular.

Once (i) and (ii) are achieved, Kato-Niziol's desingularization of (X_1, Z_1) completes the proof.

Recall (Kato) : An fs noetherian log scheme X is called *log regular* if at each geom. pt $\overline{x} \to x \in X$, $\operatorname{Spec} \mathcal{O}_{X,\overline{x}}/I_{\overline{x}}$ is regular, where $I_{\overline{x}} = \alpha(M_{\overline{x}} - \mathcal{O}^*_{X,\overline{x}})$, and $\dim \mathcal{O}_{X,\overline{x}}/I_{\overline{x}} + \operatorname{rk} \overline{M}^{\mathrm{gp}}_{\overline{x}} = \dim \mathcal{O}_{X,\overline{x}}$ (NB. $\overline{M} := M/\mathcal{O}^*$; $I_{\overline{x}}$ defines the stratum at $\overline{x}$ of the stratification of X by the rank of $\overline{M}^{\mathrm{gp}}$). A pair (X, Z) (Z closed in X) is called *log regular* if X is log regular and Z is the complement of the open subset $j : U \to X$ of triviality of the log structure ; in this case, $M_X = \mathcal{O}_X \cap j_* \mathcal{O}^*_U$. Log regular $\Rightarrow$ toric singularities. Log smooth over log regular $\Rightarrow$ log regular. (X, Z) log regular and X regular (as a scheme) $\Rightarrow Z =$ dnc.

The goal is achieved in 2 main steps :

Step 1 : Extracting the ℓ-part and making (X', Z') log regular

As $X'/G \to X$ is covering for the ℓ'-topology, WMA $Y = Y'/G$, $X = X'/G$. Fix an ℓ-Sylow H of G. Consider the factorization

$$\begin{tikzcd} X' \arrow[r, "a_1"] \arrow[d, "f'"] & X'/H \arrow[r, "a_2"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "b_1"] & Y'/H \arrow[r, "b_2"] & Y \end{tikzcd} .$$

As a_2 is covering for the ℓ'-topology, we may replace X by X'/H and Z by its inverse image in X'/H, Y by Y'/H, and then G by H, so that we may assume that *G is an ℓ-group.*

Apply induction assumption to $(Y, T := T'/G)$. Get a family (Y_i, T_i) uniformizing (Y, T) ($(Y_i \to Y)$ covering for the ℓ'-topology, Y_i regular connected, $T_i =$ sncd). Take "normalized pull-back" by b, i. e. let $Y'_i =$ normalization of a component of $Y' \times_Y Y_i$,. Replace Y by Y_i, Y' by Y'_i, G by the decomposition group G_i of Y'_i, and take pull-back of the other data by $Y_i \to Y$, $Y'_i \to Y'$. Working separately over each Y_i, and changing notation, WMA that in diagram (*), $Y = Y'/G$ is regular connected, $T = T'/G$ is a sncd, $X = X'/G$. As Y' is normal integral, and G acts freely on $Y' - T'$ and is an ℓ-group, and $T = T'/G$ is a sncd, Y' admits a unique log structure which makes $Y' \to Y$ a finite Kummer étale cover of (Y, T) of group G. In particular, (Y', T') is log regular. Then, the local structure of nodal curves implies that $(X', f'^{-1}(T') \cup D)$ is log smooth over (Y', T') (with open subset of triviality in X' contained in $f'^{-1}(Y' - T')$), and in particular log regular. For simplicity we will assume that $Z' = f'^{-1}(T') \cup D$ (the general case follows easily from this special one).

Step 2 : making the action of G very *tame.*

As G is an ℓ-group, G acts *tamely* on X', i. e. the inertia group at each x in X' is of order invertible in $k(x)$. If the quotient $(X = X'/G, Z = Z'/G)$ was log regular, then, by Kato-Niziol's log desingularization, we would be home. However, the tameness of the action of G is not enough to ensure that (X, Z) is log regular. For example, if k is alg. closed of char. $\neq 2$, V a vector space of dim. 2 over k, with trivial log structure, and $G = \mathbb{Z}/2\mathbb{Z} = \{1, \tau\}$ acting by $\tau(v) = -v$, then V/G is singular (a quadratic cone).

To overcome this Gabber introduces the following notion :

Def 1 X noetherian, separated, fs log regular, with G finite, acting admissibly on X. Action of G called *very tame* if :

- $\forall x \in X$, $|G_x|$ is invertible in $k(x)$, where $G_x =$ inertia group at x,
- G acts generically freely,
- For all geom. pt $\overline{x} \mapsto x \in X$, G_x acts *trivially* on $\overline{M}_{\overline{x}}$ and on the connected stratum of the stratification by $\mathrm{rk}\, \overline{M}^{\mathrm{gp}}$ containing x.

Lemma 2. Suppose G acts very tamely on a (noetherian, separated, fs) log regular (X, Z). Then G acts freely on $X - Z$, $(X/G, Z/G)$ is log regular, and $X \to X/G$ is a Kummer étale cover of group G (ramified along Z/G).

Back to Step 2. To complete it we need the following th. :

Th. 5 (Gabber). Let G act tamely and generically freely on a (noetherian, separated, fs) log regular (Y, T). Then there exists a G-equivariant modification $p : Y' \to Y$, with a log regular str. (Y', T') (acted on by G) such that $Y' - T' \subset p^{-1}(Y - T)$, and G acts *very* tamely on (Y', T').

Apply th. 5 to the action of G on $(X', Z') = (Y, T)$. Get a commutative diagram

$$\begin{tikzcd}
(X'', Z'') \arrow[r] \arrow[d, "p"] & (X''/G, Z''/G) \arrow[d, "q"] \\
(X', Z') \arrow[r] & (X, Z) = (X'/G, Z'/G)
\end{tikzcd} ,$$

where $(X''/G, Z''/G)$ is log regular (by Lemma 2) and q is a modification. Finally apply Kato-Niziol's desingularization to $(X''/G, Z''/G)$: find log blow-up $(X''', Z''') \to (X''/G, Z''/G)$ with X''' *regular* and Z''' a sncd in X'''.

About the proof of th. 5. Use Bierstone-Milman "functorial" desingularization in char. 0 to get a G-equivariant map of log schemes $p_1 : (Y_1, T_1) \to (Y, T)$ with Y_1 regular, T_1 a sncd, and p_1 projective, birational. So WMA Y regular, T a sncd in Y. To make the action very tame, several pbs must be fixed :

(a) Replacing Y by suitable blow-ups of intersections of components of T (and T by the corresponding inverse images) make the actions of the inertia groups G_x trivial on $\overline{M}_x$.

(b) There may be some fixed points loci of G (or of nontrivial subgroups of G) in $Y - T$. By blowing them up and enlarging T accordingly, one makes G act freely on $Y - T$.

(c) It remains to make the inertia groups act trivially on the strata. This requires a delicate argument using canonical desingularizations again.

Gabber proves the following important :

Complement to Th. 5 : if (Y, T) is log smooth over some base S with trivial G-action, then one can ensure that (Y', T') and $(Y'/G, T'/G)$ are log smooth over S. Combined with de Jong, this has the following consequences :

Cor. 1. Let X be separated and of finite type over a field k, $Z \subset X$ a nowhere dense closed subset, ℓ a prime $\neq \mathrm{char}(k)$. Then there exists a finite extension k' of k of order prime to ℓ and an ℓ'-alteration $p : X' \to X$ over $\mathrm{Spec}\, k' \to \mathrm{Spec}\, k$ with X' smooth over k' and $p^{-1}(Z)$ the support of a sncd in X'.

(See comment after th. 2 for the definition of an ℓ'-alteration.)

Cor. 2. Let S be a trait, X separated and of finite type over S, ℓ a prime invertible on S, $Z \subset X$ a nowhere dense closed subset. Then there exists a finite extension S'/S of degree prime to ℓ and an ℓ'-alteration $p : X' \to X$ over $S' \to S$, such that X' is regular,

T' the support of a sncd, $Z' = p^{-1}(Z)$ a subdivisor of T', with (X', T') iog smooth over S' (with its standard log structure).

(Idea : use [dJ, 5.16] to get plurinodal fibrations, then use Gabber-Vidal [Vi 4.4.4 + rem] to get equivariant semistable model. Then divide by ℓ-Sylow, using th. 5.)

3. Other applications of uniformization to étale cohomology

3.1. *New proof of absolute purity.* First part of old proof reduces to proving punctual purity for regular schemes of finite type over an excellent trait. Then, instead of K-theoretic arguments, use *refined uniformizaton*, namely Cor. 2 above to reduce to X regular *and log smooth over* S. In this case, punctual purity follows from $\Lambda_X = Rj_*^{\mathrm{ket}}\Lambda_{X_\eta}$ [I, 2.3].

3.2. *Affine Lefschetz.* Key case :

X local henselian excellent, $\dim(X) = d$, U open affine in X, ℓ invertible on X. Then : $\mathrm{cd}_\ell(U) \leq d + \mathrm{cd}_\ell(k)$ (k the residue field).

Implies : for X integral, field of fractions K, $\mathrm{cd}_\ell(K) = d + \mathrm{cd}_\ell(k)$. (M. Artin's conjecture [SGA 4 X 3.1]).

See Orgogozo's talk for variants *à la* Kato for p-dimension.

Uses only *weak uniformization.*

3.3. *Dualizing complexes.* Exist over any excellent noetherian X having a dimension function, stable under $f^!$, Λ_X dualizing if X regular ($\Lambda = \mathbb{Z}/n\mathbb{Z}$, n invertible on X).

Uses absolute purity, a theory (Gabber) of Gysin maps for globally smoothable locally complete intersection morphisms, and ths. 1 and 4.

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