On uniform bound of the maximal subgroup of the inertia group acting unipotently on ℓ-adic cohomology

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abstract

For a smooth projective variety over a local field, the action of the inertia group on the ℓ-adic cohomology group is unipotent if it is restricted to some open subgroup.

We give an ℓ-independent uniform bound of the index of the maximal open subgroup satisfying this property. This bound depends only on the Betti numbers of X and certain Chern numbers of X.

1 Background

setting

K : local field, p : the residue characteristic of K, ℓ : prime number not equal to p.
$\bar{K}$: algebraic closure of K. I_K : the inertia group of K.

Grothendieck's monodromy theorem [Appendix, 1]

For a smooth projective variety X over K, the action of I_K on $H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$ is quasi-unipotent.
Namely, there exists an open subgroup I of I_K to which the restriction of the action is unipotent.

This open subgroup I is finite index in I_K. We study about this index.

2 Main Theorem

Before we state our main theorem, we define some invariants and constants.

Invariants

X : smooth projective variety over K ,L : very ample invertible sheaf on X

define $b_i(X)$ and $c_i(X, L)$ for every natural number i

- $b_i(X) = \dim H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$
- $c_i(X, L) = f_*(c(\Omega^1_X)c(L^{\oplus i})^{-1}c_i(L^{\oplus i}))$

$c(E)$: the total ℓ-adic Chern class, $c_i(E)$: the i-th ℓ-adic Chern class of locally free sheaf E.
The map f_* is the Gysin map defined by structure map of X.

We know that

- b_i is non-negative integer and c_i is integer.
- $EP(X_j) = c_j(X, L)$ for X_j is j times hyperplane section of X defined by L.
- b_i and c_i are independent of choice of ℓ.

Constants

k : positive integer, ℓ : prime number
we define a group $G_{\ell,k}$ and a constant C_k

- $G_{\ell,k} = GL_k(\mathbb{F}_\ell)$ if $\ell \neq 2$ and $G_{2,k} = GL_k(\mathbb{Z}_2/4\mathbb{Z}_2)$.
- $C_k =$ the g.c.d. of $\sharp G_{\ell,k}$ for $\ell \neq p$

n : positive integer, $b = (b_0, \cdots, b_n) \in \mathbb{N}^{n+1}$, $c = (c_1, \cdots, c_{n-1}) \in \mathbb{Z}^{n-1}$. We define $b' = (b'_0, \cdots, b'_n) \in \mathbb{N}^{n+1}$, and a constant $C_n(b, c)$ by

- $b'_j = (-1)^j \left(c_{n-j} - 2\sum_{i=0}^{i=n-j-1}(-1)^i b_i\right)$ for $j \neq 0, n$
- $b'_j = b_j$ for $j = 0, n$
- $C_n(b, c) = \prod_{j=1}^{j=n} C_{b'_j}$

Our main theorem is the following.

Main Theorem

n : positive integer, $b \in \mathbb{N}^{n+1}$, $c \in \mathbb{Z}^{n-1}$.

X : smooth projective variety over K, L : very ample invertible sheaf on X

- $\dim X = n$
- $(b_i(X))_{i=0,\dots,n} = b$
- $(c_i(X, L))_{i=1,\dots,n-1} = c$

there exists an open subgroup I of I_K such that

- index $[I_K : I]$ divides $C_n(b, c)$
- the action of I on $H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unipotent for every i and every $\ell \neq p$
- $H^i(X_{\bar{K}}, \mathbb{Q}_p)$ is semi-stable representation of I

3 Outline of the proof

We prove the main theorem by induction on dimension of X.

key fact

(ρ, V) : ℓ-adic representation of I_K, $\dim V = n$.

Assume $\mathrm{Im}(\rho) \subset \mathrm{GL}_n(\mathbb{Z}_\ell) \subset \mathrm{GL}_n(\mathbb{Q}_\ell) \cong \mathrm{GL}(V)$.

$$\begin{cases} \rho(\sigma) \in 1 + \ell \mathrm{M}_n(\mathbb{Z}_\ell) & \ell \neq 2 \\ \rho(\sigma) \in 1 + \ell^2 \mathrm{M}_n(\mathbb{Z}_\ell) & \ell = 2 \end{cases} \Longrightarrow \rho(\sigma) \text{ is unipotent.}$$

By using this fact,

- $\{\sigma \in I_K | \rho(\sigma) \text{ is unipotent}\} = \{\sigma \in I_K | \mathrm{Tr}(\rho(\sigma); V) = n\}$
- this subset is an open subgroup of I_K, index $[I_K : I]$ dividing $C_{\ell,n}$.

Lemma

$\{(\rho_\ell, V_\ell)\}_{\ell \neq p}$: family of ℓ-adic representations of I_K such that

- $\mathrm{Tr}(\rho_\ell) = \mathrm{Tr}(\rho_{\ell'})$ for every ℓ, ℓ'
- $\dim V_\ell = n$.

Then there exists an open subgroup I of I_K

- $\rho_\ell\,|_I$ is unipotent for every $\ell \neq p$
- index $[I_K : I]$ divides C_n.

one dimensional case

X : curve, the trace of $H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$ is independent of ℓ. By using this lemma, we prove the theorem.

higher dimensional case

Take a smooth j times hyperplane section X_j of X defined by L. Then

- $H^i(X_{\bar{K}}, \mathbb{Q}_\ell) \cong H^i(X_{j,\bar{K}}, \mathbb{Q}_\ell)$ for $i < n - j$
- $H^{n-j}(X_{\bar{K}}, \mathbb{Q}_\ell) \subset H^{n-j}(X_{j,\bar{K}}, \mathbb{Q}_\ell)$
- the invariants $b_i(X_j)$ and $c_i(X_j, L)$ is determined by b and c.

ℓ-independence of alternating sum [2]

X proper smooth variety over K, $\sigma \in I_K$,

- $\sum_{i=0}^{2n}(-1)^i \mathrm{Tr}(\sigma; H^i(X_{\bar{K}}, \mathbb{Q}_\ell))$ is independent of ℓ
- $\sum_{i=0}^{2n}(-1)^i \mathrm{Tr}(\sigma; H^i(X_{\bar{K}}, \mathbb{Q}_\ell)) = \sum_{i=0}^{2n}(-1)^i \mathrm{Tr}\left(\sigma; \hat{D}_{pst}(H^i(X_{\bar{K}}, \mathbb{Q}_p))\right)$

induction step

Sequence of hyperplane sections $X \supset X_1 \supset \cdots \supset X_{n-1}$.
Construct sequence of subgroups $I \supset I_1 \supset \cdots \supset I_{n-1}$.

1. open subgroup $I_j \subset I$ such that the action on $H^i(X_{n-j,\bar{K}}, \mathbb{Q}_\ell)$ is unipotent for every i.
2. the trace of I_j on $H^{j+1}(X_{n-(j+1),\bar{K}}, \mathbb{Q}_\ell)$ is independent of ℓ.
3. open subgroup I_{j+1} such that the action on $H^{j+1}(X_{n-(j+1),\bar{K}}, \mathbb{Q}_\ell)$ is unipotent for every ℓ.

The condition on the index is satisfied because of the definition of the invariants and the constants.

4 Global case

setting

K : global field
constant $C'(b, c)$: defined similar as $C_n(b, c)$.

Global version of Main Theorem

n : positive integer, $b \in \mathbb{N}^{n+1}$, $c \in \mathbb{Z}^{n-1}$.

X : smooth projective variety over K, L : very ample invertible sheaf on X

- $\dim X = n$
- $(b_i(X))_{i=0,\dots,n} = b$
- $(c_i(X, L))_{i=1,\dots,n-1} = c$

there exists a finite extension K' of K such that

- degree of the extension K'/K divides $C'_n(b, c)$
- v : place of K', the action of I_v on $H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unipotent for v not above ℓ
- v : place of K', the action of I_v on $H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$ is semi-stable for v above ℓ

5 Reference

1. J. P-. Serre, J. Tate, Good reduction of abelian varieties, Ann. of Math. 88 (1968),492-517
2. T. Ochiai, ℓ-independence of the trace of monodromy, Math. Ann. 315 (1999),321-340