Kernel of the monodromy operator for semistable curves

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joint work with B. Chiarellotto, R. Coleman and A. Iovita

We reprove Chiarellotto's theorem that the invariant cycles for the monodromy acting on the de Rham cohomology of the generic fiber of a semistable curve are the rigid cohomology of the special fiber. This is done in an explicit way along the line of Coleman and Iovita's work. They are also able to define monodromy for coefficients: we extend such an invariant cycles sequence to the unipotent coefficients where we show that it is not always exact. We give an example of such a behaviour together with a general condition for the non-exactness.

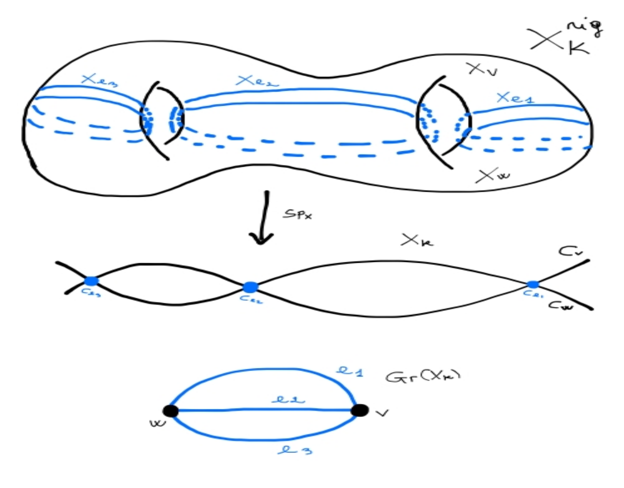
Setting and notations

- V: complete discrete valuation ring of mixed characteristic $(0,p)$, uniformizer π, fraction field K, perfect residue field k
- X/V: semistable projective curve, *i. e.* Zariski-locally étale over $\mathrm{Spec}(V[x,y]/xy-\pi)$, X_k the special fiber, X_K the generic fiber, X_K^{rig} the rigid analytic fiber. We assume that X_k is the union of at least two smooth irreducible components
- $\mathrm{sp}_X : X_K^{\mathrm{rig}} \to X_k$ Berthelot's specialization map
- Fix $X \hookrightarrow P$ an immersion of X into P a projective space of some dimension
- Let $X_k \hookrightarrow \mathscr{P}$ with $\mathscr{P}$ the formal scheme associated to P. We denote by $]X_k[_{\mathscr{P}}$ the tube of X_k in $\mathscr{P}$
- $\mathcal{E}$: overconvergent F-isocristal on X_k. It induces a module with connection (E,∇_E) on $]X_k[_{\mathscr{P}}$ and on X_K^{rig} by pullback
- $H^i_{\mathrm{rig}}(X_k/K,\mathcal{E}) := H^i_{\mathrm{dR}}(]X_k[_{\mathscr{P}},(E,\nabla_E))$
- $H^i_{\mathrm{log-crys}}(X_k,\mathcal{E})\otimes K = H^i_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E))$

A Mayer-Vietoris exact sequence [Co89], [CoIo10]

One can associate to X_k a graph $\mathrm{Gr}(X_k)$ with $\mathscr{V}$ set of vertices and $\mathscr{E}$ set of edges:

- C_v irreducible component of $X_k \leftrightarrow v$ vertex of $\mathrm{Gr}(X_k)$
- C_e a point in the intersection between C_v and C_w in $X_k \leftrightarrow e=[v,w]$ edge between v and w
- $X_v := \mathrm{sp}_X^{-1}(C_v)$ is a wide open in X_K^{rig}; $X_e := \mathrm{sp}_X^{-1}(C_e)$ is an open annulus.
- $C_v \cap C_w = C_e \Leftrightarrow X_v \cap X_w = X_e$
- $\{X_v\}_{v\in\mathscr{V}}$ is an admissible covering of X_K^{rig} consisting of Stein spaces



Given the admissible covering $\{X_v\}_{v\in\mathscr{V}}$ we can write the Mayer-Vietoris sequence:

$$\begin{tikzcd} \oplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(X_v,(E,\nabla_E)) \arrow[r,"\alpha"] & \oplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e,(E,\nabla_E)) \arrow[r] & H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E)) \arrow[dll] \\ \oplus_{v\in\mathscr{V}} H^1_{\mathrm{dR}}(X_v,(E,\nabla_E)) \arrow[r,"\beta"] & \oplus_{e\in\mathscr{E}} H^1_{\mathrm{dR}}(X_e,(E,\nabla_E)). & \end{tikzcd} \quad (1)$$

If we denote by $H^1(\mathrm{Gr}(X_k),\mathcal{E}) := \mathrm{Coker}(\alpha)$, we can deduce the following short exact sequence

$$0 \longrightarrow H^1(\mathrm{Gr}(X_k),\mathcal{E}) \xrightarrow{\gamma} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E)) \longrightarrow \mathrm{Ker}(\beta) \longrightarrow 0.$$

Monodromy operator [Co94], [CoIo99], [CoIo10]

An element $[\omega]$ in $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E))$ is represented by an hypercocycle $((\omega_v)_{v\in\mathscr{V}},(f_e)_{e\in\mathscr{E}})$, with (ω_v) in $\Omega^1_{X_v}\otimes E_{X_v}$ and f_e in E_{X_e} such that $\omega_{v|X_e}-\omega_{w|X_e} = \nabla_E(f_e)$ if $e=[v,w]$.

The space X_e is an open annulus; as usual we can define a residue map

$$\mathrm{Res} : H^1_{\mathrm{dR}}(X_e,(E,\nabla_E)) \to H^0_{\mathrm{dR}}(X_e,(E,\nabla_E))$$

which is an isomorphism of K-vector spaces.

The monodromy operator is defined as follows:

$$N_{\mathcal{E}} : H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E)) \to H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E))$$

$$[\omega] \mapsto i(\mathrm{Res}(\omega_{v|X_e})_{e=[v,w]})$$

where $[\omega]$ is represented by $((\omega_v)_{v\in\mathscr{V}},(f_e)_{e\in\mathscr{E}})$ and i is the map induced by the Mayer-Vietoris sequence

$$i : \oplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e,(E,\nabla_E)) \to H^1(Gr(X_k),\mathcal{E}) \xrightarrow{\gamma} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E))$$

$$i(f_e)_{e\in\mathscr{E}} = \Big(0, f_e/\mathrm{Imm}(\alpha)\Big)_{v\in\mathscr{V},e\in\mathscr{E}}$$

Remark 1. *If $\mathcal{E}$ is the trivial F-isocrystal, this definition of monodromy agrees with Hyodo-Kato's.*

Monodromy and rigid cohomology

By hypothesis $X_k \hookrightarrow \mathscr{P}$ with $\mathscr{P}$ a formally smooth scheme over V,

$$\begin{tikzcd} & & \mathscr{P}_K^{\mathrm{rig}} \arrow[dl,"\mathrm{sp}_{\mathscr{P}}"'] \arrow[d] \\ X_k \arrow[r] & \mathscr{P}_k \arrow[r] & \mathscr{P} \end{tikzcd}$$

If $Y_K := \mathrm{sp}_{\mathscr{P}}^{-1}(X_k) =]X_k[_{\mathscr{P}}$, the first rigid cohomology is defined as

$$H^1_{\mathrm{rig}}(X_k/K,\mathcal{E}) := H^1_{\mathrm{dR}}(Y_K,(E,\nabla_E)).$$

We can complete the above diagram as follows

$$\begin{tikzcd} X_K^{\mathrm{rig}} \arrow[d,"\mathrm{sp}_X"] & & \mathscr{P}_K^{\mathrm{rig}} \arrow[dl] \arrow[d] \\ X_k \arrow[r] & \mathscr{P}_k \arrow[r] & \mathscr{P} \end{tikzcd}$$

If we denote by $\varphi : X_K^{\mathrm{rig}} \longrightarrow Y_K$ the map induced by the immersion of X into P, we get a map

$$\varphi^* : H^1_{\mathrm{dR}}(Y_K,(E,\nabla_E)) \longrightarrow H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E)).$$

Using φ^* and $N_{\mathcal{E}}$ we can form the following sequence

$$H^1_{\mathrm{dR}}(Y_K,(E,\nabla_E)) \xrightarrow{\varphi^*} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E)) \xrightarrow{N_{\mathcal{E}}} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(E,\nabla_E)). \quad (2)$$

Lemma 2. *If $\mathcal{E}$ is an overconvergent F-isocrystal and (E,∇_E) is the coherent module with integrable connection attached to it, then the map φ^* in the sequence (2) is injective and*

$$N_{\mathcal{E}}\circ\varphi^* = 0.$$

Theorem 3 ([Ch99], [Na06]). *In the sequence (2) if $(E,\nabla) = (\mathcal{O}_X,d)$ is the module with connection induced by the trivial F-isocrystal $\mathcal{O}$, then* $\mathrm{Imm}\varphi^* = \mathrm{Ker}N_{\mathcal{O}}$.

Proof. We give a new proof using graph theory. □

Remark 4. *Theorem 3 gives an intepretation à la Fontaine of the first rigid cohomology group of a semistable curve. We can translate the exactness of the sequence in (2) in the unramified case as follows: since*

$$D_{\mathrm{st}}(H^1_{\mathrm{\acute{e}t}}(X_K\times\bar{K}),\mathbb{Q}_p) = H^1_{\mathrm{log-crys}}(X_k)\otimes K$$

$$D_{\mathrm{st}}^{N=0} = D_{\mathrm{crys}},$$

then

$$H^1_{\mathrm{rig}}(X_k/K) = D_{\mathrm{crys}}(H^1_{\mathrm{\acute{e}t}}(X_K\times\bar{K}),\mathbb{Q}_p)$$

Unipotent coefficients

We analyze the case of unipotent F-isocrystal proving that (2) falis to remain exact.
We consider the following non-split exact sequence of F-isocrystals

$$0\to\mathcal{E}\to\mathcal{G}\to\mathcal{O}\to 0 \quad (3)$$

with $\mathcal{E}$ a unipotent overconvergent F-isocrystal such that the sequence in (2) is exact.
We suppose that (3) corresponds to the element $x\neq 0\in H^1_{\mathrm{rig}}(X_k/K,\mathcal{E})$

Theorem 5. *If $\varphi^*(x) = N_{\mathcal{E}}(y)$ for $y\in H^1_{\mathrm{dR}}(X_K,(E,\nabla_E))$ and if*

$$\alpha_{\mathrm{log-crys}} : H^1_{\mathrm{log-crys}}(X_k,\mathcal{E}) \to H^1_{\mathrm{log-crys}}(X_k,\mathcal{G})$$

is the map in cohomology induced by the injective map in the exact sequence (3) then

$$\mathrm{Kernel}(N_{\mathcal{G}}) = H^1_{\mathrm{rig}}(X_k/K,\mathcal{G})\oplus K\cdot\alpha_{\mathrm{log-crys}}(y)$$

Example 6. *Let X be the Tate elliptic curve with invariant q, where $q\in\mathcal{O}_K$ with $\mathrm{ord}q=3$. Then the graph associated to X is a triangle with vertices I, II, III and edges $[I,II],[II,III],[I,III]$.*
Pick $x\in H^1_{\mathrm{rig}}(X_k/K,\mathcal{O})$ such that $\varphi^*(x)\in\mathrm{Imm}(N_{\mathcal{O}})$ and consider the corresponding extension of F-isocrystals

$$0\to\mathcal{O}\to\mathcal{G}\to\mathcal{O}\to 0.$$

The element $\varphi^*(x)\in H^1_{\mathrm{log-crys}}(X_k,\mathcal{O})$, as hypercocycle, can be written as $((0_v)_{v\in\mathscr{V}},(g_e)_{e\in\mathscr{E}})$ with $g_e\in H^0(X_e,\mathcal{O}_X)$. Making explicit calculations we characterize the vector space $\mathrm{Kernel}(N_{\mathcal{G}})/H^1_{\mathrm{rig}}(X_k/K,\mathcal{G})$ in terms of conditions on the residue of hypercocylces in $H^1_{\mathrm{log-crys}}(X_k,\mathcal{O})$.

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