On good reduction of some K3 surfaces

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Abstract

For abelian varieties over local fields, the properties of the l-adic étale cohomology of a variety as a Galois representation is enough to tell whether the variety have good reduction or not. I will show similar results for some special kind of K3 surfaces.

Notation: Let K be a local field of characteristic $(0, p)$, $p \geq 0$, with perfect residue field. Its absolute Galois group is denoted by G_K. A G_K-module is called *unramified* if the inertia subgroup I_K $(\subset G_K)$ acts trivially.

Introduction

We want to determine whether a variety over a local field has good reduction from Galois representations. First result in this direction is the following.

Theorem 1 (Néron–Ogg–Šafarevič criterion)**.** *Let E be an elliptic curve. Then E has good reduction if and only if the Tate module $T_l(E)$ is unramified for all/some prime $l \neq p$.*

Serre–Tate (1968) have extended this to abelian varieties of arbitrary dimension.

One direction of this theorem can be extended to general varieties using l-adic étale cohomology:

Theorem 2 (SGA4, Exposé XVI)**.** *If a variety X has good reduction, then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified for all i and all $l \neq p$.*

(Note that if A is an abelian variety then we have $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_l)^\vee \cong T_l(A) \otimes \mathbb{Q}_l$ as G_K-modules.)

So we naturally come to the following problem:

Problem 3. For what kind of varieties (other than abelian varities) the converse holds, i.e. the cohomology being unramified implies having good reduction?

Remark. Not all varieties satisfy this condition: for example, if C is a curve of genus ≥ 2, and if $H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{Q}_l)$ is unramified, then its Jacobian variety $J(C)$ (which has the same $H^1_{\text{ét}}$ as C) has indeed good reduction, but C itself does not have good reduction in general.

On 2001 Tetsushi Ito proved the following results of Kummer surfaces (which are special kinds of K3 surfaces):

Theorem 4 (Ito 2001, unpublished)**.** *Let K be a local field with residue characteristic $p \neq 2$ and l a prime number $\neq p$. Let X be a Kummer surface over K. Assume that X has at least one K-rational point. If $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified, then $X_{K'}$ has good reduction for some finite unramified extension K'/K.*

(A Kummer surface $\mathrm{Km}\, A$ is the minimal desingularization of the quotient of an abelian surface A by the involution $x \mapsto -x$.)

My Results

I proved that, for another kind of K3 surfaces, the unramifiedness of H^2 implies good reduction after some base change:

Theorem 5 (M.)**.** *Let K be a local field with residue characteristic $p \neq 2, 3$ and l a prime number $\neq p$. Let Y be a K3 surface over K admitting a Shioda–Inose structure of product type. If $H^2_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)$ is unramified, then $Y_{K'}$ has good reduction for some finite extension K' which is a finite extension of K of ramification index* 1, 2, 3, 4 *or* 6*.*

(A K3 surface admitting a Shioda–Inose structure of product type means that it is the quotient of an product Kummer surface $\mathrm{Km}(E_1 \times E_2)$ by a certain involution. We omit the details.)

Sketch of Proof. Relate the $H^2_{\text{ét}}$ of Y and the corresponding Kummer surface $X = \mathrm{Km}(E_1 \times E_2)$ and derive that $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified (after taking finite extension); Use Theorem 4 to obtain a model $\mathcal{X}$ of X; Then divide $\mathcal{X}$ by a certain involution and we get a (smooth proper) model $\mathcal{Y}$ of Y. □

Unlike Ito's result, I could not (at the moment) show that we can take the field extension to be unramified.

Remark. The moduli of such K3 surfaces is in one-to-one correspondence with the moduli of product abelian surfaces, and hence is 2-dimensional.

p-adic version

(Let $p > 0$.) For p-adic representations, the condition "unramified" seems to be too restrictive. Instead, "crystalline" p-adic representations are considered to be the p-adic counterpart of the notion of unramified l-adic representations ($l \neq p$).

Theorems 1 and 2 remains true if we replace l by p and "unramified" by "crystalline" (Faltings 1989, Tsuji 1999, Coleman–Iovita 1999).

I proved that Theorems 4 and 5 are also true in this p-adic situation.

Application: "singular" K3 surfaces

Recall that a K3 surface over a field of characteristic 0 is called *singular* if it has the maximum possible Picard number 20.

Corollary 6 (M.)**.** *Any singular K3 surface has potential good reduction.*

This may be considered as an analogue of the fact that any elliptic curve with complex multiplication has potential good reduction.

Forthcoming?

I think I have proved the following theorem, which treats a wider class of K3 surfaces. Details need to be checked.

Theorem 7 (?)**.** *Let X a K3 surface over K which admits a very ample line bundle L of degree $2d$ with $p > 2d + 4$ (if $p \neq 0$). If $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is [potentially] unramified, then for some finite extension K'/K,*

1. *There exists a scheme $\mathcal{X}$ proper over $\mathcal{O}_{K'}$, with generic fiber $X_{K'}$, such that its special fiber has at most RDP (rational double point) singularity.*
2. *There exists an* algebraic space *$\mathcal{X}'$ proper and smooth over $\mathcal{O}_{K'}$ with generic fiber $X_{K'}$. In other words, X has potential good reduction "as algebraic space".*

Remark. The condition on the degree of the line bundle forces p to be ≥ 11 (unless $p = 0$). If $p \geq 11$, there exists a 19-dimensional family of (X, L) satisfying the condition (e.g. for $L^2 = 2d = 4$).