About modulo p representations of p-adic reductive groups of rank 1

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- We denote by F a finite extension of $\mathbb{Q}_p$ or a Laurent series field $\mathbb{F}_q((t))$ with residue class field $\mathbb{F}_q$ of cardinality $q = p^f$. We fix a uniformizer ϖ of F and an embedding of $\mathbb{F}_q$ into a fixed algebraic closure $\overline{\mathbb{F}}_p$.
- We denote by $\mathcal{G}$ a connected reductive group which is defined, quasi-split and of rank 1 over F, and we let $G := \mathcal{G}(F)$ be the group of its rational points. We choose a maximal split torus $\mathcal{S}$ of $\mathcal{G}$, we let $\mathcal{T}$ be its centralizer in $\mathcal{G}$, and we fix a parabolic subgroup $\mathcal{B}$ (which is a Borel subgroup as $\mathcal{G}$ is quasi-split of rank 1 over F) containing $\mathcal{T}$ as a Levi subgroup. As $\mathcal{T}$ is a torus, any irreducible smooth representation of $T := \mathcal{T}(F)$ over $\overline{\mathbb{F}}_p$ is a character, and any smooth character $B \to \overline{\mathbb{F}}_p^\times$ comes from such a character.

Non-supercuspidal representations

An irreducible smooth representation of G is called *supercuspidal* if it can't be written (up to isomorphism) as a subquotient of some parabolically induced representation of G. We first have to describe all the non-supercuspidal representations of G over $\overline{\mathbb{F}}_p$, what leads to the following theorem.

Theorem 1 ([A3]). Let $\chi : B \to \overline{\mathbb{F}}_p^\times$ be a smooth character.

- The following statements are equivalent :
  - $i)$ $\mathrm{Ind}_B^G(\chi)$ is an irreducible $\overline{\mathbb{F}}_p[G]$-module;
  - $ii)$ $\mathrm{Ind}_B^G(\chi)$ is an indecomposable $\overline{\mathbb{F}}_p[B]$-module;
  - $iii)$ the character χ doesn't extend to a smooth character of G over $\overline{\mathbb{F}}_p$.
- We have the following non-split short exact sequence of $\overline{\mathbb{F}}_p[G]$-modules :

$$1 \longrightarrow \mathbf{1} \longrightarrow \mathrm{Ind}_B^G(\mathbf{1}) \longrightarrow St_G \longrightarrow 1 \,.$$

- There exists no non-trivial intertwinning between two non-supercuspidal representations of G.

Examples :

- When $\mathcal{G} = SL_2$, our irreducibility criterion reduces to $\chi \neq \mathbf{1}$.
- When $\mathcal{G} = U(2,1)$ is a quasi-split unitary group, our irreducibility criterion reduces to χ *doesn't factor through the determinant map.*

Comparison with the complex theory : We can notice some important differences with the complex theory, as for example :

- the length of the $\overline{\mathbb{F}}_p[B]$-modules defined by the non-supercuspidal representations (no case of length 3);
- the size of the intertwinning spaces (no non-trivial isomorphism);
- the lack of equivalence between cuspidality and supercuspidality (see the Steinberg representation).

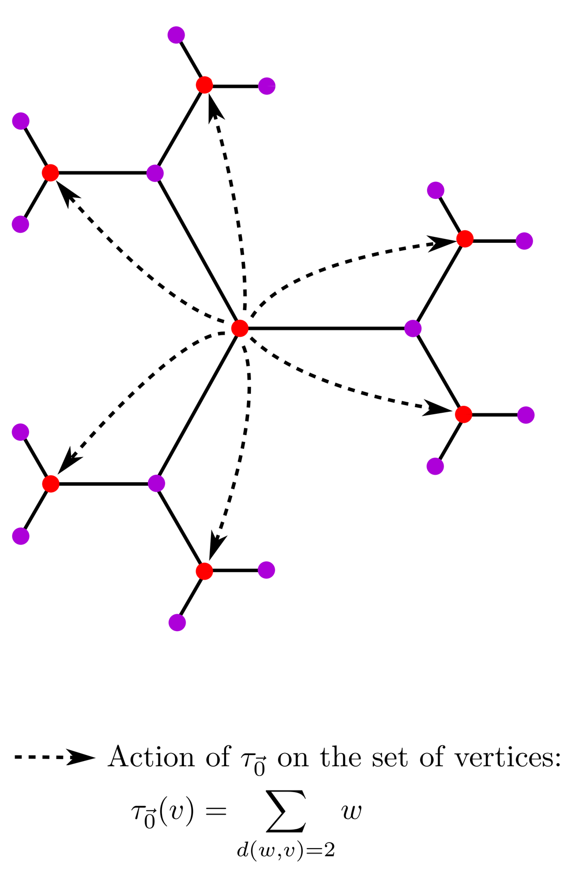

Figure 1: *Action of the Hecke operator $\tau_{\vec{0}}$ on the set of vertices of the Bruhat-Tits tree attached to $SL_2(\mathbb{F}_q)$*

Supersingularity and supercuspidality for $SL_2(F)$

We now consider the case $\mathcal{G} = SL_2$ and we fix a maximal open compact subgroup K of G. Up to isomorphism, the irreducible smooth representations of K are parametrized by the f-uplets $\vec{r} \in \{0, \dots, p-1\}^f$. If we denote by $\sigma_{\vec{r}}$ the representation attached to the f-uplet $\vec{r} = (r_0, \dots, r_{f-1})$, we have the following description of the associated spherical Hecke algebra.

Theorem 2 ([A2]). There exists an explicit Hecke operator $\tau_{\vec{r}}$ such that

$$\mathrm{End}_{\overline{\mathbb{F}}_p[G]}(\mathrm{ind}_K^G(\sigma_{\vec{r}})) = \overline{\mathbb{F}}_p[\tau_{\vec{r}}] \,.$$

Example : The operator $\tau_{\vec{0}}$ naturally acts on the set of vertices of the Bruhat-Tits tree attached to $SL_2(\mathbb{F}_q)$, as drawn on Figure 1.

As in Barthel-Livné's work [BL94, BL95], this naturally leads to the introduction of the following $\overline{\mathbb{F}}_p[G]$-modules: for any coefficient $\lambda \in \overline{\mathbb{F}}_p$, we set

$$\pi(\vec{r}, \lambda) := \frac{\mathrm{ind}_K^G(\sigma_{\vec{r}})}{(\tau_{\vec{r}} - \lambda \mathrm{Id})\left(\mathrm{ind}_K^G(\sigma_{\vec{r}})\right)} \,.$$

The following theorem explains why understanding these cokernels would provide the expected classification.

Theorem 3 ([A2]).

- Any irreducible admissible smooth representation of G over $\overline{\mathbb{F}}_p$ is a quotient of some cokernel $\pi(\vec{r}, \lambda)$.
- If $(\vec{r}, \lambda) \in \{0, \dots, p-1\}^f \times \overline{\mathbb{F}}_p^\times$ is different from $(\vec{0}, 1)$, then $\pi(\vec{r}, \lambda)$ is isomorphic to a parabolically induced representation. In particular, it has a unique (up to isomorphism) irreducible quotient.
- We have the following non-split short exact sequence of $\overline{\mathbb{F}}_p[G]$-modules :

$$1 \longrightarrow St_G \longrightarrow \pi(\vec{0}, 1) \longrightarrow \mathbf{1} \longrightarrow 1 \,.$$

In contrast, we say that an irreducible admissible smooth representation is *supersingular* when it is isomorphic to a quotient of some $\pi(\vec{r}, 0)$. This definition is justified by the following theorem, which underlines its importance in our study.

Theorem 4 ([A2]). An irreducible admissible smooth representation of G over $\overline{\mathbb{F}}_p$ is supersingular if, and only if, it is supercuspidal.

Unfortunately, these cokernels $\pi(\vec{r}, 0)$ are in general very mysterious. The only general statement we have is the following one, where we set $\alpha := \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$.

Theorem 5 ([A2]). Let $\vec{r} \in \{0, \dots, p-1\}^f$ be a parameter. There exists a representation $\pi_{\vec{r}}$ of G over $\overline{\mathbb{F}}_p$ such that we have the following non-split short exact sequence of $\overline{\mathbb{F}}_p[G]$-modules :

$$1 \longrightarrow \pi_{\vec{r}}^{\alpha} \longrightarrow \pi(\vec{r}, 0) \longrightarrow \pi_{\vec{r}} \longrightarrow 1 \,.$$

A few remarks :

- This theory doesn't depend on the choice of the maximal open compact subgroup K.
- We proved similar statements for the quasi-split (but non-split!) unramified unitary group $\mathcal{G} = U(2,1)$ [A1].

The $SL_2(\mathbb{Q}_p)$ case

A mod p semi-simple Langlands correspondence

When $F = \mathbb{Q}_p$, we choose $\varpi = p$ and keep the previous notations. We used Breuil's work [Br] about $GL_2(\mathbb{Q}_p)$ to get an explicit description of the supersingular representations of G.

Theorem 6 ([A2]).

- Any supersingular representation of $SL_2(\mathbb{Q}_p)$ is isomorphic to a representation π_r, for a unique parameter $r \in \{0, \dots, p-1\}$.
- Let π be a supersingular representation of $GL_2(\mathbb{Q}_p)$. There exists a (non-unique) parameter $r \in \{0, \dots p-1\}$ such that $\pi|_{SL_2(\mathbb{Q}_p)} \simeq \pi_r \oplus \pi_{p-1-r}$.
- For any $r \in \{0, \dots, p-1\}$, we have $\pi_r^{\alpha} \simeq \pi_{p-1-r}$.

By comparison with what exists for $GL_2(\mathbb{Q}_p)$ [Br], this necessarily leads to a mod p semi-simple Langlands correspondence for $SL_2(\mathbb{Q}_p)$ of the following form :

$$\left\{ \begin{array}{c} \textbf{packets of smooth} \\ \text{semi-simple} \\ \text{representations} \\ \text{of } SL_2(\mathbb{Q}_p) \text{ over } \overline{\mathbb{F}}_p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textbf{projectives} \\ \text{semi-simple represen-} \\ \text{tations of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \\ \text{of dimension } 2 \text{ over } \overline{\mathbb{F}}_p. \end{array} \right\}$$

Relation to the Hecke-Iwahori modules

The compact Frobenius reciprocity motivates the study of the simple right modules over some Hecke-Iwahori algebras. In this setting, we get the following result.

Theorem 7 ([A1]). Let $I(1)$ be the standard pro-p-Iwahori of G and $\mathcal{H}_S^1 := \mathrm{End}_{\overline{\mathbb{F}}_p[G]}(\mathrm{ind}_{I(1)}^G(\mathbf{1}))$.

- The map sending a smooth non-zero representation of G over $\overline{\mathbb{F}}_p$ on the space of its $I(1)$-invariants vectors defines a bijection between the isomorphism classes of non-supersingular irreducible smooth representations of $SL_2(F)$ over $\overline{\mathbb{F}}_p$ and the isomorphism classes of non-supersingular simple right $\mathcal{H}_S^1$-modules.
- This bijection extends to supersingular objets when $F = \mathbb{Q}_p$.

Description of the socle filtration (joint work with S. Morra)

The comparison of our results with Morra's work in the $GL_2(\mathbb{Q}_p)$ case highlighted striking similarities and motivated a common work in which we proved the following result.

Theorem 8 (with S. Morra, [AM]). Assume $p \neq 2$ and fix a pair $(r, \lambda) \in \{0, \dots, p-1\} \times \overline{\mathbb{F}}_p^\times$.

- The K-socle filtration of $\mathrm{Ind}_B^G(\mu_\lambda \omega^{p-1-r})$ is given by

$$\mathrm{Socfil}(\mathrm{Ind}_I^K(\chi_r^s)) \text{—} \mathrm{Socfil}(\mathrm{Ind}_I^K(\chi_{r-2}^s)) \text{—} \mathrm{Socfil}(\mathrm{Ind}_I^K(\chi_{r-4}^s)) \text{—} \dots$$

- The K-socle filtration of the Steinberg representation is given by

$$\mathrm{Sym}^{p-1}(\overline{\mathbb{F}}_p^2) \text{—} \mathrm{Socfil}(\mathrm{Ind}_I^K(\mathfrak{a})) \text{—} \mathrm{Socfil}(\mathrm{Ind}_I^K(\mathfrak{a}^2)) \text{—} \dots$$

- The K-socle filtration of the supersingular representation π_r is given by

$$\mathrm{Sym}^r(\overline{\mathbb{F}}_p^2) \text{—} \mathrm{Socfil}(\mathrm{Ind}_I^K(\chi_{-r-2}^s)) \text{—} \mathrm{Socfil}(\mathrm{Ind}_I^K(\chi_{-r-4}^s)) \text{—} \dots$$

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