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計算法 II. 変数変換つづき.

$$\int_{\frac{1}{2}x \leq y \leq 2x, x+y \leq \frac{3}{2}\pi} \cos\frac{2x-y}{3} dxdy$$

$x = 2s+t, y = s+2t$ と変数変換すると,

$$\begin{aligned} &= \int_{s\geq 0, t\geq 0, s+t\leq\frac{\pi}{2}} \cos s \cdot 3dsdt = 3\int_0^{\frac{\pi}{2}} dt \int_0^{\frac{\pi}{2}-t} \cos s ds \\ &= 3\int_0^{\frac{\pi}{2}} [\sin s]_0^{\frac{\pi}{2}-t} dt = 3\int_0^{\frac{\pi}{2}} \cos t dt = 3. \end{aligned}$$

球と円柱の共通部分.

$$\begin{aligned} & 2\int_{(x-\frac{1}{2})^2+y^2\leq\frac{1}{4}} \sqrt{1-x^2-y^2}dxdy = 2\int_{0\leq r\leq\cos\theta, -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}} \sqrt{1-r^2}rdrd\theta \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta} \sqrt{1-r^2}rdr = 4\int_0^{\frac{\pi}{2}} [-\frac{1}{3}\sqrt{1-r^2}^3]_0^{\cos\theta} d\theta \\ &= \frac{4}{3}\int_0^{\frac{\pi}{2}} (1-\sin^3\theta)d\theta = \frac{2}{3}\pi - \frac{4}{3}\int_0^{\frac{\pi}{2}} (\sin\theta - \cos^2\theta\sin\theta)d\theta \\ &= \frac{2}{3}\pi - \frac{4}{3}[-\cos\theta + \frac{1}{3}\cos^3\theta]_0^{\frac{\pi}{2}} = \frac{2}{3}\pi - \frac{8}{9}. \end{aligned}$$

3 変数関数の積分. 密度と質量.

曲面の面積。

$$S = \int_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}dxdy.$$

例　単位球の表面積 (注意：これは広義積分)

$$\begin{aligned} S &= 2\int_{x^2+y^2\leq 1} \sqrt{\left(\frac{\partial\sqrt{1-x^2-y^2}}{\partial x}\right)^2 + \left(\frac{\partial\sqrt{1-x^2-y^2}}{\partial y}\right)^2 + 1}dxdy \\ &= 2\int_{x^2+y^2\leq 1} \sqrt{\frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + 1}dxdy \\ &= 2\int_{0\leq r\leq 1, 0\leq\theta\leq 2\pi} \frac{1}{\sqrt{1-r^2}} rdrd\theta \\ &= 4\pi\int_0^1 \frac{r}{\sqrt{1-r^2}}dr = 4\pi[-\sqrt{1-r^2}]_0^1 = 4\pi \end{aligned}$$

面積の近似値　ベクトル $\begin{pmatrix} h \\ 0 \\ f_x(x,y)h \end{pmatrix}, \begin{pmatrix} 0 \\ k \\ f_y(x,y)k \end{pmatrix}$ のはる平行四辺形の面積の和.

ベクトル $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ b \end{pmatrix}$ のはる平行四辺形の面積 S．$\mathbf{c} = \begin{pmatrix} -a \\ -b \\ 1 \end{pmatrix}$ とおくと，$\mathbf{c}$ は $\mathbf{a}, \mathbf{b}$ と直交する．よって

$$S \times (\mathbf{c}\text{ の長さ}) = (\mathbf{a}, \mathbf{b}, \mathbf{c}\text{ のはる平行 6 面体の体積}) = |\det(\mathbf{a}\ \mathbf{b}\ \mathbf{c})|.$$

$$S = (1 + a^2 + b^2)/\sqrt{1 + a^2 + b^2} = \sqrt{1 + a^2 + b^2}.$$