2 Characteristic cycle

2.1 Characteristic cycles

Theorem 2.1.1. *There exists a unique way to attach a* $\mathbf{Z}$*-linear combination* $CC\mathcal{F} = \sum_a m_a C_a$ *of irreducible components* $SS\mathcal{F} = \bigcup_a C_a$ *for each constructible complex* $\mathcal{F}$ *of* Λ*-modules on a smooth scheme* X *over* k*, satisfying the following axioms:*

(1) *(normalization) For* $X = \operatorname{Spec} k$ *and* $\mathcal{F} = \Lambda$*, we have*

$$CC\Lambda = T^*_X X. \tag{2.1}$$

(2) *(additivity) For distringuished triangle* $\to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$*, we have*

$$CC\mathcal{F} = CC\mathcal{F}' + CC\mathcal{F}''. \tag{2.2}$$

(3) *(pull-back) For* $SS\mathcal{F}$*-transversal morphism* $h\colon W \to X$ *of smooth schemes over* k*, we have*

$$CCh^*\mathcal{F} = h^! CC\mathcal{F}. \tag{2.3}$$

(4) *(closed immersion) For closed immersion* $i\colon X \to P$ *of smooth schemes over* k*, we have*

$$CCi_*\mathcal{F} = i_! CC\mathcal{F}. \tag{2.4}$$

(5) *(Radon transform) For* $X = \mathbf{P}^n$ *and for the Radon transform, we have*

$$CCR\mathcal{F} = LCC\mathcal{F}. \tag{2.5}$$

Corollary 2.1.2. *(index formula) Assume that* X *is projective and smooth. Then, we have*

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X). \tag{2.6}$$

Proof. By (1), (2) and (3), if $\mathcal{F}$ is locally constant, we have

$$CC\mathcal{F} = (-1)^n \operatorname{rank} \mathcal{F} \cdot T^*_X X. \tag{2.7}$$

By (4), we may assume that $X = \mathbf{P}^n$ and $n \geqq 2$. Then, we have

$$CCR^\vee R\mathcal{F} = CC\mathcal{F} + (n-1) \cdot \chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F})[T^*_{\mathbf{P}^n}\mathbf{P}^n]. \tag{2.8}$$

By (5) and (2), we have $CC(R^\vee R\mathcal{F}) - CC\mathcal{F} = L^\vee LCC\mathcal{F} - CC\mathcal{F}$. Hence, we have $(n-1)\chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F}) = (n-1)(CC\mathcal{F}, T^*_{\mathbf{P}^n}\mathbf{P}^n)$ and (2.6).

We will deduce Theorem 2.1.1 from the following variant.

Theorem 2.1.3. *There exists a unique way to attach a* $\mathbf{Q}$*-linear combination* $CC\mathcal{F} = \sum_a m_a C_a$ *of irreducible components* $SS\mathcal{F} = \bigcup_a C_a$ *for each constructible complex* $\mathcal{F}$ *of* Λ*-modules on smooth smooth scheme* X *over* k*, satisfying the following axioms:*

(1) *(Milnor formula) Let* $f\colon X \to Y$ *be a proper morphism over* k *to a smooth curve* Y *over* k *and* $x \in X$ *be a closed point such that* f *is* $SS\mathcal{F}$*-transversal on the complement of* x*. Then, the coefficient of the fiber* $T^*_y Y$ *at* $y = f(x)$ *in* $f_\circ CC\mathcal{F}$ *is minus the Artin conductor* $-a_x Rf_*\mathcal{F}$*.*

(3) *For étale morphism* $h\colon W \to X$ *of smooth schemes over* k*, we have (2.3).*

(4) *For closed immersion* $i\colon X \to P$ *of smooth schemes over* k*, we have (2.4).*

Outline and key points of proof of theorems.

Proof of Theorem 2.1.3. We show the uniqueness. By (3), we may assume X is affine. By (4), we may assume $X = \mathbf{A}^n$. By (3), we may assume X is projective. We may take a Lefschetz pencil. Since it suffices to determine the coefficient m_a for each C_a, we may assume that $f\colon W \to L$ is C_b-transversal for $C_b \neq C_a$ and C_a-transversal except at x and is not C_a-transversal at x. Then, by (1), we have

$$m_a(C_a, df)_x = -a_x \tag{2.9}$$

and the uniqueness follows.

To show the existence, first we show that the coefficient m_a determined by (2.9) is well-defined. This follows from the (semi-)continuity of Swan conductor and the formalism of vanishing cycles over general base. Then $CC\mathcal{F}$ characterized by (2.9) satisfies the conditions (3) and (4) by standard properties of usual vanishing cycles.

Proof of the uniqueness in Theorem 2.1.1. By Corollary 2.1.2, we have the index formula (2.6) for projective and smooth X. By comparing the index formula (2.6) for proper smooth curve X and the Grothendieck-Ogg-Shafarevich formula and using (3) for étale morphism of smooth curves and (2.7), we obtain (1) in Theorem 2.1.3 for $f = 1_X\colon X \to X$.

Similarly as in the proof of Theorem 2.1.3, it is reduced to the case where X is projective and smooth. Then by taking a Lefschetz pencil, it follows from (5), (3) and (1) in Theorem 2.1.3.

Proof of the existence in Theorem 2.1.1. We deduce the existence from Theorem 2.1.3. We show that $CC\mathcal{F}$ satisfying the conditions in Theorem 2.1.3 also satisfies those in Theorem 2.1.1. The conditions (1) and (2) in Theorem 2.1.1 follow from (1) in Theorem 2.1.3. The condition (4) in Theorem 2.1.1 is the same as (4) in Theorem 2.1.3. Hence it remains to show the conditions (3), (5) and the integrality.

The condition (3) for smooth morphism is a consequence of the Thom-Sebastiani formula. The integrality in the case $p \neq 2$ or non-exceptional case in $p = 2$ follows from (1) in Theorem 2.1.3. In the exceptional case, it is reduced to the non-exceptional case using the condition (3) for $X \times \mathbf{A}^1 \to X$.

To show (3) in the case where h is an immersion, we first consider the case where X is an projective space $\mathbf{P}^n$.

Lemma 2.1.4. *Let $h\colon W \to P = \mathbf{P}^n$ be an immersion and*

$$\begin{tikzcd} W \arrow[d, "h"'] & W \times_P Q \arrow[l, "p_W"'] \arrow[r, "p_W^\vee"] \arrow[d] & P^\vee \\ P & Q \arrow[l, "p"'] & \end{tikzcd}$$

be the cartesian diagram. Let $\mathcal{G}$ be a constructible complex on $P^\vee$ micro supported on $C^\vee$ and assume that h is properly C-transversal for $C = L^\vee C^\vee$. Then, we have

$$\mathbf{P}(CCRp_{W*}p_W^{\vee*}\mathcal{G}) = \mathbf{P}(p_{W!}p_W^{\vee!}CC\mathcal{G}).$$

Proof. Since the characteristic cycle is characterized by the Milnor formula, it suffices to show that $p_{W!}p_W^{\vee!}CC\mathcal{G}$ satisfies the Milnor formula for $Rp_{W*}p_W^{\vee*}\mathcal{G}$ and for *smooth* morphisms $f\colon W\to Y$ to a curve defined locally on W. Since h is C-transversal, $p_W^\vee\colon Q\times_{\mathbf{P}}W\to\mathbf{P}^\vee$ is $C^\vee$-transversal by Lemma **??**.2 and $p_W^*\mathcal{G}$ is micro supported on $p_W^\vee C^\vee$. Since $p_W^\vee\colon Q\times_{\mathbf{P}}W\to\mathbf{P}^\vee$ is smooth outside $\mathbf{P}(C_W)$, we have $CCp_W^{\vee*}\mathcal{G}=p_W^{\vee\circ}CC\mathcal{G}$ outside $\mathbf{P}(C_W)$ as (3) is already proved for smooth morphisms.

Assume that f is smooth and has only isolated characteristic point. Then, by Lemma **??**.2, the composition fp_W is $p^\vee C$-transversal outside the inverse images of the characteristic points. Further it is $p^\vee C$-transversal outside of finitely many closed points in the inverse images by Lemma **??**.3 and these points are not contained in $\mathbf{P}(C_W)$ by Lemma **??**.1. Hence the assertion follows.

Lemma 2.1.4 implies also $\mathbf{P}(CCh^*\mathcal{F})=\mathbf{P}(h^!CC\mathcal{F})$. Since the coefficient of the 0-section is determined by the generic rank as in (2.7), we deduce (3) in the case $X=\mathbf{P}$. In the general case, since the assertion is local, we may assume that there exists an open subscheme $U\subset\mathbf{P}$ and a cartesian diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "i"] \\ V \arrow[r, "g"] & U\subset P \end{tikzcd}$$

where $i\colon X\to U$ and $g\colon V\to U$ are closed immersions of smooth subschemes meeting transversely. Then, since h is properly C-transversal, g is properly $i_\circ C$-transversal. Hence the case where $X=\mathbf{P}$ implies $CCg^*i_*\mathcal{F}=g^!CCi_*\mathcal{F}=g^!i_!CC\mathcal{F}$. This implies $j_!CCh^*\mathcal{F}=CCj_*h^*\mathcal{F}=j_!h^!CC\mathcal{F}$ and (2.3).

We show (5). The case $W=\mathbf{P}$ in Lemma 2.1.4 means the projectivization

$$\mathbf{P}(CCR\mathcal{F})=\mathbf{P}(LCC\mathcal{F}) \tag{2.10}$$

of (5). Hence it remains to show that the coefficients of the 0-section in $CCR\mathcal{F}=LCC\mathcal{F}$ are the same. Similarly as in the proof of Corollary 2.1.2, this is equivalent to the index formula (2.6) for $X=\mathbf{P}^n$. To prove this, we introduce the characteristic class.

2.2 Characteristic class

We identify the Chow group of the projective completion $\mathbf{P}(T^*X\oplus\mathbf{A}^1_X)$ by the canonical isomorphism

$$\mathrm{CH}_\bullet(X)=\bigoplus_{i=0}^n\mathrm{CH}_i(X)\to\mathrm{CH}_n(\mathbf{P}(T^*X\oplus\mathbf{A}^1_X)). \tag{2.11}$$

For a constructible complex $\mathcal{F}$ on X with the characteristic cycle $CC\mathcal{F}=\sum_a m_aC_a$, we define the characteristic class

$$cc_X(\mathcal{F})\in\mathrm{CH}_\bullet(X) \tag{2.12}$$

to be the class of $\sum_a m_a\bar{C}_a\in\mathrm{CH}_n(\mathbf{P}(T^*X\oplus\mathbf{A}^1_X))$.

Let $K(X,\Lambda)$ denote the Grothendieck group of the category of constructible complexes of Λ-modules on X. By the additivity, we have a morphism

$$cc_X\colon K(X,\Lambda)\to\mathrm{CH}_\bullet(X) \tag{2.13}$$

sending the class $\mathcal{F}$ to $cc_X\mathcal{F}$. In characteristic 0, we recover the MacPherson Chern class.

The pull-back by the immersion $\mathbf{P}(T^*X) \to \mathbf{P}(T^*X \oplus \mathbf{A}^1_X)$ and the push-forward by $\mathbf{P}(T^*X \oplus \mathbf{A}^1_X) \to X$ induce an isomorphism

$$\mathrm{CH}_n(\mathbf{P}(T^*X \oplus \mathbf{A}^1_X)) \to \mathrm{CH}_{n-1}(\mathbf{P}(T^*X)) \oplus \mathrm{CH}_n(X).$$

For $A = \sum_a m_a C_a$, the images of $\bar{A} = \sum_a m_a \bar{C}_a$ is the pair of $\mathbf{P}(A) = \sum_a m_a \mathbf{P}(C_a)$ and the coefficient of the 0-section.

End of Proof of Theorem 2.1.3. Under (2.10), the equality (2.5) is equivalent to the condition that the diagram

$$\begin{array}{ccc} K(\mathbf{P}^n, \Lambda) & \xrightarrow{cc_{\mathbf{P}^n}} & \mathrm{CH}_\bullet(\mathbf{P}^n) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle L} \\ K(\mathbf{P}^{n\vee}, \Lambda) & \xrightarrow{cc_{\mathbf{P}^{n\vee}}} & \mathrm{CH}_\bullet(\mathbf{P}^{n\vee}) \end{array} \tag{2.14}$$

gets commutative after composed with the projection $\mathrm{CH}_\bullet(\mathbf{P}^{n\vee}) \to \mathrm{CH}_n(\mathbf{P}^{n\vee})$ and also to the commutativity of the diagram (2.14) itself.

We prove the commutativity of (2.14) (CD n) and the index formula (2.6) for $\mathbf{P}^n$ (IF n) by a simultaneous induction on n along the diagram; (IF $n-1$) $\Rightarrow$ (CD n) $\Rightarrow$ (IF n). For $n \leqq 1$, the commutativity of (2.14) is obvious. For $n = 0$, the index formula follows from (2.7). For $n = 1$, this is nothing but the Grothendieck-Ogg-Shafarevich formula.

We prove (IF $n-1$) $\Rightarrow$ (CD n). Let $i\colon H \to \mathbf{P}^n$ be the immersion of a hyperplane. Then, the right square in

$$\begin{array}{ccccc} K(\mathbf{P}^n, \Lambda) & \xrightarrow{cc_{\mathbf{P}^n}} & \mathrm{CH}_\bullet(\mathbf{P}^n) & \xrightarrow{i^!} & \mathrm{CH}_{n-1}(H) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle \deg} \\ K(\mathbf{P}^{n\vee}, \Lambda) & \xrightarrow{cc_{\mathbf{P}^{n\vee}}} & \mathrm{CH}_\bullet(\mathbf{P}^{n\vee}) & \longrightarrow & \mathrm{CH}_n(\mathbf{P}^{n\vee}) = \mathbf{Z} \end{array} \tag{2.15}$$

is commutative. Hence it suffices to show that the long rectangle is commutative. For $\mathcal{F}$ on $\mathbf{P}^n$, the generic rank of $R\mathcal{F}$ equals the Euler number $\chi(H_{\bar{k}}, \mathcal{F})$ for a generic H. Hence the composition via lower left sends the class of $\mathcal{F}$ to $\chi(H_{\bar{k}}, \mathcal{F})$. By (3) for the immersion $i\colon H \to \mathbf{P}^n$ and (IF $n-1$), we have $\chi(H_{\bar{k}}, \mathcal{F}) = (CCi^*\mathcal{F}, T^*_H H) = \deg i^! cc_{\mathbf{P}^n}\mathcal{F}$ and the long rectangle is commutative.

We prove (CD n) $\Rightarrow$ (IF n). Let $\chi\colon K(\mathbf{P}^n, \Lambda) \to \mathbf{Z}$ be the morphism sending the class of $\mathcal{F}$ to the Euler number $\chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F})$. We show that there is a commutative diagram

$$\begin{array}{ccc} K(\mathbf{P}^n, \Lambda) & \xrightarrow{cc_{\mathbf{P}^n}} & \mathrm{CH}_\bullet(\mathbf{P}^n) \\ & \searrow^{\chi} & \downarrow \\ & & \mathbf{Z}. \end{array} \tag{2.16}$$

Since $cc_{\mathbf{P}^n}$ is a surjection, it suffices to show that $cc_{\mathbf{P}^n}\mathcal{F} = 0$ implies $\chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F}) = 0$. By (2.8), (CD n) and the assumption $cc_{\mathbf{P}^n}\mathcal{F} = 0$ imply $\chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F}) = 0$ for $n - 1 \neq 0$. Thus, there exists a unique morphism $\mathrm{CH}_\bullet(\mathbf{P}^n) \to \mathbf{Z}$ making the diagram (2.16) commutative. We show that the morphism $\mathrm{CH}_\bullet(\mathbf{P}^n) \to \mathbf{Z}$ equals the degree mapping. This is reduced to the case where $\mathcal{F} = \Lambda_{\mathbf{P}^i}, i = 0, \ldots, n$ generating $\mathrm{CH}_\bullet(\mathbf{P}^n) = \mathbf{Z}^{n+1}$.