Normal and core reduction numbers [1]

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Throughout this note, let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed residue field $k = \overline{k} \cong A/\mathfrak{m}$ unless otherwise specified. Then there exists a resolution of singularity $Y \to \operatorname{Spec} A$. Then $p_g(A) = \ell_A(H^1(Y, \mathcal{O}_Y))$ is called the geometric genus of A, which is independent on the choice of resolution of singularities. This invariant plays a key role in our argument.

1. Geometric genus and normal reduction number

Throughout this section, let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain with algebraically closed residue field k, and let $I \subset A$ be an $\mathfrak{m}$-primary integrally closed ideal. Then there exists a resolution of singularity $X \to \operatorname{Spec} A$ and an anti-nef cycle Z on X so that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and $I = H^0(\mathcal{O}_X(-Z))$. The ideal I is *represented* by Z on X which is denoted by $I = I_Z$. Then $\overline{I^n} = I_{nZ}$.

We recall the definition of normal reduction numbers. In what follows, we always assume that $I = I_Z$.

Definition 1.1 (Normal reduction number). Let Q be a minimal reduction of I, that is, $Q \subset I$ is a parameter ideal and there exists a positive integer n such that $I^{n+1} = QI^n$. Then

$$\operatorname{nr}(I) = \inf\{n \in \mathbb{Z} \mid \overline{I^{n+1}} = Q\overline{I^n}\},$$

is independent on the choice of Q by Huneke [6, Theorem 4.5] and so we call it the *normal reduction number* of I. Moreover, we can define

$$\operatorname{nr}(A) = \max\{\operatorname{nr}(I) \mid I \text{ is a } \mathfrak{m}\text{-primary integrally closed ideal of } A\},$$

which is called the *normal reduction number* of A.

Remark 1.2. Put $\bar{r}(I) = \inf\{n \in \mathbb{Z} \mid \overline{I^{N+1}} = Q\overline{I^N} \ (\forall N \geq n)\}$. In general, Lemma 2.1 and Lemma 2.3 imply $\operatorname{nr}(I) = \bar{r}(I)$ in our case. But we do *not* know whether equality holds true for higher-dimensional case.

The notion of "core" was introduced by Rees and Sally [18], and their properties have been studied by Corso-Ulrich, Huneke–Swanson, Huneke–Trung, Hyry–Smith, Polini–Ulrich and so on; e.g. [1, 2, 7, 8, 9]. The *core* of I is defined as follows:

$$\operatorname{core}(I) = \bigcap_{Q\,:\,\text{a reduction of } I} Q$$

In general, it is not so easy to calculate $\operatorname{core}(I)$, but in the case of stable ideals, it is easy to compute.

[1]This is not in final form. The detailed version will be submitted to elsewhere for publication.

Lemma 1.3 ([2, 8, 9])**.** *If $I^2 = QI$ holds true for some minimal reduction Q of I, then* $\mathrm{core}(I) = (Q : I)I$*.*

Let us introduce the following notion.

Definition 1.4 (**Normal core reduction number**)**.** Let I be an $\mathfrak{m}$-primary ideal of A. Then *the core reduction number* (resp. *the normal core reduction number*) is defined by

$$\begin{aligned} \mathrm{cr}(I) &= \min\{n \in \mathbb{Z} \mid I^{n+1} \subset \mathrm{core}(Q)\}, \\ \mathrm{ncr}(I) &= \min\{n \in \mathbb{Z} \mid \overline{I^{n+1}} \subset \mathrm{core}(Q)\}, \end{aligned}$$

respectively. Moreover, we define

$$\mathrm{ncr}(A) = \sup\{\mathrm{ncr}(I) \mid I \text{ is an } \mathfrak{m}\text{-primary ideal with } \overline{I} = I\ \},$$

which is called *the normal core reduction number* of A.

The main aim of this talk is to evaluate $\mathrm{nr}(A)$, $\mathrm{ncr}(A)$ in terms of geometric invariants.

Example 1.5. Let A be as above. Then

(1) $\mathrm{nr}(A) = 0$ if and only if A is regular (see [3]).
(2) $\mathrm{nr}(A) = 1$ if and only if A is a *rational singularity* which is not regular (see [10]).
(3) If A is an elliptic singularity, then $\mathrm{nr}(A) = 2$. How about the converse? (see Okuma [11])

The following theorem is motivated by the previous example.

Theorem 1.6. *For any $\mathfrak{m}$-primary integrally closed ideal $I \subset A$ with $r = \mathrm{nr}(I)$, we have*

$$p_g(A) \geq \binom{r}{2} + \ell_A(H^1(X, \mathcal{O}_X(-rZ))).$$

In particular, $p_g(A) \geq \binom{\mathrm{nr}(A)}{2} \geq \binom{\mathrm{ncr}(A)}{2}$.

In the next section, we give a proof of this theorem.

2. Proof of Main theorem

Throughout this section, let $I = I_Z$ be an $\mathfrak{m}$-primary integrally closed ideal in a a two-dimensional excellent normal local domain $(A, \mathfrak{m})$ with algebraically closed residue field k. For a given ideal I, we define a function $q\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ as follows:

$$q(k) := q(kI) := \ell_A(H^1(X, \mathcal{O}_X(-kZ)).$$

By definition, we put $q(0) = p_g(A)$ and $q(I) = q(1I)$. Note that $q(nI) = q(\overline{I^n})$ for every integer $n \geq 1$.

Let us recall the following fundamental properties of $q(kI)$.

Lemma 2.1 ([12, 13])**.** *The following statements hold.*

(1) $0 \leq q(I) \leq p_g(A)$*. If $q(I) = p_g(A)$ holds true, then I is said to be a p_g-ideal.*
(2) *The function $q(\cdot I)$ is decreasing: $q(kI) \geq q((k+1)I)$ for every integer $k \geq 1$.*
(3) *The function $q(\cdot I)$ stabilize: there exists an integer $n_0 = n_0(I)$ ($0 \leq n_0 \leq p_g(A)$) such that $q(nI) = q(n_0 I)$ for $n \geq n_0$.*

The $\mathfrak{m}$-primary ideal I is called *good* (in the sense of Goto-Iai-Watanabe [4]) if $I^2 = QI$ and $I = Q : I$ for some (every) minimal reduction Q of I.

Example 2.2 ([12, 14])**.** Any two-dimensional excellent normal local domain over $k = A/\mathfrak{m} = \overline{k}$ admits a p_g-ideal. If, in addition, A is not regular, then A admits a good p_g-ideal.

In order to prove Theorem 1.6, we need the following lemma.

Lemma 2.3. *For any integer $n \geq 1$, we have*

$$2 \cdot q(nI) + \ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = q((n+1)I) + q((n-1)I).$$

Proof. It follows from the following exact sequence:

$$0 \to \overline{I^{n+1}}/Q\overline{I^n} \to H^1(\mathcal{O}_X(-(n-1)Z)) \to H^1(\mathcal{O}_X(-nZ))^{\oplus 2} \to H^1(\mathcal{O}_X(-(n+1)Z)) \to 0.$$

□

Proof of Theorem 1.6. Suppose $\mathrm{nr}(I) = r$. Then since $\ell_A(\overline{I^{k+1}}/Q\overline{I^k}) \geq 1$ for every $k = 1, 2, \ldots, r-1$ and $\ell(\overline{I^{r+1}}/Q\overline{I^r}) = 0$, we have

$$\begin{array}{rcl} q((r-1)I) - q(rI) & = & q(rI) - q((r+1)I), \\ q((r-2)I) - q((r-1)I) & \geq & q((r-1)I) - q(rI) + 1, \\ q((r-3)I) - q((r-2)I) & \geq & q((r-2)I) - q((r-1)I) + 1, \\ & \vdots & \\ q(0I) - q(1I) & \geq & q(1I) - q(2I) + 1. \end{array}$$

Thus if we put $a_k = q((r-k)I)$ for $k = 1, \ldots, r$, then we get

$$\begin{array}{rcl} a_k - a_{k-1} & \geq & a_{k-1} - a_{k-2} + 1 \\ & \geq & a_{k-2} - a_{k-3} + 2 \\ & \geq & \cdots \\ & \geq & \big\{a_1 - a_0\big\} + (k-1) \geq k-1. \end{array}$$

Hence

$$p_g(A) = a_r = \sum_{k=1}^{r}(a_k - a_{k-1}) + a_0 \geq \sum_{k=1}^{r}(k-1) + a_0 = \frac{r(r-1)}{2} + q(rI),$$

as required. In particular, we have $p_g(A) \geq \binom{r}{2}$.

On the other hand, for any minimal reduction Q of I, we get $\overline{I^{r+1}} = Q\overline{I^r} \subset Q$, which shows $\overline{I^{r+1}} \subset \mathrm{core}(I)$. Hence $r \geq \mathrm{ncr}(I)$. This yields that $\mathrm{nr}(I) \geq \mathrm{ncr}(I)$. Hence $\mathrm{nr}(A) \geq \mathrm{ncr}(A)$. Hence $p_g(A) \geq \binom{\mathrm{nr}(A)}{2} \geq \binom{\mathrm{ncr}(A)}{2}$. □

The above theorem gives a best possible bound. In fact, we have the following example. See the next subsection for more details.

Example 2.4. Let $r \geq 1$ be an integer. Let $A = \mathbb{C}[[x,y,z]]/(x^2+y^{2r}+z^{2r})$. Then $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = r$ and

$$p_g(A) = \binom{r}{2} = \binom{\mathrm{nr}(A)}{2} = \binom{\mathrm{ncr}(A)}{2}.$$

In particular, we consider the case of $r = 2$. Let $I \subset A$ be an $\mathfrak{m}$-primary integrally closed ideal. Then $0 \leq q(I) \leq p_g(A) = 1$ implies $q(I) = 0$ or $q(I) = 1$.

If $q(I) = 0$, then $q(2I) = q(3I) = \cdots = 0$ by Lemma 2.1. Then by Lemma 2.3 we get

$$\begin{aligned} \ell_A(\overline{I^2}/QI) &= 2 \cdot q(I) + \ell_A(\overline{I^2}/QI) = q(2I) + p_g(A) = 1, \\ \ell_A(\overline{I^{k+1}}/Q\overline{I^k}) &= 2 \cdot q(kI) + \ell_A(\overline{I^{k+1}}/Q\overline{I^k}) = q((k+1)I) + q((k-1)I) = 0 \quad \text{for } k \geq 2. \end{aligned}$$

Hence $\mathrm{nr}(I) = \bar{r}(I) = 2$.

On the other hand, if $q(I) = 1$, then I is a p_g-ideal and hence $\overline{I^{k+1}} = Q\overline{I^k}$ for every $k \geq 1$ and $q(I) = q(2I) = \cdots = p_g(A) = 1$. That is, $\mathrm{nr}(I) = \bar{r}(I) = 1$.

For instance, $\mathfrak{m} = (x,y,z)$ satisfies $q(\mathfrak{m}) = 0$ and $I = (x^2, y, z)$ satisfies $q(I) = 1$.

3. Normal reduction numbers of hypersurfaces of Fermat type

In what follows, let $R = \mathbb{C}[x,y,z]/(z^2 + x^a + y^b)$ be a hypersurface with $2 \leq a \leq b$. Put $\mathfrak{m} = (x,y,z)A$ and $r = \lfloor \frac{a}{2} \rfloor$. Then the $\mathfrak{m}$-adic completion $A = \widehat{R_{\mathfrak{m}}}$ is a two-dimensional excellent normal local domain. Put $Q = (x,y)$. This gives a minimal reduction of $\mathfrak{m}$. Also we put $F_k = \overline{\mathfrak{m}^k}$ for every integer $k \geq 1$. First we calculate $\ell_A(F_{k+1}/QF_k)$ for all $k \geq 0$. In order to do that we determine the normalization of the extended Rees algebra $\mathcal{R}'(\mathfrak{m}) = A[\mathfrak{m}t, t^{-1}]$

Lemma 3.1. *The normalization of* $\mathcal{R}' = \mathcal{R}'(\mathfrak{m}) = A[xt, yt, zt, t^{-1}]$ *is given by*

$$\overline{\mathcal{R}'} = \mathcal{R}'[zt^2, \ldots, zt^r] \cong \begin{cases} \mathbb{C}[X,Y,Z,U]/(Z^2 + X^{2r} + Y^bU^{b-2r}) & \text{if } a = 2r, \\ \mathbb{C}[X,Y,Z,U]/(Z^2 + X^{2r+1}U + Y^bU^{b-2r}) & \text{if } a = 2r+1. \end{cases}$$

Proof. Put $X = xt, Y = yt, Z = zt^r, U = t^{-1} \in Q(\mathcal{R}')$. Then $S = \mathbb{C}[xt, yt, zt, t^{-1}, zt^2, \ldots, zt^r]$ is generated by X, Y, Z and U as $\mathbb{C}$-algebra because $zt^i = ZU^{r-i}$ for each $i = 0, 1, \ldots, r-1$. Note that $a = 2r$ or $a = 2r+1$ by definition.

- <u>The case of $a = 2r$</u>

Then we have

$$Z^2 = (zt^r)^2 = z^2t^{2r} = -x^{2r}t^{2r} - y^bt^{2r} = -X^{2r} - Y^bU^{b-2r},$$

that is, $F := Z^2 + X^{2r} + Y^bU^{b-2r} = 0$ in S. Clearly, $Z^2 + X^{2r} + Y^bU^{b-2r}$ is a prime element of $\mathbb{C}[X,Y,Z,U]$ and thus $\dim \mathbb{C}[X,Y,Z,U]/(F) = 3$. On the other hand, since $\dim \overline{\mathcal{R}'} = \dim \mathcal{R}' = 3$ and S is a homomorphic image of $\mathbb{C}[X,Y,Z,U]/(F)$, we can prove that $S \cong \mathbb{C}[X,Y,Z,U]/(F)$.

So it is enough to show that S is normal. The Jacobian ideals of S is

$$\begin{aligned} J(F) &= \left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}, \frac{\partial F}{\partial U}\right) \\ &= (2rX^{2r-1},\ bY^{b-1}U^{b-2r},\ 2Z,\ (b-2r)Y^bU^{b-2r-1}) \\ &= \begin{cases} (Z,\ X^{2r-1},\ Y^{b-1}U^{b-2r},\ Y^bU^{b-2r-1}) & \text{if } b \geq 2r+1, \\ (Z,\ X^{2r-1},\ Y^{2r-1}), & \text{if } b = 2r. \end{cases} \end{aligned}$$

Since S is Cohen-Macaulay and height $J(F) = 2$, S is normal.

- The case of $a = 2r+1$

Then we have

$$Z^2 = (zt^r)^2 = -x^{2r+1}t^{2r} - z^bt^{2r} = -X^{2r+1}U - Y^bU^{b-2r},$$

that is, $F_o := Z^2 + X^{2r+1}U + Y^bU^{b-2r} = 0$ in S. Similar argument implies that $S \cong \mathbb{C}[X,Y,Z,U]/(F_o)$.

So it is enough to show that S is normal. The Jacobian ideals of S is

$$\begin{aligned} J(F_o) &= \left(\frac{\partial F_o}{\partial X}, \frac{\partial F_o}{\partial Y}, \frac{\partial F_o}{\partial Z}, \frac{\partial F_o}{\partial U}\right) \\ &= (2Z,\ (2r+1)X^{2r}U,\ bY^{b-1}U^{b-2r},\ \ X^{2r+1} + (b-2r)Y^bU^{b-2r-1}) \\ &= \begin{cases} (Z,\ X^{2r}U,\ Y^{b-1}U^{b-2r},\ X^{2r+1} + (b-2r)Y^bU^{b-2r-1}) & \text{if } b \geq 2r+2, \\ (Z,\ X^{2r}U,\ Y^{2r}U,\ X^{2r+1} + Y^{2r+1}) & \text{if } b = 2r+1. \end{cases} \end{aligned}$$

Suppose $b \geq 2r+2$ and $P \in \operatorname{Spec} K[X,Y,Z,U]$ such that

$$P \supset (Z,\ X^{2r}U,\ Y^{b-1}U^{b-2r},\ X^{2r+1} + (b-2r)Y^bU^{b-2r-1}).$$

If $U \notin P$, then $(X,Y,Z) \subset P$. Otherwise, $(X,Z,U) \subset P$. Hence height $J(F_o) \geq 2$.

Next suppose $b = 2r+1$ and $P \in \operatorname{Spec} K[X,Y,Z,U]$ such that

$$P \supset (Z,\ X^{2r}U,\ Y^{2r}U,\ X^{2r+1} + Y^{2r+1}).$$

If $U \notin P$, then $(X,Y,Z) \subset P$. Otherwise, $(Z,U) \subset P$ and $X^{2r+1} + Y^{2r+1} \in P$. Take ω as one of $(2r+1)$-th primitive roots of unity. Hence $(Z, U, X + \omega^i Y) \subset P$. Therefore height $J(F_o) \geq 2$ and S is normal. $\square$

Lemma 3.2. *We have $F_k = z\mathfrak{m}^{k-r} + \mathfrak{m}^k$ for every $k \geq 1$ and $\operatorname{nr}(\mathfrak{m}) = \overline{r}(\mathfrak{m}) = r$. Furthermore, we get*

$$\begin{cases} \ell_A(F_2/QF_1) = \ell_A(F_3/QF_2) = \cdots = \ell_A(F_r/QF_{r-1}) = 1, \\ \ell_A(F_{r+1}/QF_r) = \ell_A(F_{r+2}/QF_{r+1}) = \cdots = 0. \end{cases}$$

Proof. By the previous lemma, $A[\mathfrak{m}t][zt^2, \ldots, zt^r] = A[xt, yt, zt, \ldots, zt^r, t^{-1}] \cap A[t]$ is normal. From this, one can easily see that $F_k = z\mathfrak{m}^{k-r} + \mathfrak{m}^k$ for every $k \geq 1$, where $\mathfrak{m}^n = A$ for each $n \leq 0$.

We will show that $\ell_A(F_{k+1}/QF_k) = 1$ for each $k = 1, 2, \ldots, r-1$. For such an integer k, we have $z^2 \in \mathfrak{m}^{2r} \subset \mathfrak{m}^{k+1}$. Thus

$$z\mathfrak{m} + \mathfrak{m}^{k+1} = zQ + \mathfrak{m}^{k+1} = zQ + Q\mathfrak{m}^k = Q(zA + \mathfrak{m}^k) = QF_k.$$

It follows that $F_{k+1} = zA + QF_k$ and $z\mathfrak{m} \subset QF_k$. Hence $\ell_A(F_{k+1}/QF_k) = 1$ because $z \notin QF_k$.

Next we will show that $F_{k+1} = QF_k$ for every $k \geq r$. Since $z^2 \in \mathfrak{m}^{2r}$, we get

$$\begin{aligned} QF_k &= (x,y)(z\mathfrak{m}^{k-r} + \mathfrak{m}^k) \\ &= z(x,y)\mathfrak{m}^{k-r} + Q\mathfrak{m}^k \\ &= (z^2, xz, yz)\mathfrak{m}^{k-r} + \mathfrak{m}^{k+1} \\ &= z\mathfrak{m}^{k+1-r} + \mathfrak{m}^{k+1} = F_{k+1}, \end{aligned}$$

as required. By definition, we have $\mathrm{nr}(\mathfrak{m}) = \overline{r}(\mathfrak{m}) = r$. $\square$

By virtue of the previous lemma, we can determine $q(i\mathfrak{m})$ completely in our case.

Theorem 3.3. *Put $p = p_g(A)$. Then we have*

$$q(i\mathfrak{m}) = \begin{cases} p - i(r-1) + \binom{i}{2} & 1 \leq i \leq r-1; \\ p - \binom{r}{2} & i \geq r. \end{cases}$$

In particular, $q(\mathfrak{m}) = p - (r-1)$. Moreover, for all $n \geq r-1$, we get

$$\ell_A\big(A/\overline{\mathfrak{m}^{n+1}}\big) = 2 \cdot \binom{n+2}{2} - r \cdot \binom{n+1}{1} + \binom{r}{2}.$$

In particular, we have

$$e_0(\mathfrak{m}) = 2, \qquad e_1(\mathfrak{m}) = r, \qquad e_2(\mathfrak{m}) = \binom{r}{2}.$$

Proof. Put $k = p - q(\mathfrak{m}) \geq 0$. Then we prove the following claim.

Claim 1: $q(i\mathfrak{m}) = p - ik + \binom{i}{2}$ for all $i = 1, 2, \ldots, r$.

Use an induction on i. It is easy to check the case of $i = 1$. Now suppose $2 \leq i+1 \leq r$, and the above equation holds true for $j \leq i$. Then by assumption, we get

$$\begin{aligned} q((i+1)\mathfrak{m}) &= 2 \cdot q(i\mathfrak{m}) - q((i-1)\mathfrak{m}) + \ell_A(F_{i+1}/QF_i) \\ &= 2\left[p - ik + \binom{i}{2}\right] - \left[p - (i-1)k + \binom{i-1}{2}\right] + 1 \\ &= p - (i+1)k + \binom{i+1}{2}. \end{aligned}$$

Next we show that

Claim 2: $q((r+i)\mathfrak{m}) = p - rk + \binom{r}{2} + i(r-1-k)$ for all $i = 1, 2, \ldots$.

Use an induction on i. When $i=1$, we have

$$\begin{aligned} q((r+1)\mathfrak{m}) &= 2\cdot q(r\mathfrak{m}) - q((r-1)\mathfrak{m}) + \ell_A(F_{r+1}/QF_r) \\ &= 2\left[p - rk + \binom{r}{2}\right] - \left[p-(r-1)k + \binom{r-1}{2}\right] \\ &= p-(r+1)k + \binom{r+1}{2} - 1 \\ &= p - rk + \binom{r}{2} + (r-1-k), \end{aligned}$$

as required. Now suppose $i \geq 2$ and the above equation holds true for any $j \leq i$. Then we have

$$\begin{aligned} q((r+i+1)\mathfrak{m}) &= 2\cdot q((r+i)\mathfrak{m}) - q((r+i-1)\mathfrak{m}) + \ell_A(F_{r+i+1}/QF_{r+i}) \\ &= 2\left[p - rk + \binom{r}{2} + i(r-1-k)\right] \\ &\qquad - \left[p - rk + \binom{r}{2} + (i-1)(r-1-k)\right] \\ &= p - rk + \binom{r}{2} + (i+1)(r-1-k). \end{aligned}$$

Since $q(i\mathfrak{m})$ is stable for sufficiently large i, we obtain that $k=r-1$. Indeed, if $k \leq r-2$, then $q((k+2)\mathfrak{m}) > q((k+1)\mathfrak{m})$. On the other hand, if $k \geq r$, then $q(i\mathfrak{m})$ becomes strictly decreasing function on i. This is a contradiction. Hence $k = r-1$. Thus

$$q(i\mathfrak{m}) = \begin{cases} p - i(r-1) + \binom{i}{2} & 1 \leq i \leq r-1; \\ p - \binom{r}{2} & i \geq r. \end{cases}$$

By [13], we obtain

$$\overline{e}_1(\mathfrak{m}) = e_0(\mathfrak{m}) - \ell_A(A/\mathfrak{m}) + [p_g(A) - q(\mathfrak{m})] = 2 - 1 + [p - (p-(r-1))] = r$$

and

$$\overline{e}_2(\mathfrak{m}) = p - q(r\mathfrak{m}) = p - \left[p - \binom{r}{2}\right] = \binom{r}{2}.$$

On the other hand,

$$\ell_A(A/\overline{\mathfrak{m}^{n+1}}) = 2\cdot\binom{n+2}{2} - r\cdot\binom{n+1}{1} + p - q((n+1)\mathfrak{m}).$$

Thus $P_{\mathfrak{m}}(n) = H_{\mathfrak{m}}(n)$ if and only if $n \geq r-1$. $\square$

In the last of this section, we calculate the geometric genus of A. We regard $R = \mathbb{C}[X,Y,Z]/(Z^2 - X^a - Y^b)$ as a graded ring by $\deg Z = ab =: q_0$, $\deg X = 2b =: q_1$,

$\deg Y = 2a := q_2$. If we put $D = 2ab$, then the a-invariant of R is given by $a(R) = D - q_0 - q_1 - q_2$. Then we can calculate the geometric genus of A by

$$\begin{aligned} p_g(A) &= \sum_{n=0}^{a(R)} \dim_{\mathbb{C}} R_n \\ &= \sharp\{(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{\geq 0} \,|\, D - (q_0 + q_1 + q_2) \geq \lambda_0 q_0 + \lambda_1 q_1 + \lambda_2 q_2\}. \end{aligned}$$

In this case, we have

$$\begin{aligned} p_g(A) &= \sharp\{(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{\geq 0} \,|\, 2ab - ab - 2b - 2a \geq ab\lambda_0 + 2b\lambda_1 + 2a\lambda_2\} \\ &= \sharp\{(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{\geq 0} \,|\, ab - 2b - 2a \geq ab\lambda_0 + 2b\lambda_1 + 2a\lambda_2\}. \end{aligned}$$

Then one can easily see that $\lambda_0 = 0$. Hence

$$p_g(A) = \sharp\{(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0} \,|\, ab - 2a - 2b \geq 2b\lambda_1 + 2a\lambda_2\}. \tag{3.1}$$

Example 3.4. Let $p \geq 1$ be an integer. Let $A = \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^{6p+1})$. Then $p_g(A) = p$ and $\mathrm{nr}(\mathfrak{m}) = 1$.

Example 3.5. Let $p \geq 1$ be an integer. Let $A = \mathbb{C}[[x, y, z]]/(x^2 + y^4 + z^{4p+1})$. Then $p_g(A) = p$ and $\mathrm{nr}(\mathfrak{m}) = 2$.

4. An example of normal core reduction number

In the last of this note, we prove Example 2.4.

Proposition 4.1. *Let $r \geq 2$ be an integer, and let $A = \mathbb{C}[[x, y, z]]/(z^2 + x^{2r} + y^{2r})$. Then*

1. $p_g(A) = \binom{r}{2}$.
2. $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = r$.
3. $\mathrm{ncr}(A) = \mathrm{ncr}(\mathfrak{m}) = r$.

Proof. Put $R = \mathbb{C}[x, y, z]/(z^2 + x^{2r} + y^{2r})$.

(1) By the formula (3.1), we have

$$p_g(A) = \sharp\{(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0} \,|\, r - 2 \geq \lambda_1 + \lambda_2\} = \binom{r}{2}.$$

(2) One can easily see that $\mathrm{nr}(\mathfrak{m}) = \bar{r}(\mathfrak{m}) = r$ and our main theorem implies that $p_g(A) \geq \binom{\mathrm{nr}(I)}{2}$ for any integrally closed $\mathfrak{m}$-primary ideal and thus $\mathrm{nr}(A) \leq r$. Hence we obtain that $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = r$.

(3) By definition, we have $\mathrm{ncr}(I) \leq \mathrm{nr}(I)$ for any $\mathfrak{m}$-primary integrally closed ideal $I \subset A$. On the other hand, since $\mathfrak{m}^2 = Q\mathfrak{m}$, we have $\mathrm{core}(\mathfrak{m}) = (Q \colon \mathfrak{m})\mathfrak{m} = \mathfrak{m}^2$. Hence $\overline{\mathfrak{m}^{n+1}} = F_{n+1} \subset \mathrm{core}(\mathfrak{m}) = \mathfrak{m}^2$ if and only if $n \geq r$. Thus $\mathrm{ncr}(\mathfrak{m}) = r$.

For any $\mathfrak{m}$-primary integrally closed ideal I, since $\mathrm{nr}(I) \leq \mathrm{nr}(A) = r$, we have that $\overline{I^{r+1}} \subset Q'$ for any minimal reduction Q' of I. Hence $\mathrm{ncr}(I) \leq r = \mathrm{ncr}(\mathfrak{m})$ and thus $\mathrm{ncr}(A) = r$, as required. $\square$

Question. *The following questions are interesting.*

1. *When does* $\mathrm{ncr}(A) = \mathrm{ncr}(\mathfrak{m})$ *hold?*

(2) *When does* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m})$ *hold?*
(3) *When does* $\mathrm{ncr}(A) = \mathrm{nr}(A)$ *hold?*
(4) *When does* $\mathrm{nr}(\mathfrak{m}) = \binom{r}{2}$ *hold?*

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