

HOMOGENEOUS PRIME ELEMENTS IN NORMAL TWO-DIMENSIONAL GRADED RINGS

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ABSTRACT. This is a report on a joint work with A. Singh (University of Utah) and R. Takahashi (Nagoya University). We prove necessary and sufficient conditions for the existence of homogeneous prime elements in normal \mathbb{N} -graded rings of dimension two, in terms of rational coefficient Weil divisors on projective curves.

1. INTRODUCTION

We investigate the existence of homogeneous prime elements, equivalently, of homogenous principal prime ideals, in normal \mathbb{N} -graded rings R of dimension two. It turns out that there are elegant necessary as well as sufficient conditions for the existence of such prime ideals in terms of rational coefficient Weil divisors, i.e., \mathbb{Q} -divisors, on $\text{Proj } R$.

When speaking of an \mathbb{N} -graded ring R , we assume throughout this paper that R is a finitely generated algebra over its subring R_0 , and that R_0 is an algebraically closed field. We say that an \mathbb{N} -grading on R is *irredundant* if

$$\gcd\{n \in \mathbb{N} \mid R_n \neq 0\} = 1.$$

Relevant material on \mathbb{Q} -divisors is summarized in §2. Our main result is:

Theorem 1.1. *Let R be a normal ring of dimension 2, with an irredundant \mathbb{N} -grading, where R_0 is an algebraically closed field. Set $X := \text{Proj } R$, and let D be a \mathbb{Q} -divisor such that $R = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n$. Let d be a positive integer.*

(1) *Suppose $x \in R_d$ is a prime element. Set*

$$s := \gcd\{n \in \mathbb{N} \mid [R/xR]_n \neq 0\}.$$

Then the integers d and s are relatively prime, and the divisor sdD is linearly equivalent to a point of X . In particular, $\deg D = 1/sd$.

(2) *Conversely, suppose dD is linearly equivalent to a point P with $P \notin \text{supp}(\text{frac}(D))$. Let g be a rational function on X with*

$$\text{div}(g) = P - dD.$$

¹This is not in final form. The detailed version will be submitted elsewhere as [STW] for publication.

Then $x := gT^d$ is a prime element, and the induced grading on R/xR is irredundant.

We next record various examples.

Example 1.2. If a standard \mathbb{N} -graded ring R , as in the theorem, has a homogeneous prime element, we claim that R must be a polynomial ring over R_0 .

If $x \in R_d$ is a prime element, the theorem implies that $d = 1$. Independent of the theorem, note that R/xR is an \mathbb{N} -graded domain of dimension 1, with $[R/xR]_0$ algebraically closed, so R/xR is a numerical semigroup ring by [GW, Proposition 2.2.11]. Since it is standard graded, R/xR must be a polynomial ring. But then R is a polynomial ring as well.

Example 1.3. The hypothesis that the underlying field R_0 is algebraically closed is crucial in Theorem 1.1 and Example 1.2: the standard graded ring $\mathbb{Q}[x, y, z]/(x^2 + y^2 + z^2)$ has a homogeneous prime element x .

Example 1.4. In view of Example 1.2, the ring $R := \mathbb{C}[x, y, z]/(x^2 - yz)$, with the standard \mathbb{N} -grading, has no homogeneous prime element. However, for nonstandard gradings, there exist homogeneous primes:

Fix such a grading with $\deg x = a$, $\deg y = b$, and $\deg z = 2a - b$, where $\gcd(a, b) = 1$. Then one has a homogeneous prime element

$$\begin{aligned} & y^{2a-b} - z^b && \text{if } b \text{ is odd, or} \\ & y^{a-b/2} - z^{b/2} && \text{if } b \text{ is even,} \end{aligned}$$

which generates the kernel of the \mathbb{C} -algebra homomorphism $R \longrightarrow \mathbb{C}[t]$ with

$$x \longmapsto t^a, \quad y \longmapsto t^b, \quad z \longmapsto t^{2a-b}.$$

Example 1.5. The ring $\mathbb{C}[x, y, z]/(x^4 + y^2z + xz^2)$, with $\deg x = 4$, $\deg y = 5$, and $\deg z = 6$, has no homogeneous prime elements in view of Theorem 1.1 (1), since the corresponding \mathbb{Q} -divisor has degree $2/15$ by Proposition 2.2.

Example 1.6. Consider $\mathbb{C}[x, y, z]/(x^2 + y^3 + z^6)$, with $\deg x = 3$, $\deg y = 2$, and $\deg z = 1$. Then (z) is the unique homogeneous principal prime ideal: the corresponding \mathbb{Q} -divisor has degree 1, again by Proposition 2.2.

Example 1.7. Set $R := \mathbb{C}[x, y, z]/(x^2 - y^3 + z^7)$, with $\deg x = 21$, $\deg y = 14$, and $\deg z = 6$. Then the corresponding \mathbb{Q} -divisor has degree $1/42$ by Proposition 2.2, so the degree of a homogeneous prime element must divide 42. In view of the degrees of the generators of R , the possibilities are 6, 14, 21, and 42, and indeed there are prime elements with each of these degrees, namely z , y , x , and $y^3 - \lambda x^2$ for scalars $\lambda \neq 0, 1$, see also Example 3.3.

2. RATIONAL COEFFICIENT WEIL DIVISORS

We review the construction of normal graded rings in terms of \mathbb{Q} -divisors; this is work of Dolgačev [Do], Pinkham [Pi], and Demazure [De]. Let X be a normal projective variety. A \mathbb{Q} -divisor on X is a \mathbb{Q} -linear combination of irreducible subvarieties of X of codimension one. Let $D = \sum n_i V_i$ be such a divisor, where $n_i \in \mathbb{Q}$, and V_i are distinct. Set

$$\lfloor D \rfloor := \sum \lfloor n_i \rfloor V_i,$$

where $\lfloor n \rfloor$ is the greatest integer less than or equal to n . We define

$$\mathcal{O}_X(D) := \mathcal{O}_X(\lfloor D \rfloor).$$

The divisor D is *effective*, denoted $D \geq 0$, if each n_i is nonnegative. The *support* of the fractional part of D is the set

$$\text{supp}(\text{frac}(D)) := \{V_i \mid n_i \notin \mathbb{Z}\}.$$

Let $K(X)$ denote the field of rational functions on X . Each $g \in K(X)$ defines a Weil divisor $\text{div}(g)$ by considering the zeros and poles of g with appropriate multiplicity. As these multiplicities are integers, it follows that for a \mathbb{Q} -divisor D one has

$$\begin{aligned} H^0(X, \mathcal{O}_X(\lfloor D \rfloor)) &= \{g \in K(X) \mid \text{div}(g) + \lfloor D \rfloor \geq 0\} \\ &= \{g \in K(X) \mid \text{div}(g) + D \geq 0\} = H^0(X, \mathcal{O}_X(D)). \end{aligned}$$

A \mathbb{Q} -divisor D is *ample* if ND is an ample Cartier divisor for some $N \in \mathbb{N}$. In this case, the *generalized section ring* $R(X, D)$ is the \mathbb{N} -graded ring

$$R(X, D) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)) T^n,$$

where T is an element of degree 1, transcendental over $K(X)$.

Theorem 2.1 ([De, 3.5]). *Let R be an \mathbb{N} -graded normal domain that is finitely generated over a field R_0 . Let T be a homogeneous element of degree 1 in the fraction field of R . Then there exists a unique ample \mathbb{Q} -divisor D on $X := \text{Proj } R$ such that*

$$R_n = H^0(X, \mathcal{O}_X(nD)) T^n \quad \text{for each } n \geq 0.$$

The following result is due to Tomari:

Proposition 2.2 ([To, Proposition 2.1]). *For R and D as in the theorem above, one has*

$$\lim_{t \rightarrow 1} (1-t)^{\dim R} P(R, t) = (\deg D)^{\dim R - 1},$$

where $P(R, t)$ is the Hilbert series of R .

3. EXAMPLES

We don't give the proof of the Main Theorem here. Instead, we give some examples.

Example 3.1. Take $\mathbb{P}^1 := \text{Proj } \mathbb{C}[u, v]$, with points parametrized by u/v , and set

$$D := \frac{1}{2}(0) + \frac{1}{2}(\infty) - \frac{1}{2}(1)$$

Then $R := R(\mathbb{P}^1, D)$ is the \mathbb{C} -algebra generated by

$$x := \frac{u-v}{v}T^2, \quad y := \frac{u-v}{u}T^2, \quad z := \frac{(u-v)^2}{uv}T^3,$$

i.e., R is the hypersurface $\mathbb{C}[x, y, z]/(z^2 - xy(x-y))$, with $\deg x = 2 = \deg y$, and $\deg z = 3$. Note that $\deg D = 1/2$, and that $2D$ is an integral divisor. Theorem 1.1 (2) shows that

$$\bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD - P))T^n$$

is a prime ideal for $P \in \mathbb{P}^1 \setminus \{0, \infty, 1\}$. Indeed, for $P = [\lambda : 1]$ with $\lambda \neq 0, 1$, the displayed ideal is the prime $(x - \lambda y)R$. These are precisely the homogeneous principal prime ideals of R , with the points $0, \infty$, and 1 that belong to $\text{supp}(\text{frac}(D))$ corresponding respectively to the ideals xR, yR and $(x-y)R$ that are not prime.

Remark 3.2. Let D be a \mathbb{Q} -divisor on \mathbb{P}^1 such that $\deg D = 1/d$ where d is a positive integer, and dD is integral. Then the ring $R := R(\mathbb{P}^1, D)$ has infinitely many distinct homogeneous principal prime ideals: all points of \mathbb{P}^1 are linearly equivalent, so for each point P there exist a rational function g with

$$\text{div}(g) = P - dD,$$

and Theorem 1.1 (2) implies that $gT^d R$ is a prime ideal for each point P with

$$P \in \mathbb{P}^1 \setminus \text{supp}(\text{frac}(D)).$$

This explains the infinitely many prime ideals in Example 3.1, and also in Example 3.3 below; the latter, moreover, has homogeneous prime elements of different degrees:

Example 3.3. On $\mathbb{P}^1 := \text{Proj } \mathbb{C}[u, v]$, consider the \mathbb{Q} -divisor

$$D := \frac{1}{2}(\infty) - \frac{1}{3}(0) - \frac{1}{7}(1).$$

Then $R := R(\mathbb{P}^1, D)$ is the ring $\mathbb{C}[x, y, z]/(x^2 - y^3 + z^7)$, where

$$z := \frac{u^2(u-v)}{v^3}T^6, \quad y := \frac{u^5(u-v)^2}{v^7}T^{14}, \quad x := \frac{u^7(u-v)^3}{v^{10}}T^{21}.$$

For each point $P = [\lambda : 1]$ in $\mathbb{P}^1 \setminus \{0, \infty, 1\}$, i.e., with $\lambda \neq 0, 1$, one has a prime ideal

$$\bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD - P))T^n = (y^3 - \lambda x^2)R.$$

These, along with xR , yR , and zR , are precisely the homogeneous principal prime ideals.

Example 3.4. Set X to be the elliptic curve $\text{Proj } \mathbb{C}[u, v, w]/(v^2w - u^3 + w^3)$. Then

$$\text{div}(v/w) = P_1 + P_2 + P_3 - 3O,$$

where $O = [0 : 1 : 0]$ is the point at infinity, and

$$P_1 = [1 : 0 : 1], \quad P_2 = [\theta : 0 : 1], \quad P_3 = [\theta^2 : 0 : 1],$$

for θ a primitive cube root of unity. Take

$$D := \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 - O.$$

The ring $R := R(X, D)$ has generators

$$x := \frac{w}{v}T^2, \quad y := \frac{w}{v}T^3, \quad z := \frac{uw}{v^2}T^4,$$

so $R = \mathbb{C}[x, y, z]/(x^6 + y^4 - z^3)$. Since $\deg D = 1/2$, the only possibility for homogeneous prime elements are in degree 2. Indeed,

$$2D = P_1 + P_2 + P_3 - 2O = \text{div}(v/w) + O,$$

and $O \notin \text{supp}(\text{frac}(D))$, so $(w/v)T^2 = x$ is a prime element; note that xR is the *unique* homogeneous principal prime ideal of R , in contrast with Examples 3.1 and 3.3.

Example 3.5. With X and O as in the previous example, note that

$$\text{div}(u/w) = Q_1 + Q_2 - 2O,$$

where

$$Q_1 = [0 : i : 1], \quad Q_2 = [0 : -i : 1].$$

Consider the \mathbb{Q} -divisor

$$D = \frac{1}{2}Q_1 + \frac{1}{2}Q_2 - \frac{1}{2}O.$$

Then the ring $R := R(X, D)$ has generators

$$x := \frac{w}{u}T^2, \quad y := \frac{w}{u}T^3, \quad z := \frac{w}{u}T^4, \quad t := \frac{vw^2}{u^3}T^6,$$

and presentation

$$R = \mathbb{C}[x, y, z, t]/(y^2 - xz, x^6 - z^3 + t^2).$$

Again, since $\deg D = 1/2$, the only possibility for homogeneous prime elements are in degree 2. We see that

$$2D = Q_1 + Q_2 - O = \operatorname{div}(u/w) + O.$$

However, since $O \in \operatorname{supp}(\operatorname{frac}(D))$, Theorem 1.1 (2) does not apply. Indeed, $(w/u)T^2 = x$ is not a prime element. The key point is that X is not rational, and there does not exist a point P , linearly equivalent to $2D$, with $P \notin \operatorname{supp}(\operatorname{frac}(D))$.

4. RATIONAL SINGULARITIES

Let H be a numerical semigroup. For \mathbb{F} a field and t an indeterminate, set

$$\mathbb{F}[H] := \mathbb{F}[t^n \mid n \in H].$$

Question 4.1. Does $\mathbb{F}[H]$ deform to a normal \mathbb{N} -graded ring, i.e., does there exist a normal \mathbb{N} -graded ring R , with $x \in R$ homogeneous, such that $R/xR \cong \mathbb{F}[H]$?

Question 4.2. For which H does there exist R , as above, such R has rational singularities?

The following is a partial answer:

Proposition 4.3. *Let R be a normal ring of dimension 2, with an irredundant \mathbb{N} -grading, where $R_0 = \mathbb{F}$ is an algebraically closed field. Suppose x_0 is a homogeneous prime element that is part of a minimal reduction of R_+ , and that the induced grading on R/x_0R is irredundant. Then the following are equivalent:*

- (1) *The ring R has rational singularities.*
- (2) *There exist minimal \mathbb{F} -algebra generators x_0, \dots, x_r for R , with x_i homogeneous, and*

$$(4.3.1) \quad r + \deg x_0 > \deg x_1 > \dots > \deg x_r = r.$$

Example 4.4. Consider the \mathbb{Q} -divisor

$$D := \frac{5}{7}(0) - \frac{4}{7}(\infty)$$

on $\mathbb{P}^1 := \operatorname{Proj} \mathbb{C}[u, v]$, with points parametrized by u/v . Then $R := R(\mathbb{P}^1, D)$ has generators

$$w := \frac{v^2}{u^2}T^3, \quad x := \frac{v^3}{u^3}T^5, \quad y := \frac{v^4}{u^4}T^7, \quad z := \frac{v^5}{u^5}T^7.$$

The relations are readily seen to be the size two minors of the matrix

$$\begin{pmatrix} w & x & z \\ x & y & w^3 \end{pmatrix}.$$

Each point $P = [\lambda : 1]$ with $\lambda \neq 0$ gives a prime ideal

$$\bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD - P))T^n = (y - \lambda z)R,$$

and these are precisely the homogeneous principal prime ideals of R .

For example,

$$R/(y - z)R = \mathbb{C}[t^3, t^5, t^7].$$

Since $(y - z, w)R$ is a minimal reduction of R_+ and the grading on $R/(y - z)R$ is irredundant, Proposition 4.3 applies. The ring R has rational singularities since $a(R) = -3$, and the inequalities (4.3.1) indeed hold since

$$3 + \deg(y - z) > \deg y > \deg x > \deg w = 3.$$

Example 4.5. Take R as in Example 1.6, i.e., $R := \mathbb{C}[x, y, z]/(x^2 + y^3 + z^6)$, with $\deg x = 3$, $\deg y = 2$, and $\deg z = 1$. Then z is a prime element such that the induced grading on R/zR is irredundant; z is also part of the minimal reduction $(z, y)R$ of R_+ . Since $a(R) = 0$, the ring R does not have rational singularities; likewise, (4.3.1) does not hold since

$$2 + \deg z \not> \deg x.$$

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