DERIVATIONS AND CLOSED POLYNOMIALS IN POLYNOMIAL RINGS

TAKANORI NAGAMINE

Abstract. In this paper, we study closed polynomials over an integral domain of characteristic zero and give a criterion for a non-constant polynomial to be a closed polynomial.

1. Introduction

Let R be an integral domain with unit and let $R[\mathbf{X}] := R[x_1, \ldots, x_n]$ be the polynomial ring in n variables over R. We denote by $Q(R)$ the quotient field of R. A non-constant polynomial $f \in R[\mathbf{X}] \setminus R$ is a *closed polynomial* if the ring $R[f]$ is integrally closed in $R[\mathbf{X}]$. An R-linear map $D : R[\mathbf{X}] \to R[\mathbf{X}]$ is an *R-derivation* on $R[\mathbf{X}]$ if $D(fg) = fD(g) + gD(f)$ for $f, g \in R[\mathbf{X}]$. By using terms of derivations and their kernels, we can understand closed polynomials. The following result gives us a relation between closed polynomials and derivations and is a generalization of a part of [1, Theorem 1].

Theorem 1.1. (cf. [2, Theorem 3.1]) *Let R be an integral domain and $K := Q(R)$. For a non-constant polynomial $f \in R[\mathbf{X}] \setminus R$ satisfying $K[f] \cap R[\mathbf{X}] = R[f]$, the following conditions are equivalent.*

(1) *f is a closed polynomial.*
(2) *There are no polynomials $g \in K[\mathbf{X}]$ with $K[f] \subsetneq K[g]$.*

If the characteristic of R equals zero, then the following condition (3) *is equivalent to the condition* (1)*.*

(3) *There exist an R-derivation D on $R[\mathbf{X}]$ such that* $\operatorname{Ker} D = R[f]$.

Furthermore, closed polynomials relate the Jacobian conjecture as below. Let k be a field of characteristic zero and let $k[\mathbf{X}] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. For polynomials $f_1, \ldots, f_n \in k[\mathbf{X}]$, let $F := (f_1, \ldots, f_n)$. Then F defines a k-endomorphism on $k[\mathbf{X}]$ by $F(x_i) = f_i$ for $1 \le i \le n$. We define the *Jacobian matrix* of F with respect to $x_1, \ldots, x_n$ by

Key words and phrases. Closed polynomials, Derivations.

$$J(F) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \in k[\mathbf{X}].$$

Now we consider the following two conditions:

(A) F defines a k-automorphism on $k[\mathbf{X}]$.

(B) $\det J(F)$ belongs to $k \setminus \{0\}$.

Jacobian conjecture says that the implication "(B) $\Rightarrow$ (A)" holds true. If $n = 1$, then this conjecture is true. In the case where $n \geq 2$, however, this conjecture is still open. The following result gives us a relation between closed polynomials and this conjecture.

Proposition 1.2. *Let k be a field of characteristic zero. For polynomials $f_1, \ldots, f_n \in k[\mathbf{X}]$, let $F := (f_1, \ldots, f_n)$. If $\det J(F) \in k \setminus \{0\}$, then these polynomials $f_1, \ldots, f_n$ are closed polynomials.*

In this paper, we give a criterion for a polynomial $f \in R[\mathbf{X}]$ to be a closed polynomial, in the case where R is an arbitrary integral domain of characteristic zero. The main result in this paper is Theorem 2.4. As a corollary of this theorem, we get Proposition 1.2.

2. Criteria for closed polynomials

Let R be an integral domain and let $R[\mathbf{X}] = R[x_1, \ldots, x_n]$ be the polynomial ring in n variables over R. For a polynomial $f \in R[\mathbf{X}]$,

$$\hat{f} := \gcd(f_{x_1}, \ldots, f_{x_n}),$$

where f_{x_i} is the partial derivative of f with respect to x_i and we take the greatest common divisor of $f_{x_1}, \ldots, f_{x_n}$ as polynomials in $Q(R)[\mathbf{X}]$. Now we represent $f \in R[\mathbf{X}]$ as follows:

$$f = \sum_{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n} u_{\mathbf{a}} x_1^{a_1} \cdots x_n^{a_n},$$

where $u_{\mathbf{a}} \in R$ and $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n$. We define the *support set* of f by $\mathrm{Supp}(f) := \{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n \mid u_{\mathbf{a}} \neq 0\}$. For $\mathbf{w} = (w_1, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$, we define the *weighted degree* of f with respect to $\mathbf{w}$ by the maximal element of the set $\{\mathbf{a} \cdot \mathbf{w} \mid \mathbf{a} \in \mathrm{Supp}(f)\}$, where $\mathbf{a} \cdot \mathbf{w} = a_1 w_1 + \cdots + a_n w_n$ and denote by $\deg_{\mathbf{w}}(f)$. Note that the weighted degree of the zero-polynomial is $-\infty$. Also, we denote simply $\deg(f)$ by the weighted degree of f with respect to $(1, \ldots, 1)$.

Remark 2.1. For any $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^n$, the weighted degree of polynomials with respect to $\mathbf{w}$ is a degree function on $R[\mathbf{X}]$. That is, for $f, g \in R[\mathbf{X}]$, the following conditions are satisfied.

(1) $\deg_{\mathbf{w}}(f) = -\infty$ if and only if $f = 0$.
(2) $\deg_{\mathbf{w}}(fg) = \deg_{\mathbf{w}}(f) + \deg_{\mathbf{w}}(g)$.
(3) $\deg_{\mathbf{w}}(f+g) \leq \max\{\deg_{\mathbf{w}}(f), \deg_{\mathbf{w}}(g)\}$.

Definition 2.2. Let $f \in R[\mathbf{X}]$ and $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^n$. Assume that $\deg_{\mathbf{w}}(f) \geq 2$. Then we denote by $N_{\mathbf{w}}(f)$ the smallest positive prime dividing $\deg_{\mathbf{w}}(f)$.

Example 2.3. For $f = x^9 + x^6y^2 + x^3y^4 \in \mathbb{Z}[x, y]$, we can easily see that $\mathrm{Supp}(f) = \{(9, 0), (6, 2), (3, 4)\}$. Then,

(1) for $\mathbf{u} = (1, 1)$, $\deg_{\mathbf{u}}(f) = \deg(f) = 9$ and $N_{\mathbf{u}}(f) = 3$,
(2) for $\mathbf{v} = (0, 1)$, $\deg_{\mathbf{v}}(f) = 4$ and $N_{\mathbf{v}}(f) = 2$,
(3) for $\mathbf{w} = (1, 2)$, $\deg_{\mathbf{w}}(f) = 11$ and $N_{\mathbf{w}}(f) = 11$.

In general, for given a polynomial $f \in R[\mathbf{X}] \setminus R$, it is difficult to understand whether f is a closed polynomial or not. The following gives a sufficient condition for f to be a closed polynomial and is the main theorem in this paper.

Theorem 2.4. (cf. [3, Proposition 3.11]) *Let R be an integral domain of characteristic zero and let $f \in R[\mathbf{X}] \setminus R$ be a non-constant polynomial such that $Q(R)[f] \cap R[\mathbf{X}] = R[f]$. If there exists $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^n$ such that $\deg_{\mathbf{w}}(f) = 1$ or*

$$\deg_{\mathbf{w}}(f) \geq 2 \textit{ and } \deg_{\mathbf{w}}(\hat{f}) < \frac{N_{\mathbf{w}}(f) - 1}{N_{\mathbf{w}}(f)} \deg_{\mathbf{w}}(f),$$

then f is a closed polynomial.

To show this theorem, we prepare the following lemma.

Lemma 2.5. *Let R be an integral domain. Let $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^n$ and let $f, g \in R[\mathbf{X}] \setminus R$ with $f \in R[g]$. Assume that $\deg_{\mathbf{w}}(f) > 0$ and $f = u(g)$ for a polynomial $u(t) \in R[t]$ in one variable t of degree $m \geq 1$. Then the following assertions hold true.*

(1) *$\deg_{\mathbf{w}}(f) = m \deg_{\mathbf{w}}(g)$. In particular, m divides $\deg_{\mathbf{w}}(f)$.*
(2) *If the characteristic of R equals zero, then*

$$\deg_{\mathbf{w}}(\hat{f}) \geq \frac{m-1}{m} \deg_{\mathbf{w}}(f).$$

Proof. **(1)** For $u_0 \in R \setminus \{0\}$ and $u_1, \ldots, u_m \in R$,

$$f = u(g) = u_0 g^m + u_1 g^{m-1} + \cdots + u_{m-1} g + u_m.$$

Since $\deg_{\mathbf{w}}(f) > 0$, $\deg_{\mathbf{w}}(g) > 0$. This implies that $\deg_{\mathbf{w}}(g^i) \geq \deg_{\mathbf{w}}(g^j)$ if $i \geq j$. So,

$$\deg_{\mathbf{w}}(f) = \deg_{\mathbf{w}}(u(g)) = \deg_{\mathbf{w}}(u_0 g^m) = m \deg_{\mathbf{w}}(g).$$

(2) Since $f = u(g)$, $f_{x_i} = u'(g)g_{x_i}$ for $1 \leq i \leq n$, where $u'(t) = du/dt$. This implies that each f_{x_i} is divided by $u'(g)$, so $u'(g)$ divides $\hat{f}$ as a polynomial defined over $Q(R)$. Therefore $\deg_{\mathbf{w}}(\hat{f}) \geq \deg_{\mathbf{w}}(u'(g))$. On the other hand, since the characteristic of R equals zero, $mu_0 \neq 0$. Therefore $\deg_{\mathbf{w}} u'(g) = (m-1)\deg_{\mathbf{w}}(g)$, so we have

$$\deg_{\mathbf{w}}(\hat{f}) \geq \deg_{\mathbf{w}}(u'(g)) = (m-1)\deg_{\mathbf{w}}(g) = \frac{m-1}{m}\deg_{\mathbf{w}}(f).$$

□

Now, we start the proof of Theorem 2.4.

Proof of Theorem 2.4. Set $K := Q(R)$. By Theorem 1.1, we enough to show that for $g \in K[\mathbf{X}]$ with $K[f] \subset K[g]$, $K[f] = K[g]$.

Let $g \in K[\mathbf{X}]$ with $K[f] \subset K[g]$. Since $f \in K[g]$, there exists $u(t) \in K[t]$ of degree m such that $f = u(g)$. We write $u(t)$ as

$$u(t) = u_0t^m + u_1t^{m-1} + \cdots + u_{m-1}t + u_m,$$

for some $u_i \in K$ and $u_0 \neq 0$. By Lemma 2.5 (1), $\deg_{\mathbf{w}}(f) = m\deg_{\mathbf{w}}(g)$. We enough to show that $m = 1$. Indeed, if $m = 1$, then $f = u_0g + u_1$. This implies $g \in K[f]$, so $K[f] = K[g]$.

If $\deg_{\mathbf{w}}(f) = 1$, then obviously $m = 1$. On the other hand, we suppose that $\mathbf{w} \in (\mathbb{Z}_{\geq 0})^n$ satisfies $\deg_{\mathbf{w}}(f) \geq 2$ and

$$\deg_{\mathbf{w}}(\hat{f}) < \frac{N_{\mathbf{w}}(f)-1}{N_{\mathbf{w}}(f)}\deg_{\mathbf{w}}(f).$$

Since the characteristic of R equals zero, by Lemma 2.5 (2),

$$\deg_{\mathbf{w}}(\hat{f}) \geq \frac{m-1}{m}\deg_{\mathbf{w}}(f).$$

By comparing the above two inequalities, we have $N_{\mathbf{w}}(f) > m$. By using Lemma 2.5 (1) again, we see that m divides $\deg_{\mathbf{w}}(f)$. But the number $N_{\mathbf{w}}(f)$ is the smallest positive prime dividing $\deg_{\mathbf{w}}(f)$, hence $m = 1$. Therefore f is a closed polynomial.

□

Next, we prove Proposition 1.2 by using Theorem 2.4.

Proof of Proposition 1.2. Suppose that $\det J(F) \in k \setminus \{0\}$, where $F = (f_1, \ldots . f_n)$, $f_i \in k[\mathbf{X}]$ and k is a field of characteristic zero. Then there exist $g_{ij} \in k[\mathbf{X}]$ such that

$$\frac{\partial f_i}{\partial x_j} = g_{ij}\hat{f}_i$$

for $1 \leq i, j \leq n$. Then we have

$$\begin{aligned}
\det J(F) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}} \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) g_{1\sigma(1)} \hat{f}_1 \cdots g_{n\sigma(n)} \hat{f}_n \\
&= (\hat{f}_1 \cdots \hat{f}_n) \cdot \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) g_{1\sigma(1)} \cdots g_{n\sigma(n)},
\end{aligned}$$

where S_n is the symmetric group on n elements. For each permutation $\sigma \in S_n$, $\operatorname{sgn}(\sigma)$ denotes the signature of σ. Since $\det J(F) \in k \setminus \{0\}$, $\hat{f}_i \in k \setminus \{0\}$, so $\deg(\hat{f}_i) = 0$ for $1 \leq i \leq n$. Therefore $\hat{f}_i$ satisfies the inequality of Theorem 2.4 for $\mathbf{w} = (1, \ldots, 1)$ if $\deg(f_i) \geq 2$. Otherwise $\deg(f_i) = 1$. By Theorem 2.4, f_i is a closed polynomial for $1 \leq i \leq n$. □

Proposition 2.6. *Let k be a field of characteristic zero. For a non-constant polynomial $f \in k[\mathbf{X}] \setminus k$, the following conditions are equivalent.*

(1) $\deg(\hat{f}) = \deg(f) - 1$.

(2) *There exist $r_1, \ldots, r_n \in k$ with $(r_1, \ldots, r_n) \neq (0, \ldots, 0)$ such that $f \in k[r_1 x_1 + \cdots + r_n x_n]$.*

Proof. **(1) ⇒ (2)** There exist $r_1, \ldots, r_n \in k[\mathbf{X}]$ such that $f_{x_i} = r_i \hat{f}$ for $1 \leq i \leq n$. We may assume that $f_{x_1} \neq 0$. Then

$$d - 1 = \deg(\hat{f}) \leq \deg(f_{x_1}) \leq d - 1,$$

so we have $\deg(f_{x_1}) = d - 1 = \deg(\hat{f})$ and $r_1 \in k \setminus \{0\}$. For $1 \leq i \leq n$ with $f_{x_i} \neq 0$, using the same argument, we have $r_i \in k \setminus \{0\}$. On the other hand, for $1 \leq i \leq n$ with $f_{x_i} = 0$, we have $r_i = 0$. So r_i is either a non-zero constant polynomial or 0 for $1 \leq i \leq n$. Set $g := r_1 x_1 + \cdots + r_n x_n$. By Theorem 2.4, g is a closed polynomial because $\deg(g) = 1$. Therefore, by Theorem 1.1, there exists a k-derivation D on $k[\mathbf{X}]$ such that $\operatorname{Ker} D = k[g]$. Then

$$\begin{aligned}
D(f) &= D(x_1) f_{x_1} + \cdots + D(x_n) f_{x_n} \\
&= D(x_1) r_1 \hat{f} + \cdots + D(x_n) r_n \hat{f} \\
&= D(g) \cdot \hat{f} = 0.
\end{aligned}$$

Therefore $f \in \operatorname{Ker} D = k[g]$.

(2) ⇒ (1) Set $g := r_1 x_1 + \cdots + r_n x_n$. Since $f \in k[g]$, there exists $u(t) \in k[t]$ of degree $\deg(f)$ with $f = u(g)$. Then $f_{x_i} = r_i u'(g)$ for $1 \leq i \leq n$, where $u'(t) = du(t)/dt$. Then $\deg(u'(g)) = \deg(f) - 1$ and $u'(g)$ divides $\hat{f}$. So we have

$$\deg(u'(g)) \leq \deg(\hat{f}) \leq \deg(f) - 1.$$

Therefore $\deg(\hat{f}) = \deg(f) - 1$. □

Remark 2.7. In the proof of Proposition 2.6, we use a fundamental result on derivations. For an integral domain R, let D be an R-derivation on $R[\mathbf{X}]$. The we can represent D as the following form:

$$D = D(x_1)\frac{\partial}{\partial x_1} + \cdots + D(x_n)\frac{\partial}{\partial x_n}.$$

Corollary 2.8. *Let k be a field of characteristic zero. For a non-constant polynomial $f \in k[\mathbf{X}] \setminus k$ of degree prime, the following conditions are equivalent.*

(1) *f is a closed polynomial.*
(2) $\deg(\hat{f}) < \deg(f) - 1$.

Proof. **(1) ⇒ (2)** Suppose that $\deg(\hat{f}) = \deg(f) - 1$. By Proposition 2.6, there exist $r_1, \ldots, r_n \in k$ with $(r_1, \ldots, r_n) \neq (0, \ldots, 0)$ such that $f \in k[g]$, where $g := r_1x_1 + \cdots + r_nx_n$. Since $\deg(f)$ is prime, especially $\deg(f) \geq 2$, $k[f] \subsetneq k[g]$. By Theorem 1.1, f is not a closed polynomial.

(2) ⇒ (1) Suppose that $\deg(\hat{f}) < \deg(f) - 1$. Since $\deg(f)$ is prime, $N_{\mathbf{w}}(f) = \deg(f)$, where $\mathbf{w} = (1, \ldots, 1)$. Then

$$\frac{N_{\mathbf{w}}(f) - 1}{N_{\mathbf{w}}(f)}\deg(f) = \frac{\deg(f) - 1}{\deg(f)}\deg(f) = \deg(f) - 1.$$

Therefore we have

$$\deg(\hat{f}) < \deg(f) - 1 = \frac{N_{\mathbf{w}}(f) - 1}{N_{\mathbf{w}}(f)}\deg(f).$$

By Theorem 2.4, f is a closed polynomial.

□

From this, when you want to check the closedness of polynomial of degree prime, we only have to calculate $\hat{f}$.

References

[1] I. V. Arzhantsev and A. P. Petravchuk, Closed polynomials and saturated subalgebras of polynomial algebras, Ukrainian Math. J., **59** (2007), 1783–1790.

[2] H. Kojima and T. Nagamine, Closed polynomials in polynomial rings over integral domains, J. Pure Appl. Algebra, **219** (2015), 5493–5499.

[3] T. Nagamine, Derivations having divergence zero and closed polynomials over domains, J. Algebra, **462** (2016), 67–76.

(T. Nagamine) Graduate School of Science and Technology, Niigata University, 8050 Ikarashininocho, Nishi-ku, Niigata 950-2181, Japan

E-mail address: t.nagamine14@m.sc.niigata-u.ac.jp