Telescope conjecture for homotopically smashing t-structures over commutative noetherian rings

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Tsutomu Nakamura (University of Tokyo)

j.w.w. Michal Hrbek (Czech Academy of Science)

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§1. R : ring $\quad \mathcal{T}, \mathcal{F} \subset \operatorname{Mod} R$

$(\mathcal{T}, \mathcal{F})$ is a torsion pair if

(i) $\mathcal{T} \cap \mathcal{F} = \{0\}$;

(ii) $\mathcal{T}$ is closed under quotient objects and $\mathcal{F}$ is closed under subobjects;

(iii) $^{\exists} 0 \to \underset{\substack{\mathbin{\rotatebox[origin=c]{90}{$\in$}} \\ \mathcal{T}}}{L} \to M \to \underset{\substack{\mathbin{\rotatebox[origin=c]{90}{$\in$}} \\ \mathcal{F}}}{N} \to 0$: exact $\quad {}^{\forall} M \in \operatorname{Mod} R$.

($\mathcal{T}^{\perp} = \{ N \in \operatorname{Mod} R \mid \operatorname{Hom}_R(L, N) = 0 \ {}^{\forall} L \in \mathcal{T} \} = \mathcal{F}$, $\quad \mathcal{T} = {}^{\perp}\mathcal{F}$)

A torsion pair $(\mathcal{T}, \mathcal{F})$ is hereditary if $\mathcal{T}$ is closed under subobjects.

Gabriel (1962) R: commutative noetherian

$$\left\{\begin{matrix}\text{hereditary torsion} \\ \text{pairs in Mod } R\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{specialization-closed} \\ \text{subsets of Spec} R\end{matrix}\right\}$$

$$\begin{matrix} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \{M \in \operatorname{Mod} R \mid \operatorname{Supp} M \subseteq W\} & \longleftarrow\!\shortmid & W \end{matrix}$$

Göbel–Shelah (1982)

Torsion pairs in Mod $\mathbb{Z}$ form a proper class.

§2.

$C(\operatorname{Mod} R) \ni X = (\cdots \to X^i \xrightarrow{d_X^i} X^{i+1} \to \cdots)$

$\downarrow$

$K(\operatorname{Mod} R)$ $\quad \operatorname{Hom}_{K(\operatorname{Mod} R)}(X, Y) := \operatorname{Hom}_{C(\operatorname{Mod} R)}(X, Y) / \underset{\text{homotopy equiv.}}{\sim}$

$f: X \to Y$

$f \sim 0 \iff \exists \{s^i\}_{i \in \mathbb{Z}}$ s.t.

$f^i = d_Y^{i-1} \circ s^i + s^{i+1} \circ d_X^i$

$\downarrow$

$X \xrightarrow{f} \underbrace{Y \to C(f) \to X[1]}_{\text{degreewise split exact}}$: triangle

$D(\operatorname{Mod} R)$

$\operatorname{Hom}_{D(\operatorname{Mod} R)}(X, Y) := \varinjlim \operatorname{Hom}_{K(\operatorname{Mod} R)}(X, Y')$

$\{\cdots \to Y' \to Y'' \to \cdots\}$

quism $\nwarrow \circlearrowleft \nearrow$ quism

Y

$$K_{ac}(\mathrm{Mod}\,R) \underset{\exists}{\overset{\perp}{\rightleftarrows}} K(\mathrm{Mod}\,R) \underset{\perp}{\overset{\exists\ \text{left adjoint}}{\rightleftarrows}} K_{ac}(\mathrm{Mod}\,R)^{\perp} = \{\text{K-injective complexes}\}$$

$$\begin{array}{ccccccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ C(f)[-1] & \longrightarrow & X & \xrightarrow[\exists\ \text{K-inj. resol. (quasi-isom)}]{f} & I & \longrightarrow & \underset{\text{acyclic (exact) complex}}{C(f)} \end{array}$$

$$D(\mathrm{Mod}\,R) = K(\mathrm{Mod}\,R)/K_{ac}(\mathrm{Mod}\,R) \cong K_{ac}(\mathrm{Mod}\,R)^{\perp}$$

<u>Def</u> $\mathcal{T}$: triangulated category (e.g. $D(\mathrm{Mod}\,R)$)

A (Bousfield) localization functor on $\mathcal{T}$ is a triangulated functor $L : \mathcal{T} \longrightarrow \mathcal{T}$ endowed with a natural transformation $\eta : \mathrm{id}_{\mathcal{T}} \longrightarrow L$ s.t. $L\eta$ is invertible and $L\eta = \eta L$.

$$\operatorname{Ker} L \underset{}{\overset{\text{inc}}{\underset{\perp}{\rightleftarrows}}} \mathcal{T} \underset{\text{inc}}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \operatorname{Im} L$$

$$(\ \mathcal{T}/\operatorname{Ker} L \cong \operatorname{Im} L\ )$$

<u>Ex</u> R : comm. noeth. ring, $\mathfrak{a} \subset R$: ideal

$$\Lambda^{\mathfrak{a}} = \varprojlim_{n \geq} (- \otimes_R R/\mathfrak{a}^n)$$

$$\begin{array}{ccc} L\Lambda^{\mathfrak{a}} : D(\mathrm{Mod}\,R) & \longrightarrow & D(\mathrm{Mod}\,R) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \underset{\substack{\text{K-proj. resol} \\ \text{(K-flat)}}}{P \xrightarrow{\sim} X} & & \Lambda^{\mathfrak{a}} P \xrightarrow{\sim} L\Lambda^{\mathfrak{a}} X \end{array}$$

is a localization functor on $D(\mathrm{Mod}\,R)$.

Def $\mathcal{T}$: triangulated category with arbitrary direct sums (e.g. $D(\mathrm{Mod}\,R)$)

$X \in \mathcal{T}$ is compact if

$$\bigoplus_i \mathrm{Hom}_{\mathcal{T}}(X, Y_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(X, \bigoplus_i Y_i)$$

for any direct sum $\bigoplus_i Y_i$ in $\mathcal{T}$.

$\mathcal{T}$ is compactly generated if

"$\mathcal{T}$ is generated by S"

$\exists\, S$: (small) set of compact objects s.t. $\mathrm{Loc}_{\mathcal{T}} S = \mathcal{T}$.

the smallest localizing subcat. contaning S

triangulated subcat. closed under $\bigoplus$

$$\left(\mathrm{Loc}_{\mathcal{T}} S = \mathcal{T} \iff \left(\bigcup_{i \in \mathbb{Z}} S[i] \right)^{\perp} = \{0\} \right)$$

Ex R : any ring

$$\left\{ \begin{array}{c} \text{compact objects} \\ \text{in } D(\mathrm{Mod}\,R) \end{array} \right\} = \left\{ X \in D(\mathrm{Mod}\,R) \;\middle|\; \begin{array}{l} X \cong {}^{\exists}P : \text{bounded complex} \\ \text{of f.g. proj. modules} \end{array} \right\}$$

$$\begin{array}{c} \| \\ D(\mathrm{Mod}\,R)^c \end{array} \cong K^b(\mathrm{proj}\,R)$$

essentially small

$\mathrm{Loc}\, D(\mathrm{Mod}\,R)^c = \mathrm{Loc}\{R\} = D(\mathrm{Mod}\,R)$: compactly generated

A generalized version of an original question by Ravenel (1984)

Q (Telescope conjecture)

$\mathcal{T}$: compactly generated triangulated category

If a localization functor $L : \mathcal{T} \to \mathcal{T}$ is smashing

(i.e. L commutes with $\overset{\infty}{\oplus}$), is $\mathrm{Ker}\, L$ generated by

compact objects in $\mathcal{T}$? "The converse is true."

Yes, for $D(\mathrm{Mod}\, R)$ over a comm. noeth. ring R,
by Neeman (1992) :

$$\left\{\begin{array}{c}\text{kernels of}\\ \text{localzation}\\ \text{functors}\end{array}\right\}$$

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{localizing subcategories}\\ \text{of } D(\mathrm{Mod}\, R)\end{array}\right\} & \xleftrightarrow{1-1} & \left\{\text{subsets of } \mathrm{Spec}\, R\right\} \\
\cup & & \cup \\
\left\{\begin{array}{c}\text{smashing subcategories}\\ \text{of } D(\mathrm{Mod}\, R)\end{array}\right\} & \xleftrightarrow{1-1} & \left\{\begin{array}{c}\text{specialization-closed}\\ \text{subsets of } \mathrm{Spec}\, R\end{array}\right\} \\
\Psi & & \Psi \\
\mathrm{Loc}\left\{ K(\underline{x}) \;\middle|\; \begin{array}{l} V(\underline{x}) \subseteq W \\ \underline{x} = x_1, \ldots, x_n \end{array} \quad x_i \in R \right\} & \longleftarrow & W
\end{array}$$

$$\left\{\begin{array}{c}\text{kernels}\\ \text{of smashing}\\ \text{localizations}\end{array}\right\}$$

Def (Beĭlinson–Bernstein–Deligne 1982)

$\mathcal{T}$: triangulated category

$\mathcal{U}, \mathcal{V} \subset \mathcal{T}$: full subcategories

aisle coaisle

$(\mathcal{U}, \mathcal{V}[1])$ is a t-structure in $\mathcal{T}$ if

(i) $\mathrm{Hom}_{\mathcal{T}}(U, V) = 0 \quad {}^{\forall}U \in \mathcal{U}, {}^{\forall}V \in \mathcal{V}$;

(ii) ${}^{\exists}\underset{\mathcal{U}}{\underset{\cap}{U}} \longrightarrow X \longrightarrow \underset{\mathcal{V}}{\underset{\cap}{V}} \longrightarrow U[1] \quad {}^{\forall}X \in \mathcal{T}$;

(iii) $\mathcal{U}[1] \subset \mathcal{U}$.

A t-structure $(\mathcal{U}, \mathcal{V}[1])$ is stable if $\mathcal{U}[-1] \subset \mathcal{U}$.

$\{$stable t-structurs in $\mathcal{T}\} \overset{1-1}{\longleftrightarrow} \{$localization functors on $\mathcal{T}\}$

Ex $(D^{\leq 0}(\mathrm{Mod}\,R), D^{\geq 0}(\mathrm{Mod}\,R))$ the standard t-structure

$$\tau^{\leq 0}X \to X \to \tau^{>0}X \to (\tau^{\leq 0}X)[1]$$

$$\tau^{\leq 0}X = (\cdots \to X^{-1} \to X^{0} \to \mathrm{Im}\,d_X^0 \to 0) \in D^{\leq 0}(\mathrm{Mod}\,R) \text{ : aisle}$$

$$\tau^{>0}X = (0 \to \mathrm{Coker}\,d_X^0 \to X^2 \to X^3 \to \cdots) \in D^{>0}(\mathrm{Mod}\,R) \text{ : coaisle}$$

$$\tau^{\leq 0} : D(\mathrm{Mod}\,R) \longrightarrow D^{\leq 0}(\mathrm{Mod}\,R)$$
$$\tau^{<0} : D(\mathrm{Mod}\,R) \longrightarrow D^{>0}(\mathrm{Mod}\,R)$$

$$D^{\leq 0}(\mathrm{Mod}\,R) \cap D^{\geq 0}(\mathrm{Mod}\,R) \cong \mathrm{Mod}\,R$$

<u>Fact</u> (BBD)

The heart $\mathcal{U} \cap \mathcal{V}[1]$ of a t-structure $(\mathcal{U}, \mathcal{V})$ in $\mathcal{T}$ is an abelian category with the exact structure induced by $\mathcal{T}$.

e.g. {"perverse sheaves"}

= (the heart of a t-structure in some derived category)

$(\mathcal{U}, \mathcal{V}[1])$: t-structure in $\mathcal{T}$

$$\begin{tikzcd} \mathcal{H} = \mathcal{U} \cap \mathcal{V}[1] \arrow[r, hook] \arrow[d] & \mathcal{T} \\ D^b(\mathcal{H}) \arrow[ur, dotted, "\exists ?"'] \arrow[u, phantom, "\circlearrowleft", shift left=8] & \end{tikzcd}$$

"realization functor"

↑

triangulated / derived equivalence?

⇝

tilting theory

When is $\mathcal{H}$ Grothendieck?

⇝

"homotopically smashing t-structure"

§ 3. $\mathcal{A}$: Grothendieck category

$\mathrm{Cat} \ni I$: small category (e.g. $I = \{1 \to 2 \to 3 \to \cdots\}$)

$$\begin{tikzcd} \mathcal{A} \arrow[r] & \mathcal{A}^I \arrow[l, bend right=60, "\mathrm{colim}_I"'] \arrow[l, bend left=60, "\lim_I"] \end{tikzcd} \qquad \begin{tikzcd} C(\mathcal{A}) \arrow[r] & C(\mathcal{A}^I) \cong C(\mathcal{A})^I \arrow[l, bend right=50, "\mathrm{colim}_I"'] \arrow[l, bend left=50, "\lim_I"] \end{tikzcd}$$

$$\begin{tikzcd} D(\mathcal{A}) \arrow[r] & D(\mathcal{A}^I) \arrow[l, bend right=50, "\mathrm{Lcolim}_I"'] \arrow[l, bend left=50, "\mathrm{Rlim}_I"] \arrow[r, "(\neq \text{ usually})"'] & D(\mathcal{A})^I \end{tikzcd}$$

Grothendieck 1983

Cisinski 2003

Groth 2013

Stovicek 2014

A (strong and stable) derivator is a (2-)functor

$$\begin{array}{ccc} \mathbb{D} : \mathrm{Cat}^{op} & \longrightarrow & \mathrm{CAT} \\ \Psi & & \Psi \\ I & \longmapsto & \mathbb{D}(I) : \text{triangulated} \end{array}$$

s.t.

$$\begin{tikzcd} \mathbb{D}(\mathbb{1}) \arrow[r] & \mathbb{D}(I) \arrow[l, bend right=50, "{}^{\exists}\mathrm{hocolim}_I"'] \arrow[l, bend left=50, "{}^{\exists}\mathrm{holim}_I"] \arrow[r, "\substack{\text{ess.}\\ \text{surj.}\\ \text{full}}"'] & \mathbb{D}(\mathbb{1})^I \end{tikzcd} \quad (\mathbb{1} = \{1\} \leftarrow I)$$

+ several conditions.

(e.g. $\mathbb{D}(I) := D(\mathcal{A}^I)$)

$\mathbb{D}$ is called compactly generated if $\mathbb{D}(1)$ is compactly generated.

the underlying triangulated category

Def $\mathcal{T}$: triangulated category with arbitrary direct sums

A t-structure $(\mathcal{U}, \mathcal{V}[1])$ in $\mathcal{T}$ is smashing if $\mathcal{V}$ is closed under direct sums.

A t-structure $(\mathcal{U}, \mathcal{V}[1])$ in $\mathcal{T}$ is compactly generated if $\mathcal{V} = S^{\perp}$ for some set S of compact objects in $\mathcal{T}$.

$$\begin{tikzcd}
\{\text{smashing t-structures in } \mathcal{T}\} & \{\text{smashing localizations on } \mathcal{T}\} \arrow[l, hook'] \\
\left\{\begin{array}{l}\text{compactly generated}\\ \text{t-structures in } \mathcal{T}\end{array}\right\} \arrow[u, hook] & \left\{\begin{array}{l}\text{localizations } L \text{ on } \mathcal{T} \text{ with}\\ \operatorname{Ker} L \text{ generated by a set}\\ \text{of compact objects in } \mathcal{T}\end{array}\right\} \arrow[l, hook'] \arrow[u, hook, "\overset{?}{=}\ \text{T.C.}"']
\end{tikzcd}$$

Rem

$$\begin{tikzcd}
\{\text{torsion pairs in } \operatorname{Mod}\mathbb{Z}\} \arrow[d, hook, "\exists"'] \\
\{\text{smashing t-structures in } D(\operatorname{Mod}\mathbb{Z})\}
\end{tikzcd}$$

$\{$smashing t-structures in $D(\operatorname{Mod}\mathbb{Z})\}$ is a proper class.

Def (Saorín–Šťovíček–Virill 2017)

$\mathbb{D}$: (strong and stable) derivator

A t-structure $(\mathcal{U}, \mathcal{V}[1])$ in $\mathbb{D}(\mathbb{1})$ is homotopically smashing if $\mathcal{V}$ is closed under directed homotopy colimits, i.e, homotopy colimits with respect to directed sets.

Fact (SŠV, Laking 2020)

$\mathbb{D}$: compactly generated derivator

$(\mathcal{U}, \mathcal{V}[1])$: non-degenerate t-structure in $\mathbb{D}(\mathbb{1})$

$$\bigcap_{i \in \mathbb{Z}} \mathcal{U}[i] = 0 = \bigcap_{i \in \mathbb{Z}} \mathcal{V}[i]$$

$(\mathcal{U}, \mathcal{V}[1])$ is homotopically smashing.

$\iff (\mathcal{U}, \mathcal{V}[1])$ is smashing and $\mathcal{U} \cap \mathcal{V}[1]$ is Grothendieck.

$$\begin{tikzcd}
\{\text{smashing t-structures in } \mathbb{D}(\mathbb{1})\} & \\
\left\{\begin{array}{c}\text{homotopically smashing} \\ \text{t-structures in } \mathbb{D}(\mathbb{1})\end{array}\right\} \arrow[u, hook] \arrow[r, leftarrow, hook] & \{\text{smashing localizations on } \mathbb{D}(\mathbb{1})\} \\
\left\{\begin{array}{c}\text{compactly generated} \\ \text{t-structures in } \mathbb{D}(\mathbb{1})\end{array}\right\} \arrow[u, hook, "\overset{?}{=}"'] \arrow[r, leftarrow, hook] & \left\{\begin{array}{c}\text{localizations } L \text{ on } \mathbb{D}(\mathbb{1}) \text{ with} \\ \operatorname{Ker} L \text{ generated by a set} \\ \text{of compact objects in } \mathbb{D}(\mathbb{1})\end{array}\right\} \arrow[u, hook, "\overset{?}{=}\ \text{T.C.}"']
\end{tikzcd}$$

Q For which compactly generated derivator $\mathbb{D}$, is every homotopically smashing t-structure in $\mathbb{D}(\mathbb{1})$ compactly generated ?

$$\begin{array}{rccc} \mathbb{D}_{\mathrm{Mod}\,R} : & \mathrm{Cat}^{op} & \longrightarrow & \mathrm{CAT} \\ & \Psi & & \Psi \\ & I & \longmapsto & D(\mathrm{Mod}\,R^I) \end{array} \qquad \mathbb{D}_{\mathrm{Mod}\,R}(\mathbb{1}) = D(\mathrm{Mod}\,R)$$

Rem $(\mathcal{U}, \mathcal{V}[1])$: t-structure in $D(\mathrm{Mod}\,R)$

$(\mathcal{U}, \mathcal{V}[1])$: homotopically smashing.

$\iff \mathcal{V}$ is closed under directed limits defined in $C(\mathrm{Mod}\,R)$.

§4. R : comm. noeth. ring

Thm Every homotopically smashing t-structure in $D(\mathrm{Mod}\,R)$ is compactly generated.

Sketh of proof

$$\left\{ \begin{array}{l} \text{compactly generated} \\ \text{t-structures in } D(\mathrm{Mod}\,R) \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \phi : \mathbb{Z} \to 2^{\mathrm{Spec}\,R} \;\middle|\; \begin{array}{l} \phi(i) : \text{sp.-cl. } \forall i \\ i \leq j \Rightarrow \phi(i) \supseteq \phi(j) \end{array} \right\}$$

$$\begin{array}{ccc} \Psi & & \Psi \\ (\mathcal{U}_\phi, \mathcal{V}_\phi[1]) & \longleftarrow & \underline{\phi} \\ \| \quad\quad \| & & \text{sp-filtration} \\ {}^\perp(S^\perp) \quad S^\perp & & \end{array}$$

$S = \{ K(\underline{x})[-i] \mid i \in \mathbb{Z},\ V(\underline{x}) \subset \phi(i),\ \underline{x} = x_1, \dots, x_n,\ x_j \in R \}$

by Alonso Tarrío–Jeremías López–Saorín 2010.

Take a homotopically smashing t-structure $(\mathcal{U}, \mathcal{V}[1])$ in $D(\mathrm{Mod}\, R)$.

$\phi(i) := \{ \mathfrak{p} \in \mathrm{Spec}\, R \mid E(R/\mathfrak{p})[-i] \notin \mathcal{V} \}$: sp.-closed

closed under directed (homotopy) limits

$\Longrightarrow$ ϕ is an sp.-filtration,

$\mathcal{V}_\phi$ is the smallest coaisle containing $\bigcup_{i \in \mathbb{Z}} \{ E(R/\mathfrak{p})[-i] \mid E(R/\mathfrak{p})[-i] \in \mathcal{V} \}$, and

$\mathcal{V}_\phi \subset \mathcal{V}$.

ETS: $\mathcal{V} \ni X \Rightarrow X \in \mathcal{V}_\phi$

$\inf X := \{ i \mid H^i X \neq 0 \}$ $\quad$ $\kappa(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$

Use "$\inf R\Gamma_\mathfrak{p} X_\mathfrak{p} = \inf R\mathrm{Hom}_R(\kappa(\mathfrak{p}), X_\mathfrak{p})$"

by Foxby–Iyengar 2003

$$\mathcal{V} \ni X \Longrightarrow R\Gamma_\mathfrak{p} X_\mathfrak{p} \in \mathcal{V}_\phi \Longrightarrow X \in \mathcal{V}_\phi$$

$R\Gamma_\mathfrak{p} X_\mathfrak{p} \cong \Gamma_\mathfrak{p} I_\mathfrak{p}$

K-inj. resol. $X \longrightarrow I$: comp. of injective modules

closed under directed (homotopy) limits

$\square$

Cor. R : comm. noeth. ring

$$
\begin{array}{ccc}
\left\{\begin{array}{l}\text{homotopically smashing}\\ \text{t-structures in } D(\mathrm{Mod}\,R)\end{array}\right\} & \longleftrightarrow & \left\{\text{smashing localizations on } D(\mathrm{Mod}\,R)\right\} \\
\| & & \| \\
\left\{\begin{array}{l}\text{compactly generated}\\ \text{t-structures in } D(\mathrm{Mod}\,R)\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{l}\text{localizations } L \text{ on } D(\mathrm{Mod}\,R) \text{ with}\\ \operatorname{Ker} L \text{ generated by a set}\\ \text{of compact objects in } D(\mathrm{Mod}\,R)\end{array}\right\}
\end{array}
$$

Rem (AJS) ϕ : sp.-filtration

$$\mathcal{U}_\phi = \{ X \in D(\mathrm{Mod}\,R) \mid \operatorname{Supp} H^i X \subseteq \phi(i) \quad \forall i \}$$

$$\mathcal{V}_\phi = \{ X \in D(\mathrm{Mod}\,R) \mid R\Gamma_{\phi(i)} X \in D^{>i}(\mathrm{Mod}\,R) \quad \forall i \}$$