

# Telescope conjecture for homotopically smashing t-structures over commutative noetherian rings

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§1. Torsion pairs in module categories

§2. Derived categories and t-structures

§3. Homotopically smashing t-structures

§4. Main result

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§1.  $R$ : ring  $\mathcal{T}, \mathcal{F} \subset \text{Mod } R$

$(\mathcal{T}, \mathcal{F})$  is a torsion pair if

(i)  $\mathcal{T} \cap \mathcal{F} = \{0\}$ ;

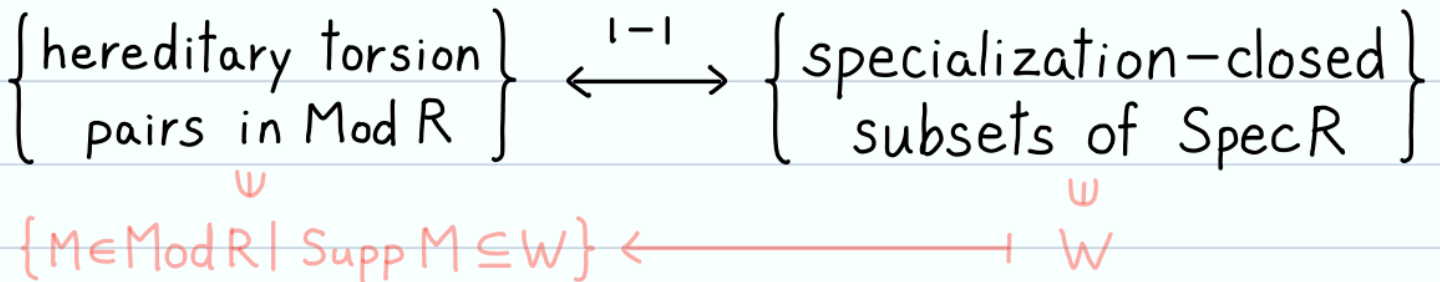
(ii)  $\mathcal{T}$  is closed under quotient objects and  
 $\mathcal{F}$  is closed under subobjects;

(iii)  $\exists 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  : exact  $\forall M \in \text{Mod } R$ .  
 $\mathcal{T}$   $\mathcal{F}$

$(\mathcal{T}^\perp = \{N \in \text{Mod } R \mid \text{Hom}_R(L, N) = 0 \forall L \in \mathcal{T}\} = \mathcal{F}, \mathcal{T} = {}^\perp \mathcal{F})$

A torsion pair  $(T, F)$  is hereditary if  $T$  is closed under subobjects.

Gabriel (1962)  $R$ : commutative noetherian



Göbel–Shelah (1982)

Torsion pairs in  $\text{Mod } \mathbb{Z}$  form a proper class.

§ 2.

$$C(\text{Mod } R) \ni X = (\dots \rightarrow X^i \xrightarrow{d_X^i} X^{i+1} \rightarrow \dots)$$

$\downarrow$

$$K(\text{Mod } R) \quad \text{Hom}_{K(\text{Mod } R)}(X, Y) := \text{Hom}_{C(\text{Mod } R)}(X, Y) / \sim$$

homotopy equiv.

$\downarrow$

$$X \xrightarrow{f} Y \rightarrow C(f) \rightarrow X[1] : \text{triangle}$$

└──────────┘  
degree-wise split exact

$$f: X \rightarrow Y$$

$$f \sim 0 \iff \exists \{s^i\}_{i \in \mathbb{Z}} \text{ s.t.}$$

$$f = d_Y^{i-1} \circ s^i + s^{i+1} \circ d_X^i$$

$D(\text{Mod } R)$

$$\text{Hom}_{D(\text{Mod } R)}(X, Y) := \varinjlim \text{Hom}_{K(\text{Mod } R)}(X, Y')$$

$$\left\{ \dots \rightarrow Y' \rightarrow Y'' \rightarrow \dots \right\}$$

quism  $\swarrow \Omega \nearrow$  quism  
 $Y$

$$\begin{array}{c}
 \text{\textcircled{3} left adjoint} \\
 K_{ac}(\text{Mod } R) \xrightleftharpoons[\text{\textcircled{3}}]{\perp} K(\text{Mod } R) \xrightleftharpoons[\perp]{\text{\textcircled{3}}} K_{ac}(\text{Mod } R)^\perp = \{K\text{-injective complexes}\} \\
 \begin{array}{ccccccc}
 \Psi & & \Psi & & \Psi & & \\
 C(f)[-1] & \longrightarrow & X & \xrightarrow{f} & I & \longrightarrow & C(f) \\
 & & \text{\textcircled{3} } & & & & \text{acyclic (exact) complex} \\
 & & K\text{-inj. resol.} & & & & \\
 & & \text{(quasi-isom)} & & & & 
 \end{array}
 \end{array}$$

$$D(\text{Mod } R) = K(\text{Mod } R) / K_{ac}(\text{Mod } R) \cong K_{ac}(\text{Mod } R)^\perp$$

Def  $\mathcal{T}$ : triangulated category (e.g.  $D(\text{Mod } R)$ )

A (Bousfield) localization functor on  $\mathcal{T}$  is a triangulated functor  $L: \mathcal{T} \rightarrow \mathcal{T}$  endowed with a natural transformation  $\eta: \text{id}_{\mathcal{T}} \rightarrow L$  s.t.  $L\eta$  is invertible and  $L\eta = \eta L$ .

$$\text{Ker } L \xrightleftharpoons[\perp]{\text{inc}} \mathcal{T} \xrightleftharpoons[\text{inc}]{\perp} \text{Im } L$$

$$(\mathcal{T} / \text{Ker } L \cong \text{Im } L)$$

Ex  $R$ : comm. noeth. ring,  $\mathfrak{a} \subset R$ : ideal

$$\Lambda^{\mathfrak{a}} = \varprojlim_{n \geq 1} (- \otimes_R R/\mathfrak{a}^n)$$

$L\Lambda^{\mathfrak{a}}: D(\text{Mod } R) \rightarrow D(\text{Mod } R)$  is a localization functor on  $D(\text{Mod } R)$ .

$$\begin{array}{ccc}
 \Psi & & \Psi \\
 P \xrightarrow{\sim} X & & \Lambda^{\mathfrak{a}} P \xrightarrow{\sim} L\Lambda^{\mathfrak{a}} X \\
 \text{K-proj. resol} & & \\
 \text{(K-flat)} & & 
 \end{array}$$

Def  $\mathcal{T}$ : triangulated category with arbitrary direct sums  
(e.g.  $D(\text{Mod } R)$ )

$X \in \mathcal{T}$  is compact if

$$\bigoplus_i \text{Hom}_{\mathcal{T}}(X, Y_i) \xrightarrow{\sim} \text{Hom}_{\mathcal{T}}(X, \bigoplus_i Y_i)$$

for any direct sum  $\bigoplus_i Y_i$  in  $\mathcal{T}$ .

$\mathcal{T}$  is compactly generated if

$\exists S$ : (small) set of compact objects s.t.  $\text{Loc}_{\mathcal{T}} S = \mathcal{T}$ .  
"T is generated by S"

the smallest localizing subcat. containing S  
triangulated subcat. closed under  $\bigoplus$

$$\left( \text{Loc}_{\mathcal{T}} S = \mathcal{T} \Leftrightarrow \left( \bigcup_{i \in \mathbb{Z}} S[i] \right)^{\perp} = \{0\} \right)$$

Ex  $R$ : any ring

$$\left\{ \begin{array}{l} \text{compact objects} \\ \text{in } D(\text{Mod } R) \end{array} \right\} = \left\{ X \in D(\text{Mod } R) \mid \begin{array}{l} X \cong P: \text{bounded complex} \\ \text{of f.g. proj. modules} \end{array} \right\}$$

$$\begin{array}{c} \text{!!} \\ D(\text{Mod } R)^c \cong K^b(\text{proj } R) \end{array}$$

essentially small

$\text{Loc } D(\text{Mod } R)^c = \text{Loc}\{R\} = D(\text{Mod } R)$ : compactly generated

A generalized version of an original question by Ravenel  
(1984)

Q (Telescope conjecture)

$T$ : compactly generated triangulated category

If a localization functor  $L: T \rightarrow T$  is smashing

(i.e.  $L$  commutes with  $\bigoplus^\infty$ ), is  $\text{Ker } L$  generated by

compact objects in  $T$ ?

"The converse is true."

Yes, for  $D(\text{Mod } R)$  over a comm. noeth. ring  $R$ ,  
by Neeman (1992):

// { kernels of localization functors }

$$\left\{ \begin{array}{c} \text{localizing subcategories} \\ \text{of } D(\text{Mod } R) \end{array} \right\} \xleftrightarrow{|-|} \left\{ \begin{array}{c} \text{subsets of } \text{Spec } R \end{array} \right\}$$

U

U

$$\left\{ \begin{array}{c} \text{smashing subcategories} \\ \text{of } D(\text{Mod } R) \end{array} \right\} \xleftrightarrow{|-|} \left\{ \begin{array}{c} \text{specialization-closed} \\ \text{subsets of } \text{Spec } R \end{array} \right\}$$

//

W

W

$$\left\{ \begin{array}{c} \text{kernels} \\ \text{of smashing} \\ \text{localizations} \end{array} \right\} \text{Loc} \left\{ \begin{array}{c} K(\mathfrak{x}) \mid V(\mathfrak{x}) \subseteq W \\ \mathfrak{x} = \mathfrak{x}_1, \dots, \mathfrak{x}_n \quad \mathfrak{x}_i \in R \end{array} \right\} \xleftrightarrow{|-|} W$$

Def (Beilinson-Bernstein-Deligne 1982)

$\mathcal{T}$ : triangulated category

$\mathcal{U}, \mathcal{V} \subset \mathcal{T}$ : full subcategories

$\uparrow$   $\uparrow$   
aisle coaisle

$(\mathcal{U}, \mathcal{V}[1])$  is a t-structure in  $\mathcal{T}$  if

(i)  $\text{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0 \quad \forall \mathcal{U} \in \mathcal{U}, \forall \mathcal{V} \in \mathcal{V}$ ;

(ii)  $\exists \mathcal{U} \rightarrow \mathcal{X} \rightarrow \mathcal{V} \rightarrow \mathcal{U}[1] \quad \forall \mathcal{X} \in \mathcal{T}$ ;  
 $\uparrow \quad \uparrow$   
 $\mathcal{U} \quad \mathcal{V}$

(iii)  $\mathcal{U}[1] \subset \mathcal{U}$ .

A t-structure  $(\mathcal{U}, \mathcal{V}[1])$  is stable if  $\mathcal{U}[-1] \subset \mathcal{U}$ .

{stable t-structures in  $\mathcal{T}$ }  $\xleftrightarrow{!-1}$  {localization functors on  $\mathcal{T}$ }

Ex  $(D^{\leq 0}(\text{Mod } R), D^{\geq 0}(\text{Mod } R))$  the standard t-structure

$$\tau^{\leq 0} X \rightarrow X \rightarrow \tau^{> 0} X \rightarrow (\tau^{\leq 0} X)[1]$$

$\tau^{\leq 0} X = (\dots \rightarrow X^{-1} \rightarrow X^0 \rightarrow \text{Im} d_x^0 \rightarrow 0) \in D^{\leq 0}(\text{Mod } R)$  : aisle

$\tau^{> 0} X = (0 \rightarrow \text{Coker} d_x^0 \rightarrow X^2 \rightarrow X^3 \rightarrow \dots) \in D^{> 0}(\text{Mod } R)$  : coaisle

$$\tau^{\leq 0} : D(\text{Mod } R) \rightarrow D^{\leq 0}(\text{Mod } R)$$

$$\tau^{< 0} : D(\text{Mod } R) \rightarrow D^{> 0}(\text{Mod } R)$$

$$D^{\leq 0}(\text{Mod } R) \cap D^{\geq 0}(\text{Mod } R) \cong \text{Mod } R$$

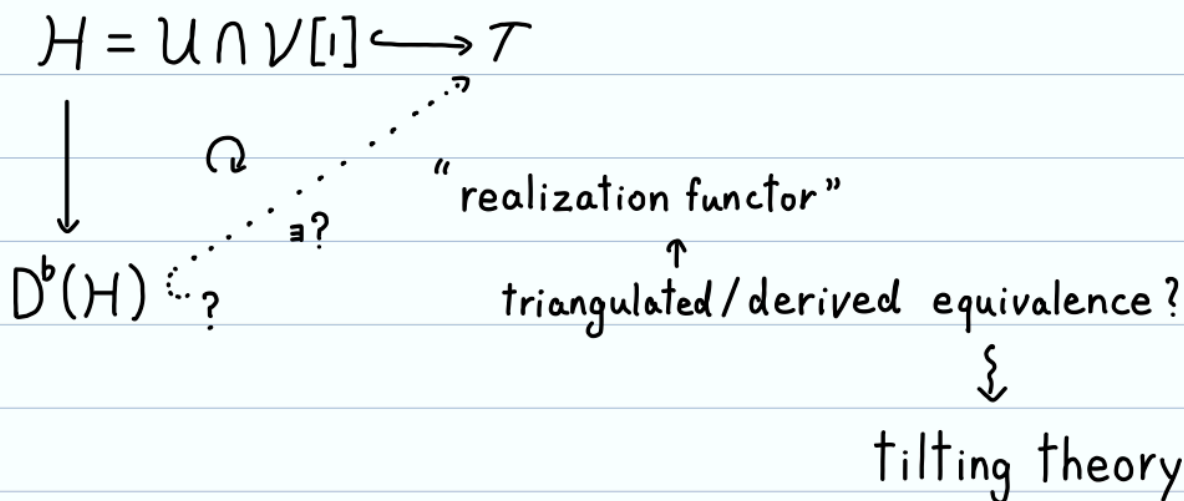
Fact (BBD)

The heart  $\mathcal{U} \cap \mathcal{V}[1]$  of a t-structure  $(\mathcal{U}, \mathcal{V})$  in  $\mathcal{T}$  is an abelian category with the exact structure induced by  $\mathcal{T}$ .

e.g. {"perverse sheaves"}

= (the heart of a t-structure in some derived category)

$(\mathcal{U}, \mathcal{V}[1])$  : t-structure in  $\mathcal{T}$



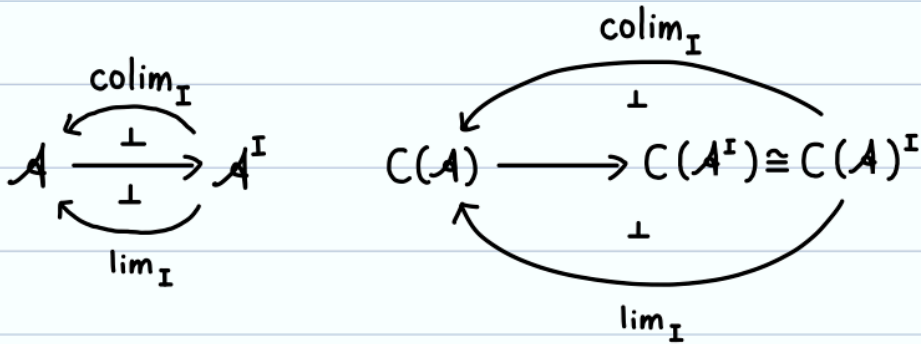
When is  $\mathcal{H}$  Grothendieck?

$\downarrow$

"homotopically smashing t-structure"

### § 3. $\mathcal{A}$ : Grothendieck category

$\text{Cat} \ni I$  : small category (e.g.  $I = \{1 \rightarrow 2 \rightarrow 3 \rightarrow \dots\}$ )

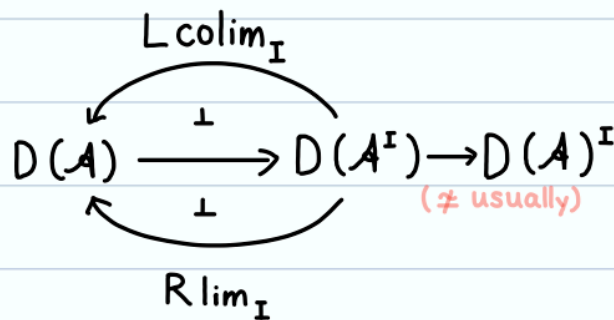


Grothendieck 1983

Cisinski 2003

Groth 2013

Stovicek 2014



A (strong and stable) derivator is a (2-)functor

$$\mathbb{D} : \text{Cat}^{\text{op}} \longrightarrow \text{CAT}$$

$$\begin{array}{ccc} \cup & & \cup \\ I & \longmapsto & \mathbb{D}(I) : \text{triangulated} \end{array}$$

s.t.

$$\mathbb{D}(\mathbb{1}) \xrightleftharpoons[\text{holim}_I]{\text{hocolim}_I} \mathbb{D}(I) \xrightarrow[\text{ess. surj. full}]{\text{ess. surj. full}} \mathbb{D}(\mathbb{1})^I \quad (\mathbb{1} = \{1\} \leftarrow I)$$

+ several conditions.

(e.g.  $\mathbb{D}(I) := \mathbb{D}(\mathcal{A}^I)$ )

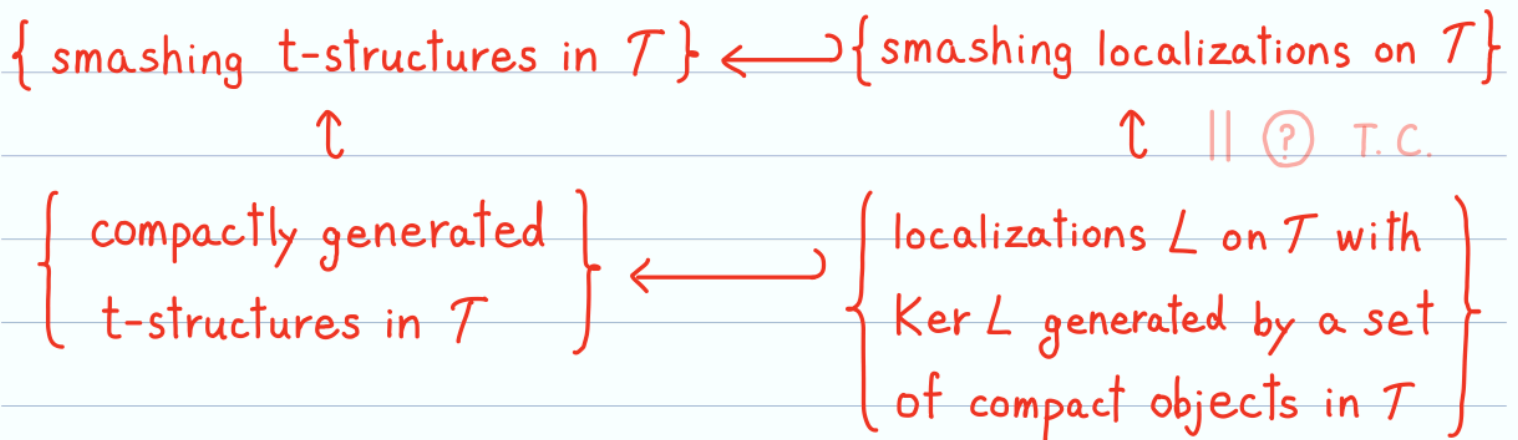


$\mathbb{D}$  is called compactly generated if  $\mathbb{D}(\mathbf{1})$  is compactly generated.  
 $\uparrow$   
 the underlying triangulated category

Def  $\mathcal{T}$ : triangulated category with arbitrary direct sums

A t-structure  $(\mathcal{U}, \mathcal{V}[i])$  in  $\mathcal{T}$  is smashing if  $\mathcal{V}$  is closed under direct sums.

A t-structure  $(\mathcal{U}, \mathcal{V}[i])$  in  $\mathcal{T}$  is compactly generated if  $\mathcal{V} = S^\perp$  for some set  $S$  of compact objects in  $\mathcal{T}$ .



Rem

$\left\{ \text{torsion pairs in } \text{Mod } \mathbb{Z} \right\}$

$\exists \int$

$\left\{ \text{smashing t-structures in } \mathbb{D}(\text{Mod } \mathbb{Z}) \right\}$  is a proper class.

# Def (Saorín - Šťovíček - Virill 2017)

$\mathbb{D}$ : (strong and stable) derivator

A t-structure  $(\mathcal{U}, \mathcal{V}[i])$  in  $\mathbb{D}(\mathbb{1})$  is homotopically smashing if  $\mathcal{V}$  is closed under directed homotopy colimits, i.e., homotopy colimits with respect to directed sets.

## Fact (SŠV, Laking 2020)

$\mathbb{D}$ : compactly generated derivator

$(\mathcal{U}, \mathcal{V}[i])$ : non-degenerate t-structure in  $\mathbb{D}(\mathbb{1})$

$$\bigcap_{i \in \mathbb{Z}} \mathcal{U}[i] = 0 = \bigcap_{i \in \mathbb{Z}} \mathcal{V}[i]$$

$(\mathcal{U}, \mathcal{V}[i])$  is homotopically smashing.

$\Leftrightarrow (\mathcal{U}, \mathcal{V}[i])$  is smashing and  $\mathcal{U} \cap \mathcal{V}[i]$  is Grothendieck.

{ smashing t-structures in  $\mathbb{D}(\mathbb{1})$  }

↑

{ homotopically smashing  
t-structures in  $\mathbb{D}(\mathbb{1})$  }

↔ { smashing localizations on  $\mathbb{D}(\mathbb{1})$  }

↑ || (?)

↑ || (?) T.C.

{ compactly generated  
t-structures in  $\mathbb{D}(\mathbb{1})$  }

↔

{ localizations  $\mathcal{L}$  on  $\mathbb{D}(\mathbb{1})$  with  
Ker  $\mathcal{L}$  generated by a set  
of compact objects in  $\mathbb{D}(\mathbb{1})$  }

Q For which compactly generated derivator  $\mathbb{D}$ ,  
 is every homotopically smashing t-structure in  $\mathbb{D}(\mathbb{1})$   
 compactly generated?

$$\mathbb{D}_{\text{Mod } R} : \text{Cat}^{\text{op}} \longrightarrow \text{CAT} \quad \mathbb{D}_{\text{Mod } R}(\mathbb{1}) = D(\text{Mod } R)$$

$$\underbrace{\quad}_{\mathbb{I}} \longmapsto \underbrace{\quad}_{D(\text{Mod } R^{\mathbb{I}})}$$

Rem  $(\mathcal{U}, \mathcal{V}[i])$ : t-structure in  $D(\text{Mod } R)$

$(\mathcal{U}, \mathcal{V}[i])$ : homotopically smashing.

$\Leftrightarrow \mathcal{V}$  is closed under directed limits defined in  $C(\text{Mod } R)$ .

§4.  $R$ : comm. noeth. ring

Thm Every homotopically smashing t-structure in  
 $D(\text{Mod } R)$  is compactly generated.

Sketch of proof

$$\left\{ \begin{array}{l} \text{compactly generated} \\ \text{t-structures in } D(\text{Mod } R) \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \phi: \mathbb{Z} \rightarrow 2^{\text{Spec } R} \\ \phi(i): \text{sp.-cl. } \forall i \\ i \leq j \Rightarrow \phi(i) \supseteq \phi(j) \end{array} \right\}$$

$$\underbrace{\quad}_{\mathbb{U}} \quad \longleftarrow \quad \underbrace{\quad}_{\mathbb{V}}$$

$$\left( \mathcal{U}_{\phi}, \mathcal{V}_{\phi}[i] \right) \quad \longleftarrow \quad \underline{\phi}$$

$$\parallel \quad \parallel$$

$${}^{\pm}(S^{\pm}) \quad S^{\pm} \quad \text{sp-filtration}$$

$$S = \{K(x)[-i] \mid i \in \mathbb{Z}, V(x) \subset \phi(i), x_1 = x_1, \dots, x_n, x_j \in R\}$$

by Alonso Tarrío-Jeremias López-Saorín 2010.

Take a homotopically smashing t-structure  $(\mathcal{U}, \mathcal{V}[i])$  in  $D(\text{Mod } R)$ .

$$\phi(i) := \{\mathfrak{p} \in \text{Spec } R \mid E(R/\mathfrak{p})[-i] \notin \mathcal{V}\} : \text{sp.-closed}$$

closed under directed (homotopy) limits

$\Rightarrow \phi$  is an sp.-filtration,

$\mathcal{V}_\phi$  is the smallest coaisle containing  $\bigcup_{i \in \mathbb{Z}} \{E(R/\mathfrak{p})[-i] \mid E(R/\mathfrak{p})[-i] \in \mathcal{V}\}$ , and

$$\mathcal{V}_\phi \subset \mathcal{V}.$$

$$\text{ETS: } \mathcal{V} \ni X \Rightarrow X \in \mathcal{V}_\phi$$

$$\text{inf } X := \{i \mid H^i X \neq 0\} \quad X(\mathfrak{p}) := R_{\mathfrak{p}}/_{\mathfrak{p}} R_{\mathfrak{p}}$$

Use "inf  $R_{\mathfrak{p}}^{\Gamma} X_{\mathfrak{p}} = \text{inf } R\text{Hom}_R(X(\mathfrak{p}), X_{\mathfrak{p}})$ "

by Foxby-Iyengar 2003

$$\mathcal{V} \ni X \implies R_{\mathfrak{p}}^{\Gamma} X_{\mathfrak{p}} \in \mathcal{V}_\phi \implies X \in \mathcal{V}_\phi$$

$\cong$

closed under directed  
(homotopy) limits

K-inj. resol.  $\Gamma_{\mathfrak{p}}^{\Gamma} I_{\mathfrak{p}}$

$X \rightarrow I$  : comp. of injective modules

□

Cor.  $R$ : comm. noeth. ring

$\left\{ \begin{array}{l} \text{homotopically smashing} \\ \text{t-structures in } D(\text{Mod } R) \end{array} \right\} \longleftrightarrow \left\{ \text{smashing localizations on } D(\text{Mod } R) \right\}$

||

$\left\{ \begin{array}{l} \text{compactly generated} \\ \text{t-structures in } D(\text{Mod } R) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{localizations } L \text{ on } D(\text{Mod } R) \text{ with} \\ \text{Ker } L \text{ generated by a set} \\ \text{of compact objects in } D(\text{Mod } R) \end{array} \right\}$

||

Rem (AJS)  $\phi$ : sp.-filtration

$$U_\phi = \{ X \in D(\text{Mod } R) \mid \text{Supp } H^i X \subseteq \phi(i) \ \forall i \}$$

$$V_\phi = \{ X \in D(\text{Mod } R) \mid R\Gamma_{\phi(i)} X \in D^{>i}(\text{Mod } R) \ \forall i \}$$