Two normal reduction numbers

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Minimal reduction

Assumption (★)

Throughout this talk,

- K is an algebraically closed field.
- $(A, \mathfrak{m}, K)$ is an exlellent normal local domain containing $K \cong A/\mathfrak{m}$ or
- A is a graded K-algebra $A = \bigoplus_{n \geq 0} A_n$, $\mathfrak{m} = \bigoplus_{n \geq 1} A_n$ and $K = A_0$.
- $\dim A = 2$ and A is **not** regular.
- I is an $\mathfrak{m}$-primary ideal.

Then there exists a parameter ideal $Q \subset I$ s.t.

$$I^{n+1} = QI^n \ (\exists n \geq 0).$$

Then Q is called a minimal reduction of I.

Reduction number

$\boxed{\text{Let } Q \text{ be a minimal reduction of } I.}$

- $r_Q(I)$ is called the reduction number of I with respect to Q.

$$\begin{aligned} r_Q(I) &= \min\left\{r \geq 0 \,\middle|\, I^{r+1} = QI^r\right\} \\ &= \min\left\{r \geq 0 \,\middle|\, I^{N+1} = QI^N \ (\forall N \geq r)\right\}. \end{aligned}$$

- $r(I)$ is called the reduction number of I.

$$r(I) = \min\left\{r_Q(I) \,\middle|\, Q \text{ is a minimal reduction of } I\right\}.$$

- I is said to be stable if $I^2 = QI$.

Let t be an indeterminate over A.

- The Rees algebra of I is

$$\mathcal{R}(I) = A[It] = \bigoplus_{n \geq 0} I^n t^n \subset A[t].$$

- The extended Rees algebra of I is

$$\mathcal{R}'(I) = A[It, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} I^n t^n \subset A[t, t^{-1}],$$

where $I^n = A$ for every $n \leq 0$.

- The associated graded ring of I is

$$G(I) = \mathcal{R}(I)/I\mathcal{R}(I) \cong \mathcal{R}'(I)/t^{-1}\mathcal{R}'(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}.$$

Assume (★).

Theorem 1.1 (Goto-Shimoda, 80's)

I is stable $\iff \mathcal{R}(I)$ is Cohen-Macaulay.

- We call $\overline{\mathcal{R}}(I) := \bigoplus_{n\geq 0} \overline{I^n}t^n = \overline{\mathcal{R}(I)}$ the normal Rees algebra of I.
- $\mathcal{R}(I)$ is called normal if $\overline{\mathcal{R}(I)} = \mathcal{R}(I)$.
  Moreover, I is called normal.
- We call $\overline{G}(I) = \bigoplus_{n\geq 0} \overline{I^n}/\overline{I^{n+1}}$ the normal associated graded ring of I.

Normal Hilbert function

Assume (★). Let $I \subset A$ be an $\mathfrak{m}$-primary ideal.

Definition 1.2 (Rees, 60's)

- The normal Hilbert fucntion is
$$\overline{H}_I(n) := \ell_A(A/\overline{I^n}).$$
- The normal Hilbert polynomial is
$$\overline{P}_I(n) = \overline{e}_0(I)\binom{n+1}{2} - \overline{e}_1(I)\binom{n}{1} + \overline{e}_2(I).$$

  such that $\overline{P}_I(n) = \overline{H}_I(n)$ for large enough n.
  Then $\overline{e}_i(I)$ is called the i^{th} normal Hilbert coefficient of I.

Normal Hilbert coefficients

Assume (★).

Theorem 1.3 (Huneke, 1987)

Let $I \subset A$ be an $\mathfrak{m}$-primary integrally closed ideal. Then

1. $\overline{e}_0(I) = e_0(I)$.
2. $\overline{e}_1(I) = e_0(I) - \ell_A(A/I) + \displaystyle\sum_{n=1}^{\infty} \ell_A(\overline{I^{n+1}}/Q\overline{I^n})$.
3. $\overline{e}_2(I) = \displaystyle\sum_{n=1}^{\infty} n \cdot \ell_A(\overline{I^{n+1}}/Q\overline{I^n})$.

Corollary 1.4

Let I be as above.

1. $\overline{e}_1(I) \geq e_0(I) - \ell_A(A/I)$.
2. $\overline{e}_2(I) \geq 0$.

Two normal reduction numbers

Assume (★) and $\bar{I} = I$ and $\sqrt{I} = \mathfrak{m}$.

Lemma 1.5 (Huneke, 1987)

Let Q, Q' be minimal reductions of I. Then for every $n \geq 1$,

$$\overline{I^{n+1}} = Q\overline{I^n} \iff \overline{I^{n+1}} = Q'\overline{I^n}.$$

Definition 1.6

- The (small) normal reduction number of I is

$$\mathrm{nr}(I) := \min\{r \geq 1 \mid \overline{I^{r+1}} = Q\overline{I^r}\}.$$

- The (big) normal reduction number of I is

$$\bar{\mathrm{r}}(I) := \min\{r \geq 1 \mid \overline{I^{N+1}} = Q\overline{I^N}\ (\forall N \geq r)\}.$$

Assume (★) and $\overline{I} = I$ and $\sqrt{I} = \mathfrak{m}$.

Definition 1.7 (OWY)

I is a p_g-ideal $\overset{\text{def}}{\iff} \mathcal{R}(I)$ is Cohen-Macaulay and normal.

$$I \text{ is a } p_g\text{-ideal} \iff \boxed{\overline{I^2} = I^2 = QI,\ \overline{I^3} = I^3 = QI^2, \ldots}$$

$$\implies \mathrm{nr}(I) = 1$$

$$\implies I \text{ is stable}$$

Characterization of p_g-ideals via normal reduction numbers

Assume ($\bigstar$) and $\overline{I} = I$ and $\sqrt{I} = \mathfrak{m}$.

Theorem 1.8 (OWY)

The following conditions are equivalent:

1. I *is a* $\boldsymbol{p_g}$*-ideal.*
2. $\bar{r}(I) = 1$ (*then* $\mathrm{nr}(I) = 1$).
3. I *is stable and normal.*
4. $\overline{e}_1(I) = e_0(I) - \ell_A(A/I)$.
5. $\overline{e}_2(I) = 0$.
6. $\overline{G}(I)$ *is Cohen-Macaulay with* $a(\overline{G}) < 0$.
7. $\overline{\mathcal{R}}(I)$ *is Cohen-Macaulay .*

In progress (j.w.with M.E.Rossi)

Assume (★) and $\overline{I} = I$ and $\sqrt{I} = \mathfrak{m}$.

Theorem 1.9 (OWY with Rossi)

The following conditions are equivalent:

1. $\bar{r}(I) = 2$ *and* $\ell_A(\overline{I^2}/QI) = 1$.
2. $\bar{e}_1(I) = e_0(I) - \ell_A(A/I) + 1$ *and* $\mathrm{nr}(I) = \bar{r}(I)$.
3. $\bar{e}_2(I) = 1$.
4. $\overline{G}(I)$ *is Cohen-Macaulay,* $a(\overline{G}) = 0$ *and* $\ell_A([H^2_{\mathfrak{M}}(\overline{G})]_0) = 1$.

When this is the case, $\overline{\mathcal{R}}(I)$ *is a Buchsbaum ring with* $\ell_A(H^2_{\mathfrak{M}}(\overline{R})) = 1$.

The final example in this talk gives an example of I such that

- $\bar{e}_1(I) = e_0(I) - \ell_A(A/I) + 1$ and
- $1 = \mathrm{nr}(I) < \bar{r}(I) = g + 1$, where $g \geq 3$.

$\boxed{\text{Find many } p_g\text{-ideals.}}$

$$\Downarrow$$

$\boxed{\text{Calculate normal reduction numbers for ideals}}$

Two normal reduction numbers of rings

In order to study the relationship between singularities and normal reduction numbers of ideals, we define the normal reduction number of rings.

Definition 2.1

- The (small) normal reduction number of A is

$$\mathrm{nr}(A) := \max\{\mathrm{nr}(I) \mid \overline{I} = I,\ \sqrt{I} = \mathfrak{m}\}.$$

- The (big) normal reduction number of A is

$$\bar{\mathrm{r}}(A) := \max\{\bar{\mathrm{r}}(I) \mid \overline{I} = I,\ \sqrt{I} = \mathfrak{m}\}.$$

- If $r = \bar{\mathrm{r}}(A) < \infty$, then we have

$$\overline{I^{r+1}} \subset Q$$

for any ideal I and its minimal reduction Q.

Main Problems

Problem 1

Assume (★).

- Determine $\mathbf{nr}(A)$ and $\bar{\mathrm{r}}(A)$.
- What is the difference between $\mathbf{nr}(A)$ and $\bar{\mathrm{r}}(A)$?
- Determine $\mathbf{nr}(I)$ and $\bar{\mathrm{r}}(I)$ for any integrally closed $\mathfrak{m}$-primary ideal of A.
- Find an ideal I which satisfies $\mathbf{nr}(I) < \bar{\mathrm{r}}(I)$.

The main purpose of this talk is to give partial answers to these problems using the theory of singularity.
Especially, we give concrete examples of ideals I in g^{th} Veronese subring A of a Brieskorn hypersurface $B_{2,2g+2,2g+2}$ which satisfies

$$1 = \mathbf{nr}(I) < \bar{\mathrm{r}}(I) = g + 1.$$

Brieskorn hypersurfaces

Let a, b, c be integers with $2 \leq a \leq b \leq c$. Let K be a field of characteristic p which does **not** divide abc. Put $L = \mathrm{LCM}\{a, b, c\}$.

Definition 2.2

- Brieskorn hypersurfaces $B_{a,b,c}$ is

$$B = B_{a,b,c} := K[X, Y, Z]/(X^a + Y^b + Z^c).$$

This is a graded K-algebra with $\deg(X) = L/a$, $\deg(Y) = L/b$ and $\deg(Z) = L/c$.

The a-invariant of B is defined by

$$a(B) = \max\{n \in \mathbb{Z} \mid [H^2_{\mathfrak{m}}(B)]_n \neq 0\}.$$

In fact,

$$\begin{aligned} a(B) &= \deg(X^2 + Y^b + Z^c) - (\deg(X) + \deg(Y) + \deg(Z)) \\ &= L - L/a - L/b - L/c. \end{aligned}$$

Normal reduction number of the maximal ideal of $B_{a,b,c}$

Theorem 2.3

Let $A = \widehat{B_{a,b,c}}$ be a Brieskorn hypersurface, and put $Q = (y, z)A$, $n_k = \lfloor \frac{kb}{a} \rfloor$ for $k = 1, 2, \ldots, a-1$. Then $\mathfrak{m} = \overline{Q} = (x, y, z)A$ and

- $\overline{\mathfrak{m}^n} = Q^n + xQ^{n-n_1} + \cdots + x^{a-1}Q^{n-n_{a-1}}$ for every $n \geq 1$.

  In particular,
- $\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = n_{a-1}$.
- $\mathcal{R}(\mathfrak{m})$ is normal if and only if $\bar{\mathrm{r}}(\mathfrak{m}) = a-1$.
- $\mathfrak{m}$ is a p_g-ideal if and only if $a = 2$, $b = 2, 3$.

Definition 2.4

- The k^{th} Veronese subring A of B is

$$A = B^{(k)} := \bigoplus_{n \geq 0} B_{kn}.$$

This can be regarded as a graded K-algebra with $A_n = B_{kn}$.

- $[H^i_{\mathfrak{m}}(A)]_n \cong [H^i_{\mathfrak{m}B}(B)]_{kn}$.

For instance, $A = K[x,y]^{(3)} = K[x^3, x^2y, xy^2, y^3]$ is the 3^{rd} Veronese subring and

$$\begin{aligned} A_1 &= Kx^3 + Kx^2y + Kxy^2 + Ky^3 \\ A_2 &= Kx^6 + Kx^5y + \cdots + Ky^6 \end{aligned}$$

Cycle and ideal

Assume (★).

Fact 2.5

Let $I \subset A$ be an **$\mathfrak{m}$-primary integrally closed ideal**. Then there exists a resolution of singularities $f\colon X \to \operatorname{Spec} A$ with $E = f^{-1}(\mathfrak{m}) = \bigcup_{i=1}^{m} E_i$ and an anti-nef cycle $Z = \sum_{i=1}^{m} a_i E_i$ on X such that

$$I\mathcal{O}_X = \mathcal{O}_X(-Z), \qquad I = H^0(X, \mathcal{O}_X(-Z)).$$

Then I is said to be represented on X and write $I = I_Z$.

- If $I = I_Z$, $I' = I_{Z'}$ then $\overline{II'} = I_{Z+Z'}$.
- In particular, if $I = I_Z$, then $\overline{I^n} = I_{nZ}$.

Assume (★).

Definition 2.6

Let $f\colon X \to \operatorname{Spec} A$ with $E = f^{-1}(\mathfrak{m}) = \bigcup_{i=1}^{m} E_i$ be a resolution of singularities. Then

$$p_g(A) = \ell_A(H^1(X, \mathcal{O}_X))$$

is called the geometric genus of A,

where $\ell_A(W)$ denotes the length of W as an A-module.

- $p_g(A) = \dim_K [H^2_{\mathfrak{m}}(A)]_{\geq 0}$.
- $p_g(B_{a,b,c}) = \displaystyle\sum_{i=0}^{a(B)} \dim_K B_i$ is given by

$$p_g(B_{a,b,c}) = \sharp\left\{(t_0, t_1, t_2) \in \mathbb{Z}_{\geq 0}^{\oplus 3} \,\middle|\, abc - bc - ac - ab \geq bct_0 + cat_1 + abt_2\right\}.$$

A sequence $q(nI)$

Assume (★) and let $I = I_Z$.
Put $q(nI) = \ell_A(H^1(X, \mathcal{O}_X(-nZ)))$ for every $n \geq 0$.

Theorem 2.7 (OWY (cf. Huneke))

- $q(0I) = p_g(A)$.
- $q(kI) \geq q((k+1)I)$ *for every* $k \geq 0$.
- *If* $q(nI) = q((n+1)I)$, *then* $q((n+1)I) = q((n+2)I)$.
- $q(nI) = q(\infty I)$ *for every* $n \geq p_g(A)$, *where*

$$q(\infty I) = \lim_{n \to \infty} q(nI)$$

Normal reduction numbers and $q(nI)$

Assume ($\bigstar$) and $I = I_Z$.

Proposition 2.8

For every integer $n \geq 0$*, we have*

$$2 \cdot q(nI) + \ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = q((n-1)I) + q((n+1)I).$$

Proposition 2.9

- $\mathrm{nr}(I) = \min\left\{n \in \mathbb{Z}_{\geq 0} \,\middle|\, q((n-1)I) - q(nI) = q(nI) - q((n+1)I)\right\}.$
- $\bar{\mathrm{r}}(I) \quad = \min\left\{n \in \mathbb{Z}_{\geq 0} \,\middle|\, q((n-1)I) = q(nI)\right\}.$

Thus if $\overline{I^{n+1}} \neq Q\overline{I^n}$, then

$$p_g(A) = q(0 \cdot I) > q(1 \cdot I) > q(2 \cdot I) > \cdots > q((n-1)I) > q(nI) \geq 0$$

Assume $I = I_Z$.

Theorem 2.10 (Riemann-Roch formula)

$$\ell_A(A/\bar{I}) + q(I) = -\frac{Z^2 + K_X Z}{2} + p_g(A).$$

Proposition 2.11 (OWY)

1. $\bar{e}_o(I) = e_0(I) = -Z^2.$
2. $\bar{e}_1(I) = \frac{-Z^2 + K_X Z}{2} = e_0(I) - \ell_A(A/I) + (p_g(A) - q(I)).$
3. $\bar{e}_2(I) = p_g(A) - q(\infty I).$

Corollary 2.12

$q(nI) - q(\infty I) = \overline{P}_I(n) - \overline{H}_I(n)$.

(Proof.) By Riemann-Roch formula, we have

$$\ell_A(A/\overline{I^n}) + q(nI) = -\frac{n^2Z^2 + nK_XZ}{2} + p_g(A).$$

Hence

$$\begin{aligned} q(nI) &= \frac{(-Z^2)}{2}n^2 - \frac{K_XZ}{2}n + p_g(A) - \overline{H}_I(n) \\ &= \frac{\overline{e}_0}{2}n^2 - \left(\overline{e}_1 - \frac{\overline{e}_0}{2}\right)n + \overline{e}_2 + q(\infty I) - \overline{H}_I(n) \\ &= \overline{P}_I(n) - \overline{H}_I(n) + q(\infty I).// \end{aligned}$$

Theorem 2.13 (OWY)

Assume $I = I_Z$. Then TFAE

1. I *is a* p_g*-ideal.*
2. $q(I) = p_g(A)$.
3. $\bar{r}(I) = 1$.
4. I *is stable and normal.*
5. $\bar{e}_1(I) = e_0(I) - \ell_A(A/I)$.
6. $\bar{e}_2(I) = 0$.
7. $\overline{G}(I)$ *is Cohen-Macaulay with* $a(\overline{G}) < 0$.
8. $\overline{\mathcal{R}}(I)$ *is Cohen-Macaulay .*

When this is the case, $q(nI) = p_g(A)$ *for all* $n \geq 0$.

- I is a p_g-ideal $\Longrightarrow \mathrm{nr}(I) = 1 \Longrightarrow I$ is stable.

In progress (again)

Theorem 2.14 (OWY with Rossi)

The following conditions are equivalent:

1. $\overline{r}(I) = 2$ *and* $\ell_A(\overline{I^2}/QI) = 1$.
2. $q(I) = q(\infty I) = p_g(A) - 1$.
3. $\overline{e}_1(I) = e_0(I) - \ell_A(A/I) + 1$ *and* $\mathrm{nr}(I) = \overline{r}(I)$.
4. $\overline{e}_2(I) = 1$.
5. $\overline{G}(I)$ *is Cohen-Macaulay*, $a(\overline{G}) = 0$ *and* $\ell_A([H^2_{\mathfrak{M}}(\overline{G})]_0) = 1$.

When this is the case, $\overline{\mathcal{R}}(I)$ *is a Buchsbaum ring with* $\ell_A(H^2_{\mathfrak{M}}(\overline{R})) = 1$.

Rational singularity

Assume (★).

Definition 2.15

A is a rational singulaity if $p_g(A) = 0$.

Theorem 2.16 (OWY)

TFAE:

1. A *is a rational singularity.*
2. *Any* $\mathfrak{m}$*-primary integrally closed ideal is a* p_g*-ideal.*
3. $\mathrm{nr}(A) = 1$.
4. $\bar{\mathrm{r}}(A) = 1$.

Namley, the theory of p_g-ideals is a generalization of the ideal theory of rational singularities (by Lipman).

Examples of rational singulairties

Recall $B_{a,b,c} = K[X,Y,Z]/(X^a + Y^b + Z^c)$ for each $2 \le a \le b \le c$.

Ex 2.17 (Rational singularities of Brieskorn type)

$A = \widehat{B_{a,b,c}}$ is a rational singularity $\Longleftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. Namely,

$$(a,b,c) = (2,2,n),\ (2,3,3),\ (2,3,4),\ (2,3,5).$$

Fact 2.18

Any quotient singularity or a toric singularity is a rational singularity.
For instance, any Veronese subring of $B_{a,b,c}$ is also a rational singularity.

Elliptic singularity

Let $X \to \mathbf{Spec}\, A$ be a resolution of singularities,

Definition 2.19

Let Z_E be a fundamental cycle of X. Put $p_f(A) := p_a(Z_E)$, the fundamental genus of A. The ring A is called elliptic if $p_f(A) = 1$.

Theorem 2.20

$A = \widehat{B_{a,b,c}}$ is elliptic $\iff$ (a, b, c) is one of the following:

- $(a, b, c) = (2, 3, c)$, $c \geq 6$.
- $(a, b, c) = (2, 4, c)$, $c \geq 4$.
- $(a, b, c) = (2, 5, c)$, $5 \leq c \leq 9$.
- $(a, b, c) = (3, 3, c)$, $c \leq 3$.
- $(a, b, c) = (3, 4, c)$, $4 \leq c \leq 5$.

Theorem 2.21 (Okuma, OWY)

- *If* $p_f(A) = 1$ (*i.e.* A *is elliptic*), *then* $\mathrm{nr}(A) = \bar{\mathrm{r}}(A) = 2$.
- *Let* $A = \widehat{B_{a,b,c}}$ *be a Brieskorn type.*
  If $\bar{\mathrm{r}}(A) = 2$, *then* $p_f(A) = 1$, *except* $(a, b, c) = (3, 4, 6), (3, 4, 7)$.

The following question is open!

Question 2.22

If $A = B_{3,4,6}$ or $B_{3,4,7}$, then is $\bar{\mathrm{r}}(A) = 2$ or 3?

Let $g \geq 2$ be an integer, and let K be a field of $p = \mathrm{char} K$ with $p \not| 2g+2$.

Theorem 3.1 (OWY)

Let $B = K[X, Y, Z]/(X^2 - Y^{2g+2} - Z^{2g+2})$ *be a graded ring with*

$$\deg X = g+1, \deg Y = \deg Z = 1.$$

Let $A = B^{(g)}$ *be the* g^{th} *Veronese subring of* A*. Put*

$$\begin{aligned} I &= (y^g, y^{g-1}z, A_{\geq 2})A \\ &= (y^g, y^{g-1}z, y^{g-2}z^{g+2}, \ldots, z^{2g}, xy^{g-1}, xy^{g-2}z, \ldots, xz^{g-1})A. \end{aligned}$$

and $Q = (y^g - z^{2g}, y^{g-1}z)A$*. Then* $I^2 = QI$ *and*

NOTE: IB is integrally closed but $(IB)^2$ is **not** in general.

Theorem

(continue)

(1) $\overline{I^n} = I^n = QI^{n-1}$ for every $n = 1, 2, \ldots, g$. Hence $\mathrm{nr}(I) = 1$.

(2) $xy^{g^2-1} \in \overline{I^{g+1}} \setminus Q\overline{I^g}$ and $\overline{I^{g+1}} = Q\overline{I^g} + (xy^{g^2-1})$.

(3) $\overline{I^{n+1}} = Q\overline{I^n}$ for every $n \geq g+1$.
Hence $\bar{\mathrm{r}}(I) = g+1$ and $\bar{\mathrm{r}}(A) = g+1$.

(4) $q(0I) = p_g(A) = g$.
$q(nI) = g - n$ for all $n = 1, 2, \ldots, g$, and
$q(nI) = 0$ for every $n \geq g$.

(5) $\ell_A(A/I) = g$ and $e_0(I) = 4g - 2$.

(6) $\overline{e}_1(I) = 3g - 1$ and $\overline{e}_2(I) = g$.
In particular, $\overline{e}_1(I) = e_0(I) - \ell_A(A/I) + 1$.

Proof of sketch (1)

For simplicity, we assume $g = 2$.

- $B = K[X, Y, Z]/(X^2 - Y^6 - Z^6)$.
- $A = B^{(2)} = K[y^2, yz, z^2, xy, xz]$, where

$$\begin{aligned} A_1 &= Ky^2 + Kyz + Kz^2. \\ A_2 &= Ky^4 + Ky^3z + Ky^2z^2 + Kyz^3 + Kz^4 + Kxy + Kxz. \end{aligned}$$

- $I = (y^2, yz, z^4, xy, xz)$
- $Q = (y^2 - z^4, yz)$.

Then $\ell_A(A/I) = g = 2$ and thus $\bar{I} = I$.

Proof of sketch (2)

$$\boxed{I = (y^2, \; yz, \; z^4, \; xy, \; xz) \text{ and } Q = (y^2 - z^4, \; yz)}$$

Assertion

- $\overline{I^2} = I^2 = QI$.
- $xy^3 \in \overline{I^3} \setminus Q\overline{I^2}$ and $\overline{I^3} = Q\overline{I^2} + (xy^3)$.
- $\overline{I^{n+1}} = Q\overline{I^n}$ for all $n \geq 3$.
- $p_g(A) = q(0I) = 2(= g)$.
- $q(1 \cdot I) = 1$ and $q(nI) = 0$ for all $n \geq 2$.

Sketch of proof (3)

Claim 1: $p_g(A) = 2$.

$B = K[X, Y, Z]/(X^2 - Y^6 - Z^6)$ is a graded ring with $\deg x = 3$ and $\deg y = \deg z = 1$.

Thus

$$\begin{aligned} a(B) &= \deg(X^2 - Y^6 - Z^6) - (\deg x + \deg y + \deg z) \\ &= 6 - (3 + 1 + 1) = 1. \end{aligned}$$

Since $A = B^{(2)}$ and $H^2_{\mathfrak{m}}(B)^\vee = K_B = B(a(B)) = B(1)$, we have

$$[H^2_{\mathfrak{m}}(A)]_{\geq 0} = [H^2_{\mathfrak{m}}(A)]_0 = [H^2_{\mathfrak{m}}(B)]_0 \cong B_1 = Ky + Kz.$$

Hence $p_g(A) = \dim_K [H^2_{\mathfrak{m}}(A)]_{\geq 0} = 2 (= g)$.

Sketch of proof (4)

$$\boxed{I = (y^2,\ yz,\ z^4,\ xy,\ xz) \text{ and } Q = (y^2 - z^4,\ yz)}$$

Claim 2: $I^2 = QI$.

Since $I = Q + (z^4, xy, xz)$, we must show that $(z^4, xy, xz)^2 \subset QI$. For instace,

$$\begin{aligned}
(z^4)^2 &= -(y^2 - z^4)z^4 + yz \cdot yz \cdot z^2 \in QI. \\
(xy)^2 &= x^2y^2 = (y^6 + z^6)y^2 \\
&= y^8 + y^2z^6 \\
&= (y^2 - z^4)y^6 + yz(y^3z^3 + yz^5) \in QI.
\end{aligned}$$

Sketch of proof (5)

Note that $A_n = K[y,z]_{2n} \oplus xK[y,z]_{2n-3}$ as $K[y,z]^{(2)}$-modules.

Claim 3: $f_0 \in K[y,z]_{2n} \cap \overline{I^n} \Longrightarrow f_0 \in I^n$ for each $n \geq 1$

Put $I_0 = (y^2, yz, z^4)K[y,z]$. By assumption, we have

$$f_0^s + c_1 f_0^{s-1} + \cdots + f_s = 0 \quad (\exists s \geq 1, \exists c_i \in I^{in}).$$

Since $I^{in} \cap K[y,z] = I_0^{in}$ (non-trivial!), we may assume $c_i \in I_0^{in}$ for $\forall i \geq 1$.

Then $f_0 \in \overline{(y^2, yz, z^4)^n} = (y^2, yz, z^4)^n \subset I^n$ because $(y^2, yz, z^4)K[y,z]$ is normal.

Claim 4: $0 \neq f_1 \in K[y,z]_{2n-3}$, $xf_1 \in \overline{I^n} \Longrightarrow n \geq 3$

By assumption, we have $(xf_1)^2 \in \overline{I^{2n}}$.

The Claim 3 yields $(y^6+z^6)f_1^2 = (xf_1)^2 \in \overline{I^{2n}} \cap K[y,z]_{2\cdot 2n} \subset I^{2n}$.

The degree (in y and z) of any monomial in $I^{2n} = (\underbrace{y^2, yz}_{deg2}, \underbrace{z^4, xy, xz}_{deg4})^{2n}$ is

at least $4n = \deg(y^6+z^6)f_1^2$.

Hence $(y^6+z^6)f_1^2 \in (y^2, yz)^{2n}$ and the the highest power of z appearing in $(y^6+z^6)f_1^2$ is at most $2n$. Therefore $n \geq 3$.

Sketch of proof (7)

Claim 5: If $n \leq 2$, then $\overline{I^n} \cap A_n \subset I^n \cap A_n$

Any $f \in \overline{I^n} \cap A_n$ can be written as

$$f = f_0 + xf_1 \quad (\exists f_0 \in K[y,z]_{2n},\ f_1 \in K[y,z]_{2n-3})$$

Let $\sigma \in \mathrm{Aut}_{K[y,z]^{(2)}}(A)$ such that $\sigma(x) = -x$.

Then since $\sigma(I) = I$, we obtain $\sigma(f) = f_0 - xf_1 \in \overline{I^n}$.

$$\therefore f_0 = \frac{f + \sigma(f)}{2} \in \overline{I^n} \quad \text{and} \quad xf_1 = \frac{f - \sigma(f)}{2} \in \overline{I^n}.$$

By Claim 3,4, we have $f_0 \in I^n$ and $f_1 = 0$.

Therefore $f = f_0 \in I^n \cap A_n$, as required.

Sketch of proof (8)

Claim 6: $xy^3 \in \overline{I^3} \setminus Q\overline{I^2}$

$$I^3 = (y^6, y^5z, y^4z^2, y^3z^3, y^2z^7, \ldots, xy^5, xy^4z, \ldots, xz^7).$$

Since $(xy^3)^2 = x^2y^6 = (y^6 + z^6)y^6 = (y^6)^2 + (y^3z^3)^2 \in (I^3)^2$, we have $xy^3 \in \overline{I^3}$.

Assume $xy^3 \in Q\overline{I^2} = (a, b)\overline{I^2}$, where $a = y^2 - z^4$ and $b = yz$. Then $xy^3 = au + bv$ for some $u, v \in \overline{I^2}$.

Sketch of proof (8)

On the other hand, $xy^3 = (y^2 - z^4)xy + yz \cdot xz^3 = a \cdot xy + b \cdot xz^3$.

$$\begin{aligned} \therefore au + bv &= a \cdot xy + b \cdot xz^3. \\ \therefore a(u - xy) &= b(xz^3 - v). \end{aligned}$$

As a, b are regular sequence, we have

$$u - xy = bh, \quad xz^3 - v = ah \quad (\exists h \in A_1).$$

So we may asssume $u, v \in A_2$ and thus $u, v \in I^2$ by Claim 5.

Thus $xy^3 = au + bv \in QI^2 = I^3$.

This is a contradiction.

Sketch of proof (9)

We will finish the proof.

Fact 3.2 (Proposition 2.8)

1. $2 \cdot q(1 \cdot I) + \ell_A(\overline{I^2}/QI) = q(0 \cdot I) + q(2 \cdot I)$.
2. $2 \cdot q(2 \cdot I) + \ell_A(\overline{I^3}/Q\overline{I^2}) = q(1 \cdot I) + q(3 \cdot I)$.
3. $2 \cdot q(n \cdot I) + \ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = q((n-1) \cdot I) + q((n+1) \cdot I)$ $(n \geq 3)$

If $q(1 \cdot I) = q(2 \cdot I)$, then $q(2 \cdot I) = q(3 \cdot I)$ and thus $\ell_A(\overline{I^3}/Q\overline{I^2}) = 0$. This contradicts Claim 6. Hence

$$2 = p_g(A) = q(0 \cdot I) > q(1 \cdot I) > q(2 \cdot I) \geq 0.$$

Thus $q(1 \cdot I) = 1$ and $q(2 \cdot I) = 0$ (and thus $q(n \cdot I) = 0$ for all $n \geq 3$).

In particular, $\overline{I^2} = QI$, $\ell_A(\overline{I^3}/Q\overline{I^2}) = 1$ and $\overline{I^{n+1}} = Q\overline{I^n}$ for $n \geq 3$ by the above formula.

Sketch of proof (10)

If we obtain that $e_0(I) = 4g - 2$, $\ell_A(A/I) = g$, $p_g(A) = g$, $q(I) = g - 1$ and $q(\infty I) = 0$, then

$$\begin{aligned} \bar{e}_1(I) &= e_0(I) - \ell_A(A/I) + \{p_g(A) - q(I)\} \\ &= (4g-2) - g + \{g - (g-1)\} \\ &= 3g - 1. \end{aligned}$$

$$\begin{aligned} \bar{e}_2(I) &= p_g(A) - q(\infty I) \\ &= p_g(A) - q(g \cdot I) \\ &= g - 0 = g. \end{aligned}$$

Thank you very much for your attention!