Uniform positivity of F-signature and F-alpha invariants

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F-signature

Let $(A, \mathfrak{m})$ be a d-dimensional F-finite local domain of char. $p > 0$ with perfect reside field $A/\mathfrak{m}$.

Definition (Huneke–Leuschke'02)

$$s(A) := \lim_{e\to\infty} \frac{\max\{n \geq 0 \mid F^e_* A \cong A^n \oplus M_e\}}{p^{de}}$$ *F-signature*

The limit alway exists (Tucker'12)

Basic properties:

1. $0 \leq s(A) \leq 1$
2. $s(A) = 1$ iff A is regular
3. $s(A) > 0$ iff A is strongly F-regular (Aberbach–Leuschke'03)
   A is SFR iff $\forall c \in A \setminus 0, \exists e \in \mathbb{N}$ s.t. $A \to F^e_* A \xrightarrow{\cdot F^e_* c} F^e_* A$ splits as an A-module hom.

Reduction modulo p

Throughout this talk,

$$R := (\mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_r))_{(X_1, \ldots, X_n)},$$

where $f_i \in \mathbb{Z}[X_1, \ldots, X_n]$.

$$\mathbb{Z}[X_1, \ldots, X_n] \twoheadrightarrow \mathbb{F}_p[X_1, \ldots, X_n] \quad f_i \mapsto f_i \bmod p$$

$$R \bmod p := (\mathbb{F}_p[X_1, \ldots, X_n]/(f_i \bmod p))_{(X_1, \ldots, X_n)}$$

Conjecture (cf. Schwede–Smith'10)

$R \bmod p$ *is SFR for* $\forall p \gg 0 \iff X := \operatorname{Spec} R$ *is of klt type*

X is of klt type iff $\exists \Delta \geq 0$ $\mathbb{Q}$-divisor on X s.t. $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X, Δ) is klt.
($\Leftarrow$) is known to hold (T.'04).
($\Rightarrow$) is known if R is $\mathbb{Q}$-Gorenstein (Hara–Watanabe'02, Smith'97).

$s(R) := \liminf_{p\to\infty} s(R \bmod p)$ ***F*-signature in equal char. 0**

Note:

- We don't know whether the limit exists.
- $s(R) = 1$ iff R is regular.

Conjecture (Carvajal-Rojas–Schwede–Tucker'18)

If $X = \operatorname{Spec} R$ is of klt type, then $s(R) > 0$.

Conj SS ($\Leftarrow$) implies if X is of klt type, then $s(R \bmod p) > 0$ for $\forall p \gg 0$. However, the limit $\liminf_{p\to\infty} s(R \bmod p)$ may be zero!

Known cases

- $\dim R = 2$ (2-dim. klt type sing. are finite quotient sing.)
- R is (a localization of) a toric ring (Singh'05)
- R is a diagonal hypersurface (Caminata–Shideler–Tucker–Zerman'24)

Main Theorem

A ring extension $R \hookrightarrow S$ is pure iff $\forall R$-module M, $M \to M \otimes_R S$ is injective. (e.g., R is a direct summand of S)

Theorem (T.–Yamaguchi'25)

Let $R \hookrightarrow S$ be a pure local $\mathbb{C}$-algebra hom. between local ring ess. of finite type over $\mathbb{C}$. If $\operatorname{Spec} S$ is of klt type and $s(S) > 0$, then $s(R) > 0$.

Remark

- If $\operatorname{Spec} S$ is of klt type, then so is $\operatorname{Spec} R$ (Zhuang'23).
- Schoutens'05 proved the $\mathbb{Q}$-Gorenstein case using different methods (ultraproduct methods).
- Schoutens+Zhuang $\Rightarrow$ the pair case (T.-Yamaguchi'24) i.e., If $(\operatorname{Spec} S, \Delta_S)$ is of klt type, then so is $(\operatorname{Spec} R, \Delta_R)$ where Δ_S is the cycle-theoretic pullback of Δ_R

A local ring $(R, \mathfrak{m})$ essentially of finite type over $\mathbb{C}$ is a reductive quotient singularity if the completion $\widehat{R} \cong \mathbb{C}[[X_1, \ldots, X_n]]^G$, where G is a reductive group acting on $\mathbb{C}[[X_1, \ldots, X_n]]$.

Corollary 1

Conj CRST holds for reductive quotient singularities.

If $X = \operatorname{Spec} R$ is of klt type, by BCHM,

$$\Big(\bigoplus_{i \geq 0} \mathcal{O}_X(-iK_X)\Big)_{\mathcal{M}}, \quad \text{where } \mathcal{M} = \bigoplus_{i \geq 1} \mathcal{O}_X(-iK_X)$$

is essentially of finite type over $\mathbb{C}$ and Gorenstein. Since $\mathcal{O}_X = R$ is its pure subring, we have the following corollary.

Corollary 2

Conj CRST can be reduced to the case when R is Gorenstein.

Local F-alpha invariants

Key ingredients in the proof of Main Thm

- A local version of Pande's F-alpha invariants
- Ultra F-regularity originally introduced by Schoutens

Let $(A, \mathfrak{m})$ be an F-finite local domain.

Definition (T.–Yamaguchi'25, cf. Pande'23)

$$\alpha_F(A) = \inf_{0 \neq f \in \mathfrak{m}} \mathrm{fpt}(f)\overline{\mathrm{ord}}_{\mathfrak{m}}(f) \quad \text{local } F\text{-alpha invariant}$$

$$\overline{\mathrm{ord}}_{\mathfrak{m}}(f) := \max\{n \geq 0 \mid f \in \overline{\mathfrak{m}^n}\}$$

$$\begin{aligned}\mathrm{fpt}(f) &:= \sup\{t \geq 0 \mid (A, t \cdot \mathrm{div}(f)) \text{ is SFR}\} \\ &= \sup\{t \geq 0 \mid \forall c \in A \setminus \{0\}, \exists e \in \mathbb{N} \text{ s.t. } A \to F^e_* A \\ &\qquad x \mapsto F^e_*(c f^{\lceil t(p^e-1)\rceil} x^{p^e}) \text{ splits}\}\end{aligned}$$

$$R = (\mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_r))_{(X_1,\ldots,X_n)}$$

$$\alpha_F(R) := \liminf_{p\to\infty} \alpha_F(R \bmod p)$$

Basic properties:

1. $0 \le \alpha_F(R) \le 1$.
2. $\alpha_F(R) = 1$ if R is regular.

e.g.(Pande'23)
$X = (x^3 + y^3 + z^3 + w^3 = 0) \subseteq \mathbb{P}^3$ and
R is the localization of the homogeneous coordinate ring
$\Rightarrow \alpha_F(R) = 1/2$ but $\alpha(X) = 2/3$.

Proposition (cf. Pande'23)

$$s(R) > 0 \iff \alpha_F(R) > 0$$

Ultraproducts

Let $\mathcal{P}$ be the set of all primes. Let $(A_p)_{p \in \mathcal{P}}$ be a family of rings.

Definition (ultraproducts)

$$\operatorname*{ulim}_p A_p := \prod_{p \in \mathcal{P}} A_p / \sim$$

where $(a_p)_{p \in \mathcal{P}} \sim (b_p)_{p \in \mathcal{P}}$ *iff* $a_p = b_p$ *for "almost all* p*".*

$\operatorname{ulim}_p A_p$ is a ring. If A_p is an (algebraically closed) field for almost all p, then so is $\operatorname{ulim}_p A_p$.

Proposition

$$\exists \text{ non-canonical isomorphism } \operatorname*{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$$

From now on, we fix this isom.

$R = (\mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_r))_{(X_1, \ldots, X_n)}$

We've fixed the isom. $\mathbb{C} \cong \operatorname{ulim}_p \overline{\mathbb{F}_p}$.

$R \rightsquigarrow R_p$ local ring essentially of finite type over $\overline{\mathbb{F}_p}$

Similarly, $x \in R \rightsquigarrow x_p \in R_p$

Note:

1. If R is regular (resp. normal, CM, Gorenstein), so is R_p for almost all p.
2. R_p is the "base change" of $R \bmod p$ over $\overline{\mathbb{F}_p}$ for almost all p.

$R_\infty := \operatorname{ulim}_p R_p$ (non-Noetherian) $\mathbb{C}$-algebra

Definition (Ultra-Frobenii)

$\epsilon = \operatorname{ulim}_p e_p \in {}^*\mathbb{N} := \operatorname{ulim}_p \mathbb{N}$

$$F^\epsilon : R \to R_\infty \quad x \mapsto \operatorname*{ulim}_p x_p^{p^{e_p}}$$

Ultra-F-regularity

Definition (T.–Yamaguchi'25, cf. Schoutens'05)

$0 \neq f = \mathrm{ulim}_p f_p \in R_\infty,\ t = \mathrm{ulim}_p t_p \in {}^*\mathbb{R}_{\geq 0} := \mathrm{ulim}_p \mathbb{R}_{\geq 0}$
(R, f^t) *is* *ultra-F-regular* *iff*
$\forall\, \mathrm{ulim}_p c_p \in R_\infty, \exists \epsilon = \mathrm{ulim}_p e_p \in {}^*\mathbb{N}$

$$R \to F^\epsilon_* R_\infty \quad x \mapsto F^\epsilon_*(\mathrm{ulim}_p c_p f_p^{\lceil t_p(p^{e_p}-1)\rceil} x_p^{p^{e_p}})$$ is pure.

Advantages of UFR

1. If $R \hookrightarrow S$ is pure and (S, f^t) is UFR, then (R, f^t) is UFR.

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ F^\epsilon_* R_\infty & \longrightarrow & F^\epsilon_* S_\infty \end{array}$$

2. UFR ensures SFR of pairs for almost all p at once.

Proposition

Suppose $\bigoplus_{i\geq 0}\mathcal{O}_X(-iK_X)$ *is finitely generated* $(X=\operatorname{Spec} R)$.
(R, f^t) *is UFR* $\iff$ $(R, t_p\cdot \operatorname{div}(f_p))$ *is SFR for almost all* p.

$R \bmod p = (\mathbb{F}_p[X_1,\dots,X_n]/(f_i \bmod p))_{(X_1,\dots,X_n)}$
$\mathfrak{m} \bmod p$ is the maximal ideal of $R \bmod p$.

Main Lemma

Let $R\hookrightarrow S$ *be a pure local* $\mathbb{C}$*-algebra hom. between klt type sing.*
Suppose f_p *is any nonzero element of* $\mathfrak{m} \bmod p$ *for all* p. *Then*

$$\operatorname{fpt}(R \bmod p, f_p)\geq \operatorname{fpt}(S \bmod p, f_p)$$

for all but finitely many p.

Note: Purity is not preserved under reduction modulo p, i.e.,
$R \bmod p\hookrightarrow S \bmod p$ is not pure in general.

Global F-alpha invarinats

Let Z be a normal projective variety over a perfect field of char. p, $B \geq 0$ be a $\mathbb{Q}$-divisor and Γ be an ample $\mathbb{Q}$-divisor on Z.

Definition (Schwede–Smith'10)

(Z, B) is globally F-regular iff $\forall D \geq 0$ Cartier divisor on Z, $\exists e \in \mathbb{N}$ s.t. $\mathcal{O}_Z \to F^e_* \mathcal{O}_Z \to F^e_* \mathcal{O}_Z(\lceil (p^e - 1)B \rceil + D)$ splits.

Definition (Global F-split thresholds, F-alpha invariants)

Suppose (Z, B) is a GFR pair and $D \geq 0$ is a $\mathbb{Q}$-divisor on Z.

1. $\mathrm{gfst}((Z, B); D) := \sup\{t \geq 0 \mid (Z, B + tD) \text{ is GFR}\}$
2. $\alpha_F((Z, B); \Gamma) := \inf_{0 \leq E \sim_{\mathbb{Q}} \Gamma} \mathrm{gfst}((Z, B); E)$

Note: $\mathrm{fpt} \neq \mathrm{gfst}$. If $D = (x^3 + y^3 + z^3 + w^3 = 0) \subseteq \mathbb{P}^3$ and $p = 2$, then $\mathrm{fpt}(D) = 1$ but $\mathrm{gfst}(D) = 1/2$.

Let (X, Δ) be a log Fano pair over $\mathbb{C}$, i.e., X is projective and normal, $\Gamma := -(K_X + \Delta)$ is ample $\mathbb{Q}$-Cartier and (X, Δ) is klt.

Proposition (Schwede-Smith'10)

The pair $(X \bmod p, \Delta \bmod p)$, in particular $X \bmod p$, is GFR for $\forall p \gg 0$.

$$\alpha_F((X, \Delta); \Gamma) := \liminf_{p \to \infty} \alpha_F((X \bmod p, \Delta \bmod p); \Gamma \bmod p)$$

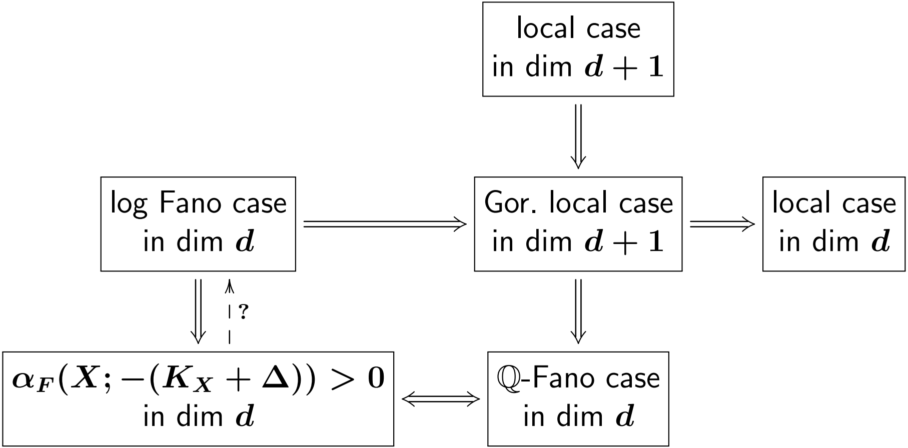
We define $\alpha_F(X; -\Gamma)$ similarly.

A projective variety Y is $\mathbb{Q}$-Fano iff $(Y, 0)$ is log Fano.

Theorem (T.–Yamaguchi'25)

1. *If $\alpha_F((X, \Delta); -(K_X + \Delta)) > 0$ for all log Fano pairs (X, Δ) over $\mathbb{C}$, then Conj CRST holds.*
2. *If $\alpha_F(Y; -K_Y) > 0$ for all $\mathbb{Q}$-Fano varieties Y over $\mathbb{C}$, then $\alpha_F(X; -(K_X + \Delta)) > 0$ for all log Fano pairs (X, Δ) over $\mathbb{C}$.*

Summary:



A key difficulty in ($\dashrightarrow$) is that the F-split region

$$\mathrm{cl}(\{(s,t) \in \mathbb{R}^2_{\geq 0} \mid (Z, sB+tD) \text{ is GFR}\})$$

is not a polytope (unlike lc region).