Introduction to Multiplier Ideals.

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- (X, D) X : normal variety, $D \geq 0$: $\mathbb{Q}$-divisor on X
  ($D = \sum d_i D_i$, $d_i \in \mathbb{Q}_{\geq 0}$, $D_i \subset X$: prime divisor)
  $K_X + D$ is $\mathbb{Q}$-cartier
  i.e. $\exists r \in \mathbb{N}$ s.t. $r(K_X + D)$ is Cartier
- $(X, \mathfrak{a}^t)$ X : $\mathbb{Q}$-Goren. normal, $\mathfrak{a} \subseteq \mathcal{O}_X$, $t > 0$
  i.e. $\exists r \in \mathbb{N}$ s.t. rK_X is cartier
- $(X, D; \mathfrak{a}^t)$ ...

Our main reference is Lazarsfeld's book [La2] for the theory of multiplier ideals.

<u>log resol.</u>

$f : \tilde{X} \to X$ log resol. of D (resp. $\mathfrak{a}$)

$\overset{\text{def}}{\Longleftrightarrow}$
- f : proper birat^{l}
- $\tilde{X}$: smooth
- $\mathrm{Supp}(f_*^{-1}D) \cup \mathrm{Exc}(f)$: SNC divisor
  ($f_*^{-1}D$: strict transform of D)

(resp. $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)$ inv., $\mathrm{Supp}\, F \cup \mathrm{Exc}(f)$ is SNC.)

Def. of multiplier ideals

- $\mathcal{J}(D) = \mathcal{J}(X, D) := f_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner \underbrace{f^*(K_X + D)}_{\frac{1}{r} f^*(r(K_X + D))} \lrcorner) \subset \mathcal{O}_X$

- $\mathcal{J}(\mathfrak{a}^t) = \mathcal{J}(X, \mathfrak{a}^t) := f_* \mathcal{O}_X(K_{\tilde{X}} - \llcorner (f^* K_X + tF) \lrcorner) \subset \mathcal{O}_X$

We can define $\mathcal{J}(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_k^{t_k})$, $\mathcal{J}(X, D; \mathfrak{a}^t)$ similarly.

(X, D) : klt (Kawamata log terminal) $\Leftrightarrow$ $\mathcal{J}(X, D) = \mathcal{O}_X$

(X, D) : klt at x $\Leftrightarrow$ $\mathcal{J}(X, D)_x = \mathcal{O}_{X,x}$

X has only lt sing. at $x \in X$ $\Leftrightarrow$ $(X, 0)$ is klt at $x \in X$
(log terminal)

- Assume X has only lt sing. at $x \in X$

$\mathrm{lct}(D; x) := \sup\{t \in \mathbb{Q} \mid \mathcal{J}(X, t \cdot D)_x = \mathcal{O}_{X,x}\}$

$\mathrm{lct}(\mathfrak{a}; x) := \sup\{t \in \mathbb{Q} \mid \mathcal{J}(X, \mathfrak{a}^t)_x = \mathcal{O}_{X,x}\}$

Basic properties

(1) $\mathcal{J}(D)$, $\mathcal{J}(\mathfrak{a}^t)$, etc. are indep. of the choice of the log resol. f. In particular,

$$\left.\begin{array}{l} X : \text{smooth} \\ \mathrm{Supp}(D) : \text{SNC} \end{array}\right. \Rightarrow \mathcal{J}(D) = \mathcal{O}_X(-\llcorner D \lrcorner)$$

(2) $D_1 \geq D_2 \Rightarrow \mathcal{J}(D_1) \subseteq \mathcal{J}(D_2)$

$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \Rightarrow \mathcal{J}(\mathfrak{a}_1^t) \subseteq \mathcal{J}(\mathfrak{a}_2^t)$

Moreover if $\mathfrak{a}_2 \subseteq \overline{\mathfrak{a}_1} \Rightarrow \mathcal{J}(\mathfrak{a}_1^t) = \mathcal{J}(\mathfrak{a}_2^t)$

$\left(\begin{array}{l} g: Y \to X : \text{normalized blow-up along } \mathfrak{a} \text{ s.t. } \mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-E) \\ \overline{\mathfrak{a}} := g_* \mathcal{O}_Y(-E) \end{array} \right)$

(3) Assume X has only lt sing.

$\Rightarrow \mathcal{J}(\mathfrak{a}) \supseteq \mathfrak{a}$

Moreover if D is a cartier int. div. $\Rightarrow \mathcal{J}(D) = \mathcal{O}(-D)$

if $\mathfrak{a}$ is of pure ht 1 $\Rightarrow \mathcal{J}(\mathfrak{a}) = \mathfrak{a}$

(☺ if $\mathfrak{a}$ is of pure ht 1 $\Rightarrow \mathfrak{a}$ is reflexive.

(4) X : $\mathbb{Q}$-Goren. affine var. , $\mathfrak{a} \subseteq \mathcal{O}_X$, $t > 0$

Fix $t < k \in \mathbb{N}$. Take general elements $x_1, \cdots, x_k \in \mathfrak{a}$

$A_i := \operatorname{div} x_i$, $D := \frac{1}{k} \sum A_i$

$\Rightarrow \mathcal{J}(\mathfrak{a}^t) = \mathcal{J}(t \cdot D)$

(5) $\operatorname{lct}(D; x), \operatorname{lct}(\mathfrak{a}; x) \in \mathbb{Q}_{>0}$

<u>Example</u>

(1) X : smooth var. of dim. n , $x \in X$, $\mathfrak{m} := \mathfrak{m}_{X,x}$

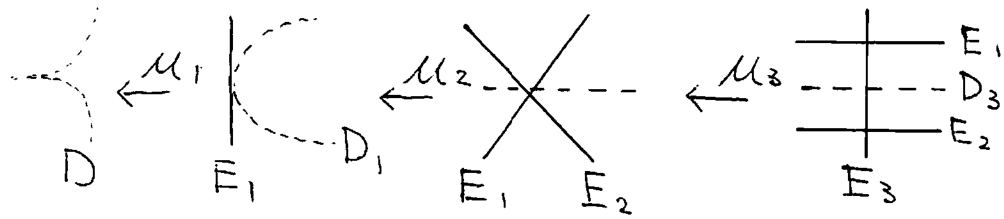
$\mathcal{J}(\mathfrak{m}^t) = \mathfrak{m}^{\lfloor t \rfloor + 1 - n}$, $\mathrm{lct}(\mathfrak{m}; x) = n$

$\because$ $f : \hat{X} \to X$: blow-up at x

$\mathfrak{m}\mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{X}}(-E)$, $K_{\hat{X}/X} = (n-1)E$

$\mathcal{J}(\mathfrak{m}^t) = f_*\mathcal{O}_{\hat{X}}((n-1-\lfloor t \rfloor)E) = \mathfrak{m}^{\lfloor t \rfloor + 1 - n}$

(2) $X = \mathbb{C}^2$, $D = (x^2 + y^3 = 0)$



$K_{X_1/X} = E_1$, $K_{X_2/X} = E_1 + 2E_2$, $K_{X_3/X} = E_1 + 2E_2 + 4E_3$

$f_1^*D = 2E_1 + D_1$, $f_2^*D = 2E_1 + 3E_2 + D_2$, $f_3^*D = 2E_1 + 3E_2 + 6E_3 + D_3$

$(f_i := \mu_i \circ \cdots \circ \mu_1 : X_i \to X)$

$\mathcal{J}(t \cdot D) = f_{3*}\mathcal{O}_{X_3}(\lceil 1-2t \rceil E_1 + \lceil 2-3t \rceil E_2 + \lceil 4-6t \rceil E_3 - \lfloor t \rfloor D_3)$

$\therefore \mathrm{lct}(D; 0) = \frac{5}{6}$, $\mathcal{J}(\frac{5}{6} \cdot D) = (x, y)$

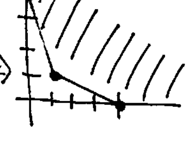
(3) $X = \mathrm{Spec}\, k[\sigma^\vee \cap M]$: affine toric var.

$\mathfrak{a} \subseteq k[\sigma^\vee \cap M]$: monomial ideal

$\mathfrak{a} \rightsquigarrow P(\mathfrak{a}) \subset M_{\mathbb{R}}$: Newton polytope of $\mathfrak{a}$

i.e. convex hull of the set of exponents of the monomials in $\mathfrak{a}$

<u>ex</u>

$\mathfrak{a} = (x^4, xy, y^4) \subset \mathbb{C}[x, y] \Rightarrow$  $P(\mathfrak{a})$

Thm. (Blickle [Bl], Hara-Yoshida [HY])

Assume $X = \operatorname{Spec} k[\sigma^\vee \cap M]$ is $\mathbb{Q}$-Goren.
i.e. $\exists r \in \mathbb{N}$ s.t. rK_X : cartier

$\exists u \in M$ s.t. $\operatorname{div} \chi^u = -rK_X$

$\omega := \frac{u}{r}$

$\Rightarrow \mathcal{J}(\mathfrak{a}^t) = \langle \chi^v \mid v + \omega \in \operatorname{Int}(t \cdot P(\mathfrak{a})) \subseteq M_{\mathbb{R}} \rangle$

Cor. (Howald [Ho1])

$X = \mathbb{C}^n$, $\mathfrak{a} \subseteq \mathbb{C}[x_1, \cdots, x_n]$: monomial ideal

$\Rightarrow \mathcal{J}(\mathfrak{a}^t) = \langle \chi^v \mid v + \mathbb{1} \in \operatorname{Int}(t \cdot P(\mathfrak{a})) \subseteq \mathbb{R}^n \rangle$

Proof

$\mu : Y \to X$: toric log resol. of $\mathfrak{a}$ s.t. $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ torus inv.

$\rightsquigarrow \mathcal{J}(\mathfrak{a}^t)$: monomial ideal

$v + \omega \in \operatorname{Int}(t \cdot P(\mathfrak{a}))$

$\Leftrightarrow v + \omega - \varepsilon v' \in t \cdot P(\mathfrak{a})$ $(0 < {}^\forall \varepsilon \ll 1, {}^\forall v' \in \operatorname{Int}(\sigma^\vee) \cap M)$

$\Leftrightarrow \mu^* \operatorname{div} \chi^v - \mu^* K_X - \varepsilon \mu^* \operatorname{div} \chi^{v'} \geq tF$

$\Leftrightarrow \mu^* \operatorname{div} \chi^v + K_Y - \lceil K_Y + \varepsilon \mu^* \operatorname{div} \chi^{v'} + \mu^* K_X + tF \rceil \geq 0$

$\lceil K_Y + \varepsilon \mu^* \operatorname{div} \chi^{v'} + \mu^* K_X + tF \rceil = \lfloor \mu^* K_X + tF \rfloor$

$(\Leftrightarrow \chi^v \in \mathcal{J}(\mathfrak{a}^t))$

$\begin{cases} ① \ K_Y = -\sum D_i < 0 \ (D_i\text{'s are all the torus inv. div}) \\ ② \ 1 > (\text{coeff. of } \varepsilon \mu^* \operatorname{div} \chi^{v'}) \geq 0 \\ ③ \ \chi^{v'} \in \omega_X \text{ i.e. } \mu^* \operatorname{div} \chi^{v'} \text{ and } K_Y \text{ have the same support} \end{cases}$ ▨

$f \in \mathbb{C}[X] \rightsquigarrow \mathfrak{a}_f \subset \mathbb{C}[X]$: ideal gen. by the mono. appearing in f

$\mathcal{J}(t\cdot \mathrm{div}(f)) \subseteq \mathcal{J}(\mathfrak{a}_f^t)$

If f is "general" $\Rightarrow \mathcal{J}(t\cdot \mathrm{div}(f)) = \mathcal{J}(\mathfrak{a}_f^t)$ $(0<\forall t<1)$

Q. How "general"?

Thm. (Howald [Ho2])

$f \in \mathbb{C}[X]$: non-degenerate.

(i.e. $f \rightsquigarrow f_\sigma$ $\quad \sigma$: face of $P(f) := P(\mathfrak{a}_f)$

df_σ is nowhere vanishing on $(\mathbb{C}^*)^n$

for $\forall$face σ of $P(f)$

$\Rightarrow \mathcal{J}(t\cdot \mathrm{div}(f)) = \mathcal{J}(\mathfrak{a}_f^t)$, $0<\forall t<1$

Ex.

$f = x_1^{d_1} + \cdots + x_n^{d_n}$ is non-degenerate. Assume $\sum_i \frac{1}{d_i} < 1$.

since $P(f) = \{(u_1, \cdots, u_n) \in \mathbb{R}^n \mid \sum_i u_i/d_i \geq 1\}$

$\mathrm{lct}(\mathrm{div}(f); 0) = \mathrm{lct}(\mathfrak{a}_f; 0) = \sum_i 1/d_i$

Vanishing thm

(i) (local vanishing)

$f : \tilde{X} \to X$: log resol. of D

(resp. $\mathfrak{a}$ s.t. $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)$)

$\Rightarrow R^j f_* \mathcal{O}_{\hat{X}}(K_{\hat{X}} - \llcorner f^*(K_X + D) \lrcorner) = 0$, $(\forall j > 0)$

(resp. $R^j f_* \mathcal{O}_{\hat{X}}(K_{\hat{X}} - \llcorner f^* K_X + tF \lrcorner) = 0$, $(\forall j > 0, \forall t > 0)$)

(ii) (Nadel vanishing)

L: cartier int. div. on X s.t. $L - D$ is nef and big

$\Rightarrow H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0$, $(\forall i > 0)$

<u>Cor.</u>

X: proj. normal var.

B: very ample divisor on X

L: cartier int. div. on X s.t. $L - D$ is nef and big

$\Rightarrow \mathcal{O}_X(K_X + L + mB) \otimes \mathcal{J}(D)$ is gl. gen. if $m \geq \dim X$

$\because$ <u>Lem.</u> (Mumford [La1, Theorem 1.8.5])

F: coherent s.t. $H^i(X, F \otimes \mathcal{O}_X(-iB)) = 0$, $\forall i > 0$

$\Rightarrow F$ is gl. gen.

Nadel $\Rightarrow H^i(X, \mathcal{O}_X(K_X + L + (m-i)B) \otimes \mathcal{J}(D)) = 0$, $\forall i > 0$

$\Rightarrow$ O.K. ∎

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Multiplier ideals and inversion of adjunction

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Recall adjunction formula
i.e.
X: smooth, $Y \subset X$: smooth divisor

$K_X + Y|_Y = K_Y$

$\overset{\text{generalize}}{\rightsquigarrow}$ $(X, S+B)$ X: normal var.
$S \subset X$: reduced divisor
$B \geq 0$: $\mathbb{Q}$-cartier on X

s.t. $\begin{cases} S \text{ has no common comp. with } \mathrm{Supp}(B). \\ K_X + S + B \text{ is } \mathbb{Q}\text{-cartier} \end{cases}$

$\nu : S^\nu \to S$: normalization

$^\exists B^\nu \geq 0$ $\mathbb{Q}$-divisor on S^ν (different of B on S^ν)

s.t. $K_{S^\nu} + B^\nu = \nu^*(K_X + S + B|_S)$

$$\begin{tikzcd} \tilde{X} \arrow[d, "f"'] & \supset & \tilde{S} \arrow[d] \arrow[dr, "g"] & \\ X & \supset & S & S^\nu \arrow[l, "\nu"] \end{tikzcd}$$

f: embedded resol.

$K_{\tilde{X}} + \tilde{S} \equiv f^*(K_X + S + B) + \sum a_i E_i$

$K_{\tilde{S}} \equiv g^*(K_{S^\nu} + B^\nu) + \sum a_i E_i|_{\tilde{S}}$

ex. S : normal cartier $\Rightarrow B^\vee = B|_S$

$$(X, S+B) \underset{\text{inv. of adj.}}{\overset{\text{adjunction}}{\rightleftarrows}} (S^\vee, B^\vee)$$

$d := \min\{a_i \mid f(E_i) \subset S\}$

$d_S := \min\{a_i \mid E_i|_{\hat{S}} \neq \phi\}$

Note

In general $d \le d_S$

Thm

(i) (Kollár [K+], Shokurov [Sh])

$d > -1 \Leftrightarrow d_S > -1$

(ii) (Kawakita [Ka])

$d \ge -1 \Leftrightarrow d_S \ge -1$

Def.

(i) $(X, D, \mathfrak{a}^t)$ X : normal, $D \ge 0$: $\mathbb{Q}$-divisor on X, $\mathfrak{a} \subseteq \mathcal{O}_X$, $t > 0$
$K_X + D$ is $\mathbb{Q}$-cartier,

$f : \tilde{X} \to X$: log resol. of $(D, \mathfrak{a})$, $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)$

$\mathcal{J}(X, D, \mathfrak{a}^t) := f_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner (f^*(K_X + D) + tF) \lrcorner) \subset \mathcal{O}_X$

(ii) $(X,S;B,\mathfrak{a}^t)$: (X,S,B) as above, $\mathfrak{a}\subseteq\mathcal{O}_X$, $t>0$

$f:\tilde{X}\to X$: log resol. of $(S+B,\mathfrak{a})$ s.t. $\tilde{S}:=f_*^{-1}S$ is smooth

$\mathrm{adj}(X,S;B,\mathfrak{a}^t):=f_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}-\lfloor f^*(K_X+S+B)+tF\rfloor+\tilde{S})\subset\mathcal{O}_X$

$\cup$

$\mathcal{J}(X,S+B,\mathfrak{a}^t)$

<u>Remark</u>

(0) $B=0$, $\mathfrak{a}=\mathcal{O}_X \Rightarrow \mathrm{adj}(X,S):=\mathrm{adj}(X,S;B,\mathfrak{a}^t)$

(i) $\mathrm{adj}(X,S;B,\mathfrak{a}^t)$ is indep. of the choice of f.

(ii) X : $\mathbb{Q}$-Goren. affine, $\mathrm{ht}\,\mathfrak{a}\geq 2$, $f\in\mathfrak{a}$ is general

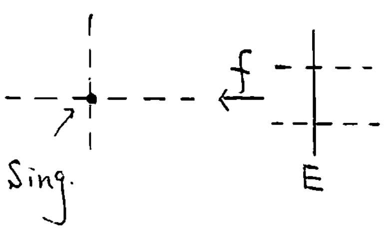
$\Rightarrow \mathcal{J}(X,\mathfrak{a})=\mathrm{adj}(X,\mathrm{div}(f))$

<u>Note</u>

- $d_S>-1 \iff \mathcal{J}(S^\nu,B^\nu)=\mathcal{O}_{S^\nu}$
  $\iff (S^\nu,B^\nu)$: klt
- $d>-1 \iff \mathrm{adj}(X,S;B)=\mathcal{O}_X$ near S
  $\iff (X,S+B)$: plt near S
  (purely log terminal)

<u>Ex</u>

(i) $X=\mathbb{C}^2$, $S=(xy=0)$, $B=0$, $\mathfrak{a}=\mathcal{O}_X$



$K_{\tilde{X}/X}=E$, $f^*S=\tilde{S}+2E$

$\mathrm{adj}(X,S)=f_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}-f^*K_X-f^*S+\tilde{S})$

$=f_*\mathcal{O}_{\tilde{X}}(-E)=(x,y)$

(ii) X: $\mathbb{Q}$-Goren. normal surface, $S \subset X$: reduced cartier divisor.

$\Rightarrow \mathrm{adj}(X,S)\mathcal{O}_S = C(S) := \mathrm{Ann}\,(\nu_*\mathcal{O}_{S^\nu}/\mathcal{O}_S)$

$$\left(\begin{array}{l} \nu : S^\nu \to S \ \text{: normalization} \\ \quad\ \ \cap \qquad \cap \\ \quad\ \ \tilde{X} \to X \text{ : embedded resol} \end{array}\right)$$

<u>Thm</u> (Restriction thm, c.f. [La2, Theorem 9.5.1])

$(X,S;B,\mathfrak{a}^t)$ as above, Assume $\mathfrak{a} \not\subset I_S = \mathcal{O}_X(-S)$

$\Rightarrow \mathrm{adj}(X,S;B,\mathfrak{a}^t)\mathcal{O}_S = \nu_* \mathcal{J}(S^\nu, B^\nu, \mathfrak{a}\mathcal{O}_{S^\nu}) \subset \mathcal{O}_S$

In particular

$d > -1 \Leftrightarrow d_S > -1$ in this case $S^\nu \cong S$ (i.e. S: normal)

<u>proof</u>

For simplicity assume $\mathfrak{a} = \mathcal{O}_X$

$$\begin{array}{ccccl} \tilde{X} & \supset & \tilde{S} & & \\ f\downarrow & & \downarrow & \overset{g}{\searrow} & S^\nu \qquad f\text{: embedded resol} \\ X & \supset & S & \swarrow_\nu & \end{array}$$

$0 \to \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner f^*(K_X+S+B)\lrcorner) \to \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner f^*(K_X+S+B)\lrcorner + \tilde{S})$

$\xrightarrow{\cdot\mathcal{O}_{\tilde{S}}} \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \llcorner g^*(K_{S^\nu}+B^\nu)\lrcorner) \to 0$

$\downarrow f_*$

$0 \to \mathcal{J}(X, S+B) \to \mathrm{adj}(X,S;B) \xrightarrow{\cdot\mathcal{O}_S} \nu^*\mathcal{J}(S^\nu, B^\nu)$

$\to R^1 f_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner f^*(K_X+S+B)\lrcorner) \underset{\uparrow}{=} 0$

local vanishing

∎

proof of Thm (ii) (Kawakita)

The question is local $\rightsquigarrow$ discuss over a germ at a closed pt. $x \in S \subset X$

($X = \operatorname{Spec} R$, R: local)

Assume $d_S \geq -1$ $\quad (\Leftrightarrow (S^\nu, B^\nu)$ is lc$)$

$\mathfrak{a}_0 := \mathrm{adj}(X, S; B) \subset \mathcal{O}_X$

$\mathfrak{a}_{n+1} := \mathrm{adj}(X, S; B, \mathfrak{a}_n^{1-\varepsilon_n})$, $0 < \varepsilon_n \ll 1$

$\mathfrak{a}_1 \mathcal{O}_S = \mathrm{adj}(X, S; B, \mathfrak{a}_0^{1-\varepsilon})\mathcal{O}_S = \nu_* \mathcal{J}(S^\nu, B^\nu, \mathfrak{a}_0 \mathcal{O}_S^{1-\varepsilon})$

$(\because d_S \geq -1,\ \mathfrak{a}_0 \mathcal{O}_S = \nu_* \mathcal{J}(S^\nu, B^\nu))$ $\quad \cup$ $\quad \mathfrak{a}_0 \mathcal{O}_S$

since $\mathfrak{a}_0 \supset \mathfrak{a}_1$,

$\mathfrak{a}_1 \mathcal{O}_S = \mathfrak{a}_0 \mathcal{O}_S$, i.e. $\mathfrak{a}_1 + I_S = \mathfrak{a}_0 + I_S$

Thus $\begin{cases} \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \\ \mathfrak{a}_0 + I_S = \mathfrak{a}_1 + I_S = \mathfrak{a}_2 + I_S = \cdots \end{cases}$

Suppose $d < -1$, $\quad (K_{\tilde{X}} + \tilde{S} \equiv f^*(K_X + S + B) + \sum a_j E_j)$

$\Rightarrow \exists E_i$: f-exc. divisor on $\tilde{X}$ s.t. $a_i < -1$.

$\mathfrak{a}_0 = \mathrm{adj}(X, S; B) \subset f_* \mathcal{O}_{\tilde{X}}(\ulcorner a_i \urcorner E_i) = f_* \mathcal{O}_{\tilde{X}}(-E_i)$

$\mathfrak{a}_1 = \mathrm{adj}(X, S; B, \mathfrak{a}_0^{1-\varepsilon}) \subset \mathrm{adj}(X, S; B, f_* \mathcal{O}_{\tilde{X}}(-E_i)^{1-\varepsilon})$

$\subset f_* \mathcal{O}_{\tilde{X}}(\ulcorner a_i - (1-\varepsilon) \urcorner E_i)$

$= f_* \mathcal{O}_{\tilde{X}}(-2E_i)$

$(\because \varepsilon \ll 1)$

$\therefore \mathfrak{a}_n \subset f_* \mathcal{O}_{\tilde{X}}(-(n+1)E_i)$, $\forall n \geq 0$

On the other hand, by Nagata's thm,

$\forall \ell \in \mathbb{N}$, $\exists k(\ell) \in \mathbb{N}$ s.t. $f_*\mathcal{O}_{\tilde{X}}(-k(\ell)E) \subset m_{x,X}^{\ell}$

$\therefore \mathfrak{a}_0 \subset \bigcap_{n \in \mathbb{N}} (\mathfrak{a}_n + I_S) \subset \bigcap_{\ell \in \mathbb{N}} (m_{x,X}^{\ell} + I_S) = I_S$

this implies $\nu_* \mathcal{J}(S^\nu, B^\nu) = 0$ contradiction.

$\therefore d \geq -1$ ▨

<u>Conj (Kollár, Shokurov)</u> (See [K+] and [Sh])

$^\forall Z \subset S$: closed subset

$d(Z) := \min\{a_i \mid f(E_i) \subset Z\}$

$d_S(Z) := \min\{a_i \mid E_i|_{\tilde{S}} \neq \phi,\ f(E_i|_{\tilde{S}}) \subset Z\}$

$d(Z) = d_S(Z)$?

($\leq$ o.K.)

<u>known case</u> (Ein-Mustaţă [EM], c.f. [EMY])

X : l.c.i , S : normal cartier

<u>Higher codimension</u>

X : $\mathbb{Q}$-Goren. normal var./$\mathbb{C}$

$Y := \sum^{k} t_i Y_i$, $t_i > 0$, $Y_i \subsetneq X$: closed subscheme

$\mathfrak{a}_i \subseteq \mathcal{O}_X$: def. ideal of Y_i

$f : \tilde{X} \to X$: log resol. of $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$

$\mathfrak{a}_i \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F_i)$

$K_{\tilde{X}/X} - \Sigma t_i F_i \equiv \Sigma a_j E_j$

(X, Y) : klt $\Leftrightarrow a_j > -1, \ \forall j$

(X, Y) : lc $\Leftrightarrow a_j \geq -1, \ \forall j$

Thm (T- [Ta1])

(X, Y) as above. Assume X is smooth

$Z \subsetneq X$: $\mathbb{Q}$-Goren. closed subvar. s.t. $Z \not\subset \cup Y_i$

$(Z, Y|_Z)$: lc $\Rightarrow (X, Y+Z)$: lc near Z

proof

For simplicity, assume $Y = 0$

$L \subset Z$: locus of lc sing.

i.e. L is defined by $\mathcal{J}(Z) = \mathcal{J}(Z, \mathcal{O}_Z) \subset \mathcal{O}_Z$

(since Z is lc, L is reduced)

$(I_Z \subset) I_L \subset \mathcal{O}_X$: def. ideal of L in X

i.e. the lift of $\mathcal{J}(Z)$

We have the following two restriction thm.

① $\mathcal{J}(Z, (\mathfrak{a}\mathcal{O}_Z)^t) \subset I_L \mathcal{J}(X, \mathfrak{a}^t)\mathcal{O}_Z, \quad \forall \mathfrak{a} \subset \mathcal{O}_X, \ \forall t > 0$

② $\mathcal{J}(Z, (\mathfrak{a}\mathcal{O}_Z)^t) \subset \mathcal{J}(X, \mathfrak{a}^t I_Z^{1-\varepsilon})\mathcal{O}_Z, \ \forall \mathfrak{a} \subset \mathcal{O}_X, \ \forall t > 0,$

$0 < \forall \varepsilon \ll 1$

(We prove these by char. $p > 0$ method, later)

$Z : lc \Rightarrow \underset{\mathcal{J}(Z)}{\underset{\|}{\mathcal{J}(Z, (I_L \mathcal{O}_Z)^{1-\varepsilon})}} \supset \underset{\mathcal{J}(Z)}{\underset{\|}{I_L \mathcal{O}_Z}} \quad , \quad 0 < {}^{\forall}\varepsilon \ll 1$

by ①, $I_L \mathcal{O}_Z \subset \mathcal{J}(Z, (I_L\mathcal{O}_Z)^{1-\varepsilon}) \underset{①}{\subset} I_L \mathcal{J}(X, I_L^{1-\varepsilon})\mathcal{O}_Z$

$\Rightarrow \mathcal{J}(X, I_L^{1-\varepsilon}) = \mathcal{O}_X$

by ②, $I_L\mathcal{O}_Z \subset \mathcal{J}(Z, (I_L\mathcal{O}_Z)^{1-\varepsilon}) \overset{②}{\subset} \mathcal{J}(X, I_L^{1-\varepsilon} I_Z^{1-\varepsilon})\mathcal{O}_Z$

since $\mathcal{J}(X, I_L^{1-\varepsilon} I_Z^{1-\varepsilon}) \supset I_Z \mathcal{J}(X, I_L^{1-\varepsilon}) = I_Z$,

$I_L \subset \mathcal{J}(X, I_L^{1-\varepsilon} I_Z^{1-\varepsilon}) \quad , \quad 0 < {}^{\forall}\varepsilon \ll 1$

$\rightsquigarrow (X, Z) : lc$ near Z ▨

Sketch of proof ② (we can prove ① similarly)

Assume $X = \operatorname{Spec} R$ $\quad ((R, \mathfrak{m})$: complete RLR of char.0)

$Z = \operatorname{Spec} S$ $\quad (S = R/I,\ I = \sqrt{I} \subset R$: unmixed) ⤵ char.p>0

ETS $\tau(S, (\mathfrak{a}S)^t) \subset \tau(R, \mathfrak{a}^t I^{1-\varepsilon})S$

$\quad {}^{\forall}\mathfrak{a} \subseteq R,\ {}^{\forall}t > 0,\ 0 < {}^{\forall}\varepsilon \ll 1$

dual

$0_{E_S}^{*(\mathfrak{a}S)^t} \supset 0_{E_R}^{*\mathfrak{a}^t I^{1-\varepsilon}} \cap E_S$
$\quad \overset{\Downarrow}{z}$

$E_S := E_S(S/\mathfrak{m}S)$, $E_R := E_R(R/\mathfrak{m})$,

$E_S \cong (0 : I)_{E_R} \subset E_R$

$\mathfrak{a}^{\lceil tq \rceil} I^{\lceil q(1-\varepsilon) \rceil} F_R^e(z) = 0 \in \mathbb{F}_R^e(E_R) \cong E_R \quad {}^{\forall}q = p^e \gg 0$

$F_R^e : E_R \to \mathbb{F}_R^e(E_R) \cong E_R$

$F_S^e : E_S \to \mathbb{F}_S^e(E_S)$

$(q = p^e)$

$\forall z \in E_S$, $F_S^e(z) = 0 \in \mathbb{F}_S^e(E_S) \Leftrightarrow (I^{[q]} : I) F_R^e(z) = 0 \in E_R$

since $I^{[q]} : I \subset I^{q-1} \subset I^{\lceil q(1-\varepsilon)\rceil}$, $q = p^e \gg 0$, $0 < {}^{\forall}\varepsilon \ll 1$

$\mathfrak{a}^{\lceil tq\rceil} (I^{[q]} : I) F_R^e(z) = 0$, $q = p^e \gg 0$,

$\Leftrightarrow (\mathfrak{a}S)^{\lceil tq\rceil} F_S^e(z) = 0$, $q = p^e \gg 0$

$\Rightarrow z \in 0_{E_S}^{*(\mathfrak{a}S)^t}$ ▨

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A char. p analog of adjoint ideals (Appendix)

$(R, \mathfrak{m})$: F-finite normal local of char. $p>0$

$f \neq 0$, $\mathfrak{a} \subset R$, $t > 0$

$\tau^{\mathrm{div}}(R, f; \mathfrak{a}^t) := \mathrm{Ann}_R\, 0_E^{*(f;\mathfrak{a}^t)}$

$E := E_R(R/\mathfrak{m}) \cong H^d_{\mathfrak{m}}(\omega_R)$

$z \in 0_E^{*(f;\mathfrak{a}^t)} \overset{\text{def}}{\Longleftrightarrow} \exists\, R \ni c \notin \forall$ min. prime of R/f

s.t. $c f^{q-1} \mathfrak{a}^{\lceil tq \rceil} z^q = 0$, $q = p^e \gg 0$

$$\left(\begin{array}{ccl} F^e : E & \longrightarrow & F^e(E) := {}^eR \otimes_R E \\ \quad\ z & \longmapsto & z^q := 1 \otimes z \end{array} \right.$$

If R is $\mathbb{Q}$-Goren.

R/f is $\mathbb{Q}$-Goren, normal

$\Rightarrow \tau(R/f, (\mathfrak{a}R/f)^t) = \tau^{\mathrm{div}}(R, f; \mathfrak{a}^t)R/f$

See [Ta3] for details

Ex

$R = k[[X, Y]]$, $f = XY$, $\mathfrak{a} = R$

$\Rightarrow \tau^{\mathrm{div}}(R, f) = (X, Y)$

$\qquad \rotatebox{90}{$=$}\ \tau^{\mathrm{div}}(R, f; R)$

Thm (T- [Ta3])

$(R, \mathfrak{m})$: normal local ring ess. of finite type $/\mathbb{C}$

$f \neq 0 \in R$, $\mathfrak{a} \subset R$, $t > 0$

$(\tilde{R}, \tilde{f}, \tilde{\mathfrak{a}})$: reduction to char. $p \gg 0$ of $(R, f, \mathfrak{a})$

$\Rightarrow \widetilde{\mathrm{adj}(\mathrm{Spec}\, R, \mathrm{div}(f); \mathfrak{a}^t)} = \tau^{\mathrm{div}}(\tilde{R}, \tilde{f}; \tilde{\mathfrak{a}}^t)$

Applications of asymptotic multiplier ideals

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local properties of multiplier ideals

(1) (Restriction thm)

X : normal $\mathbb{Q}$-Goren. var. $/\mathbb{C}$

$S \subset X$: normal $\mathbb{Q}$-Goren. cartier divisor

$\mathfrak{a} \subseteq \mathcal{O}_X$, $t > 0$. Assume $S \not\subset \mathrm{Zeroes}(\mathfrak{a})$.

$\Rightarrow \mathcal{J}(S, (\mathfrak{a}\mathcal{O}_S)^t) = \mathrm{adj}(X, S; \mathfrak{a}^t)\mathcal{O}_S \subset \mathcal{J}(X, \mathfrak{a}^t)\mathcal{O}_S$

(2) (Subadditivity) (Demailly-Ein-Lazarsfeld [DEL])

X : smooth

$\Rightarrow \mathcal{J}(\mathfrak{a}^s\mathfrak{b}^t) \subset \mathcal{J}(\mathfrak{a}^s)\mathcal{J}(\mathfrak{b}^t)$, $\forall \mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$, $\forall s, t > 0$

(More generally

$x \in X$, $\mathcal{J}(\mathfrak{a}^s\mathfrak{b}^t)_x \subset \sum_{\substack{\lambda+\mu=\dim X \\ \lambda,\mu \geq 0}} \mathcal{J}(\mathfrak{a}^s \mathfrak{m}_{X,x}^{\lambda})_x \, \mathcal{J}(\mathfrak{b}^t \mathfrak{m}_{X,x}^{\mu})_x$

$\subseteq \mathcal{J}(\mathfrak{a}^s)_x \, \mathcal{J}(\mathfrak{b}^t)_x$)

(3) (Summation) (Mustaţă [Mu])

X: smooth

$$\Rightarrow \mathcal{J}(X, (\mathfrak{a}+\mathfrak{b})^t) \subset \sum_{\substack{\lambda+\mu=t \\ \lambda,\mu\geq 0}} \mathcal{J}(X, \mathfrak{a}^\lambda)\, \mathcal{J}(X, \mathfrak{b}^\mu)$$

In particular

$$\mathcal{J}(X, (\mathfrak{a}+\mathfrak{b})^{s+t}) \subset \mathcal{J}(X, \mathfrak{a}^s) + \mathcal{J}(X, \mathfrak{b}^t)$$

<u>Sketch of proof (2)</u>

$$\begin{array}{ccccc} & X \cong \Delta & \hookrightarrow & X\times X & \\ & & \swarrow^{p_1} & & \searrow^{p_2} \\ & X & & & X \end{array}$$

since X: smooth, $\Delta \hookrightarrow X\times X$ c.I. diagonal embedding

$$\begin{array}{ccccc} \tilde{X}_1 & \leftarrow & \tilde{X}_1 \times \tilde{X}_2 & \rightarrow & \tilde{X}_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \leftarrow & X_1 \times X_2 & \rightarrow & X_2 \end{array}$$

$$\mathcal{J}(X, \mathfrak{a}^s\mathfrak{b}^t) \underset{\uparrow}{\subset} \mathcal{J}(X\times X, (p_1^{-1}\mathfrak{a})^s(p_2^{-1}\mathfrak{b})^t)\mathcal{O}_\Delta$$

repeated applications of Restriction thm

$$\mathcal{J}(X\times X, (p_1^{-1}\mathfrak{a})^s(p_2^{-1}\mathfrak{b})^t) = p_1^{-1}\mathcal{J}(X, \mathfrak{a}^s)\cdot p_2^{-1}\mathcal{J}(X, \mathfrak{b}^t)$$

$$\Rightarrow \mathcal{J}(X, \mathfrak{a}^s\mathfrak{b}^t) \subset \mathcal{J}(X, \mathfrak{a}^s)\,\mathcal{J}(X, \mathfrak{b}^t) \qquad \blacksquare$$

<u>Ex</u> (c.f. [TW])

$X = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy-z^5)$ A_4-sing.

$\mathfrak{a} = (x, y^4, y^3z, y^2z^2, yz^3, z^4)$

$\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$, $\mathcal{J}(\mathfrak{a}^{\frac{1}{2}}) = (x, y^2, yz, z^2)$

$$\rightsquigarrow \underset{\mathfrak{a}^{\frac{1}{2}}\mathfrak{a}^{\frac{1}{2}}}{\underset{\shortparallel}{\mathcal{J}(\mathfrak{a})}} \not\subset \mathcal{J}(\mathfrak{a}^{\frac{1}{2}})^2$$

$x \in \mathcal{J}(\mathfrak{a})$, $x \notin \mathcal{J}(\mathfrak{a}^{\frac{1}{2}})^2$

Sing. case (T- [Ta 2])

(2)′ (Subadditivity)

X : $\mathbb{Q}$-Goren. normal var./$\mathbb{C}$

$\Rightarrow J \cdot \mathcal{J}(\mathfrak{a}^s \mathfrak{b}^t) \subset \mathcal{J}(\mathfrak{a}^s)\mathcal{J}(\mathfrak{b}^t)$, $\forall \mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$, $\forall s, t > 0$

$$\left(\begin{array}{l} J \subseteq \mathcal{O}_X : \text{Jacobian ideal sheaf} \\ \text{We cannot replace } J \text{ by } \sqrt{J} \end{array}\right)$$

(3)′ (Summation)

X : $\mathbb{Q}$-Goren. normal var.

$$\Rightarrow \mathcal{J}(X, (\mathfrak{a}+\mathfrak{b})^t) = \sum_{\substack{\lambda+\mu=t \\ \lambda,\mu \geq 0}} \mathcal{J}(X, \mathfrak{a}^\lambda \mathfrak{b}^\mu)$$

In particular

$$J \cdot \mathcal{J}(X, (\mathfrak{a}+\mathfrak{b})^t) \subset \sum_{\substack{\lambda+\mu=t \\ \lambda,\mu \geq 0}} \mathcal{J}(X, \mathfrak{a}^\lambda)\mathcal{J}(X, \mathfrak{b}^\mu)$$

(J : Jacobian)

Sketch of proof (2)′

Assume $X = \operatorname{Spec} R$, R : complete local of char. 0

$\rightsquigarrow$ char. $p > 0$

<u>ETS</u> $J \cdot \tau(\mathfrak{a}^s \mathfrak{b}^t) \subset \tau(\mathfrak{a}^s)\tau(\mathfrak{b}^t)$

dual $\qquad \tau(\mathfrak{b}^t) := \operatorname{Ann} 0_E^{*\mathfrak{b}^t}$, $E := E_R(R/\mathfrak{m})$

$$(0_E^{*\mathfrak{a}^s\mathfrak{b}^t} : J)_E \supset (0_E^{*\mathfrak{b}^t} \underset{\substack{\vee \\ z}}{:} \tau(\mathfrak{a}^s))_E$$

$\rightsquigarrow \tau(\mathfrak{a}^s) z \in 0_E^{*\mathfrak{b}^t}$

i.e. $\exists\, c \in R^\circ$ s.t. $c\mathfrak{b}^{\lceil tq \rceil}\tau(\mathfrak{a}^s)^{[q]} z^q = 0 \in F^e(E)$,

$(R^\circ := R \setminus \bigcup_{P:\text{ minimal prime}} P)$ $\quad \forall q = p^e \gg 0$

<u>claim</u> $\exists\, d \in R^\circ$ s.t. $d\mathfrak{a}^{\lceil sq \rceil} J^{[q]} \subset \tau(\mathfrak{a}^s)^{[q]}$, $\forall q = p^e \gg 0$.

If we accept this claim

$\Rightarrow cd\,\mathfrak{a}^{\lceil sq \rceil}\mathfrak{b}^{\lceil tq \rceil} J^{[q]} z^q = 0 \in F^e(E) \quad \forall q = p^e \gg 0$

$\Rightarrow Jz \subset 0_E^{*\mathfrak{a}^s\mathfrak{b}^t}$ ▨

<u>Ex.</u>

$X, \mathfrak{a}$ as above Ex.

$J = (x, y, z^4)$, $\sqrt{J} = (x, y, z)$

$(x, y, z^4)\, \mathcal{J}(\mathfrak{a}) \subset \mathcal{J}(\mathfrak{a}^{\frac{1}{2}})^{\frac{1}{2}}$

$$\begin{array}{ccc}(x,y,z)\,\mathcal{J}(\mathfrak{a}) & \not\subset & \mathcal{J}(\mathfrak{a}^{\frac{1}{2}})^{\frac{1}{2}} \\ \cup & & \not\ni \\ xz & & \end{array}$$

<u>Asymtopic multiplier ideals</u> (See [ELS] or [La2] for details)

X : $\mathbb{Q}$-Goren. normal var.

$\mathfrak{a}_\bullet$: graded family of ideals on X

i.e. $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \mathbb{N}}$

$\mathfrak{a}_0 = \mathcal{O}_X$, $\mathfrak{a}_1 \neq 0$, $\mathfrak{a}_m \subset \mathcal{O}_X$

$\mathfrak{a}_k \cdot \mathfrak{a}_l \subset \mathfrak{a}_{k+l}$, $\forall k, l \in \mathbb{N}$

$t > 0$ fix.

$\mathcal{J}(\mathfrak{a}_{kl}^{\frac{t}{kl}}) \supset \mathcal{J}(((\mathfrak{a}_k)^l)^{\frac{t}{kl}}) = \mathcal{J}(\mathfrak{a}_k^{\frac{t}{k}})$

$\rightsquigarrow \{\mathcal{J}(\mathfrak{a}_m^{\frac{t}{m}})\}_{m \in \mathbb{N}}$ has a unique max. element w.r.t. inclusion.

Denote it by $\mathcal{J}(\mathfrak{a}_\cdot^t)$

<u>Ex.</u>

(1) $\mathfrak{a}_m := \mathfrak{a}^m$, $\mathfrak{a} \subseteq \mathcal{O}_X$

$\Rightarrow \mathcal{J}(\mathfrak{a}_\cdot^t) = \mathcal{J}(\mathfrak{a}^t)$

(2) L: linear system,

$\mathfrak{a}_m := \mathfrak{b}(|mL|)$: base ideal of $|mL|$

$\Rightarrow \mathcal{J}(\mathfrak{a}_\cdot^t) =: \mathcal{J}(t \cdot \|L\|)$

(3) $X = \operatorname{Spec} R$, $\mathfrak{p} \subset R$: prime ideal

$\mathfrak{a}_m := \mathfrak{p}^{(m)} := \mathfrak{p}^m R_\mathfrak{p} \cap R$

$\Rightarrow \mathcal{J}(\mathfrak{a}_\cdot^t) =: \mathcal{J}(t \cdot \mathfrak{p}^{(\cdot)})$

<u>Basic properties</u>

(1). $t_1 > t_2 \Rightarrow \mathcal{J}(\mathfrak{a}_\cdot^{t_1}) \subset \mathcal{J}(\mathfrak{a}_\cdot^{t_2})$

(2). $\mathfrak{a}_\cdot$, $\mathfrak{b}_\cdot$,

If $0 \neq {}^\exists c \in \mathcal{O}_X$ s.t. $c\,\mathfrak{a}_m \subset \mathfrak{b}_m$, ${}^\forall m \gg 0$

$\Rightarrow \mathcal{J}(\mathfrak{a}_\cdot^t) \subset \mathcal{J}(\mathfrak{b}_\cdot^t)$

(3). $\mathfrak{a}_k \mathcal{J}(\mathfrak{a}_\bullet^{\ell}) \subset \mathcal{J}(\mathfrak{a}_\bullet^{k+\ell})$, $k, \ell \in \mathbb{N}$

In particular, if X has only lt sing.

$(\Leftrightarrow \mathcal{J}(\mathcal{O}_X) = \mathcal{O}_X)$

$\Rightarrow \mathfrak{a}_k \subset \mathcal{J}(\mathfrak{a}_\bullet^{k})$, $\forall k \in \mathbb{N}$

(4). (Restriction) S: normal $\mathbb{Q}$-Goren Cartier divisor on X

$\mathcal{J}(S, (\mathfrak{a}_\bullet \mathcal{O}_S)^t) \subset \mathcal{J}(X, \mathfrak{a}_\bullet^t)\mathcal{O}_S \quad (\forall t > 0)$

(5). (Subadditivity)

$J \subset \mathcal{O}_X$: Jacobian ideal

$J^{\ell-1}\mathcal{J}(\mathfrak{a}_\bullet^{k\ell}) \subset \mathcal{J}(\mathfrak{a}_\bullet^{k})^{\ell}$, $k, \ell \in \mathbb{N}$

(6). (Summation)

$(\mathfrak{a}_\bullet + \mathfrak{b}_\bullet)_m := \sum_{k+\ell=m} \mathfrak{a}_k \cdot \mathfrak{b}_\ell$

$\mathcal{J}((\mathfrak{a}_\bullet + \mathfrak{b}_\bullet)^t) \subset \sum_{\lambda+\mu=t} \mathcal{J}(\mathfrak{a}_\bullet^{\lambda} \mathfrak{b}_\bullet^{\mu})$

(7). $0 \neq \exists e \subset \mathcal{O}_X$ s.t. $e\,\mathfrak{a}_m \subset \mathcal{J}(\mathfrak{b}_\bullet^{m})$, $\forall m \gg 0$

$\Rightarrow J \cdot \mathfrak{a}_m \subset \mathcal{J}(\mathfrak{b}_\bullet^{m})$, $\forall m \in \mathbb{N}$

($J \subset \mathcal{O}_X$: Jacobian ideal)

<u>short proof</u>

(2). $\mathcal{J}(\mathfrak{a}_\bullet^t) = \mathcal{J}(\mathfrak{a}_m^{\frac{t}{m}}) = \mathcal{J}(e^{\frac{t}{m}} \mathfrak{a}_m^{\frac{t}{m}})$

$\subset \mathcal{J}(\mathfrak{b}_m^{\frac{t}{m}}) \subset \mathcal{J}(\mathfrak{b}_\bullet^t)$ $m \gg 0$

(7). $eJ^{\ell}\mathfrak{a}_m^{\ell} \subset eJ^{\ell}\mathfrak{a}_{m\ell}$ use subadditivity

$\subset J^{\ell}\mathcal{J}(\mathfrak{b}_\bullet^{m\ell}) \subset \mathcal{J}(\mathfrak{b}_\bullet^{m})^{\ell}$, $\ell \gg 0$

$\Rightarrow J \cdot \mathfrak{a}_m \subset \overline{\mathcal{J}(\mathfrak{b}_\bullet^{m})} = \mathcal{J}(\mathfrak{b}_\bullet^{m})$ ▨

Symbolic powers

(Swanson [Sw]) R : normal domain

$R \supset \mathfrak{p}$: prime

$\exists\, k = k(\mathfrak{p}) \in \mathbb{N}$ s.t. $\mathfrak{p}^{(km)} \subset \mathfrak{p}^m$, $\forall m \in \mathbb{N}$

Q. What is k ?

Thm (Ein-Lazarsfeld-Smith [ELS])

R : regular affine domain $/\mathbb{C}$ (f.g. alg. over $\mathbb{C}$)

$\mathfrak{p} \subset R$: prime of ht. h

$\Rightarrow \mathfrak{p}^{(hm)} \subset \mathfrak{p}^m$, $\forall m \in \mathbb{N}$ (i.e. $k(\mathfrak{p}) = \mathrm{ht}.\,\mathfrak{p}$)

Thm (Hochster-Huneke [HH1])

R : regular ring of equal char., $\mathfrak{p} \subset R$: prime of ht h.

$\mathfrak{p}^{(hm)} \subset \mathfrak{p}^m$, $\forall m \in \mathbb{N}$

Singular case

Thm (T- [Ta2])

R : affine domain $/k$, k : perfect field of char. $p > 0$

$\mathfrak{p} \subset R$: prime of ht. h, $J \subset R$: Jacobian ideal

$\Rightarrow J^{m-1}\tau(R)\mathfrak{p}^{(hm)} \subset \mathfrak{p}^m$, $\forall m \in \mathbb{N}$

proof of ELS

$\mathfrak{p}^{(\cdot)} := \{\mathfrak{p}^{(m)}\}_{m\in\mathbb{N}}$

$\mathfrak{p}^{(hm)} \subset \mathcal{J}(hm\cdot\mathfrak{p}^{(\cdot)}) \subset \mathcal{J}(h\cdot\mathfrak{p}^{(\cdot)})^m$

ETS $\mathcal{J}(h\cdot\mathfrak{p}^{(\cdot)}) \subset \mathfrak{p}$

$$\begin{aligned}\mathcal{J}(h\cdot\mathfrak{p}^{(\cdot)})_{\mathfrak{p}} &= \mathcal{J}(h\cdot\mathfrak{p}^{(\cdot)}R_{\mathfrak{p}}) \\ &= \mathcal{J}((\mathfrak{p}R_{\mathfrak{p}})^h) \subset \mathfrak{p}R_{\mathfrak{p}} \\ &\rightsquigarrow \mathcal{J}(h\cdot\mathfrak{p}^{(\cdot)}) \subset \mathfrak{p} \quad \blacksquare\end{aligned}$$

proof of sing. case

$\mathfrak{p}^{(\cdot)} := \{\mathfrak{p}^{(m)}\}_{m\in\mathbb{N}}$

$$\begin{aligned}J^{m-1}\mathcal{J}(R)\,\mathfrak{p}^{(hm)} &\subset \mathcal{J}(hm\cdot\mathfrak{p}^{(\cdot)})J^{m-1} \\ &\subset \underbrace{\mathcal{J}(h\cdot\mathfrak{p}^{(\cdot)})}_{\hat{\mathfrak{p}}}{}^m \quad \blacksquare\end{aligned}$$

If $\mathfrak{p}$ is special

Can we get a better bound ?

Thm (Hochster-Huneke [HH2], T-Yoshida [TY])

R : regular ring of equal char.

$\mathfrak{p} \subset R$: prime of ht $h \geq 2$

If $R/\mathfrak{p}$ is F-pure or of dense F-pure type (See [Hu])

$\Rightarrow \mathfrak{p}^{(hm-1)} \subset \mathfrak{p}^m \quad \forall m \in \mathbb{N}$

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