A dual form of the sharp Nash inequality and its weighted generalization

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**Some background**

Tosio Kato was one of the most important founders of modern mathematical physics. Not only did he make great contributions himself, but he also inspired others in the field.

The subject of this talk is an example of how Kato motivated others by asking good questions. The story starts with a letter from Kato to Eric Carlen and Michael Loss, in which he asks whether it is possible to compute the sharp constant in Nash’s inequality.

Eric and Michael solved this problem in 1993 and showed, surprisingly, that every optimal function has compact support.

The unanswered question hanging in the air was *What is the dual of Nash’s inequality?* Eric and I have a solution of this problem and the result is even more surprising – as one might expect.

Let us review the situation:
**From Sobolev to Nash**

The Sobolev inequality in $\mathbb{R}^n$, ($n \geq 3$ only), is

$$\|\nabla f\|_2 \geq S_n \|f\|_{2n/(n-2)}.$$ 

This is an inequality between two (convex) integrals and has an unambiguous dual, which is the Hardy-Littlewood inequality (HLS), and which is valid for all $n$ and $0 < \lambda < n$,

$$\int \int g(x)|x - y|^{-\lambda} g(y)dx dy \leq C_n(\lambda)\|g\|_{2n/(2n-\lambda)}^2.$$ 

The special case $\lambda = n - 2$ is the dual of Sobolev, but we see that HLS covers many more cases. We learn here that it is sometimes useful to study duals because they can lead us to new mathematics. When $n = 3$, Sobolev tells us about kinetic energy, while its dual, HLS, is the story of the Coulomb potential and 'potential theory', which has quite a different flavor.

Nash’s inequality involves three integrals and is valid for all $n$.

$$C_n \|\nabla f\|_{2}^{n/(n+2)} \|f\|_{1}^{2/(n+2)} \geq \|f\|_2.$$ 

Carlen and Loss found $C_n$ and the optimizers, which always have compact support.
For $n \geq 3$, Nash's inequality can be derived from Sobolev’s inequality (but with a bad constant) by using Hölder’s inequality. Thus, Nash is weaker than Sobolev – but it is extremely useful for problems in which the $L^1$-norm is either conserved or monotone decreasing.

Kato was interested in the two-dimensional Navier Stokes equation in the vorticity formulation, which is just such a problem. Nash had applications to fluid dynamics in mind when he wrote his famous 1958 parabolic regularity paper in which his inequality first appeared. Many applications have been found in probability theory.

You have been very patient up to now, and it is time to display the dual of the Nash inequality.
The dual Nash inequality

Here it is!

\[ L_n \|g\|_2^{\frac{2n+4}{n+4}} \geq \inf_{h} \left\{ \frac{1}{2} \|(-\Delta)^{-1/2}(g - h)\|^2_2 + \|h\|_\infty \right\} . \]

What this says is, given a function \( g \in L^2(\mathbb{R}^n) \), try to minimize its Coulomb energy by subtracting another function \( h \). The price to be paid, however, is the \( L^\infty \)-norm of \( h \).

There are three topics to be discussed:

1. Where does this funny inequality come from and what is its connection to Nash?
2. Does there exist a minimizing \( h \) for this new problem and what does it look like?
3. Does there exist an optimizing \( g \) (and \( h \)) that gives the smallest value of \( L_n \)?

How is this \( g \) related to the optimizer for Nash?
Suppose we have two convex functionals, $A(f)$, $B(f)$ and $A(f) - B(f) \geq 0$, $\forall f$, as in the Sobolev inequality. We can then take the Legendre transforms:

$$A^*(g) := \sup_f \{ \int fg - A(f) \}, \quad B^*(g) := \sup_f \{ \int fg - B(f) \}.$$ 

Let $F$ be an (approximate) maximizer for $B^*(g)$, whence we have the dual inequality:

$$B^*(g) - A^*(g) \geq \int Fg - B(F) - \int Fg + A(F) \geq 0.$$ 

Thus, the dual of $A \geq B$ is $B^* \geq A^*$. Since $A, B$ are convex, the 'dual of the dual' is the original inequality $A \geq B$.)

In the case of Nash, there are 3 functionals and the right side is not convex. Help! We must combine 2 of them into one convex functional, and this will lead us to the strange construction called infimal convolution.
SECOND LAW OF THERMODYNAMICS AND INFIMAL CONVOLUTION

Let systems A and B have energy dependent entropy functions $S_A(E)$ and $S_B(E)$. These functions are concave, of course. The systems are brought into equilibrium with total energy $U$. According to the second law they distribute the energy so that the total entropy is maximized. Thus

$$S_{AB}(U) = \sup_E \{S_A(U - E) + S_B(E)\}.$$  

The amazing thing is this: Despite the supremum, the resulting $S_{AB}$ is a concave function — as required by the second law. (For convex functions everything is reversed.)

The general theorem, (1 line proof!) of which this 'convolution' is a special case, is this:  

If $F(X,Y)$ is a jointly concave function of $X,Y$ then $\sup_Y F(X,Y)$ is concave!

Let us apply this to the product $\|\nabla f\|^n/(n+2) \|f\|^{2/(n+2)}$ of functions of $f$, that appear on the 'large side' of Nash. This product is NOT a convex functional. To deal with this problem we shall first reformulate Nash.
Reformulated Nash inequality

Recall Nash:

\[ C_n \| \nabla f \|_2^{n/(n+2)} \| f \|_1^{2/(n+2)} \geq \| f \|_2. \]

By using the \( f \)-scaling properties of the various norms, we can rewrite this inequality as

\[ C_n^{(2n+4)/n} \| \nabla f \|_2^2 + \Phi(f) \geq \| f \|_2^{(2n+4)/n}, \]

where \( \Phi(f) = \begin{cases} 
0 & \| f \|_1 \leq 1 \\
\infty & \| f \|_1 > 1 
\end{cases} \), and whose Legendre transform is \( \| g \|_\infty \).

The Legendre transform of \( \| \nabla f \|_2 \) is our beloved Coulomb potential \( \| (-\Delta)^{-1/2} g \|_2^2 \).

The fundamental theorem of convex analysis is: the Legendre transform of the sum of two convex functions is the infimal convolution of the two Legendre transforms.

Conclusion: By taking the infimal convolution of these two convex functions, and scaling \( g \), we get a dual of the Nash inequality (in which both sides are convex in \( g \)):

\[ L_n \| g \|_2^{2n+4} \geq \inf_h \left\{ \frac{1}{2} \| (-\Delta)^{-1/2} (g - h) \|_2^2 + \| h \|_\infty \right\}. \]

Unfortunately, because of the \( \inf_h \), this is useless unless we can find \( h \).
Facts about $h$

This is the fun part! We cannot compute $h$ (except in one case), but we can say, more or less, what $h$ looks like.

As a preliminary step we can try to minimize $\|(-\Delta)^{-1/2}(g - h)\|_2^2$ under the condition that $\|h\|_\infty \leq c$. Call this $K(c)$ and, as a second, easy step, minimize $K(c) + c$. So let us discuss only the first step, with $c$ fixed and $|h(x)| \leq c$, $\forall x$.

It is not hard to prove (everyone here can surely do it) that a unique minimizing $h$ exists for $K(c)$. Let us then move on to the Euler-Lagrange equation for $h$, which is

$$
\psi(x) \begin{cases} 
\geq 0, & \text{if } h(x) = c \\
= 0, & \text{if } -c < h(x) < c \\
\leq 0, & \text{if } h(x) = -c
\end{cases} \quad \text{with } \psi = (-\Delta)^{-1}(g - h).
$$

An important fact about Laplacians (in the sense of distributions) is that $\Delta f = 0$ almost everywhere on the set $\{x : f(x) = 0\}$. Since $\Delta \psi = h - g$, we conclude that a.e. either

$$
\begin{array}{l}
h(x) = \pm c \quad \text{or} \quad h(x) = g(x) \quad \text{and} \quad |g(x)| < c
\end{array}
$$
We have proved that

\[ h(x) = \pm c \quad \text{or} \quad h(x) = g(x) \quad \text{and} \quad |g(x)| < c \]

(This kind of argument goes back to the 2016 ‘no-flat-spots for strictly subharmonic functions’ theorem of Frank & L.)

In case \( g \geq 0 \) one can also show that \( h \geq 0 \).

One can also easily prove that \( g(x) \geq c \Rightarrow h(x) = c \) and \( g(x) \leq -c \Rightarrow h(x) = -c \).

Another thing that one can easily prove is that \( \int h = \int g \) for any \( c > 0 \). (Otherwise the Coulomb energy would be infinite.)

Unfortunately, we cannot find a formula for \( h \) except in one special, but important case: The case in which \( g \) is a symmetric decreasing, non-negative radial function. Trivial proof!

\[
h(x) = \begin{cases} 
  c & \text{if } |x| \leq R, \\
  g(x) & \text{if } |x| > R.
\end{cases}
\]

and volume of \( \{ x : |x| < R \} = \frac{1}{c} \int g \).
The sharp constant

The sharp constant $C_n$ in Nash and $L_n$ in dual Nash (dN) are trivially related, just as are the sharp constants for Sobolev and HLS. Assume you have not read the Carlen-Loss paper for $C_n$, and let us compute $L_n$ directly. This will gives us an alternative proof of $C_n$.

Let $G$ be the maximizing $g$ in dN. By Faber-Krahn (i.e., rearrangement inequality for the Laplacian) the optimizers for N are symmetric decreasing. By the 1:1 correspondence between optimizers for N and dN, we see that $G$ also wants to be symmetric decreasing. In this case, we know the optimum $H$, as we just saw at the end of the previous slide.

Let us compute the Euler-Lagrange equation for $G$. $L_n G = (-\Delta)^{-1}(G - H)$

(Note: The variation w.r.t. $H = H_G$ vanishes since $H$ is a minimizer for $G$).

With $\phi = G - H$ we have $L_n (-\Delta)\phi = \phi$ in a ball of radius $R$, and $\phi$ satisfies Dirichlet, and also Neumann boundary conditions on the ball. This eigenvalue problem is exactly what Carlen-Loss found for Nash, and which they solved explicitly.
THE WEIGHTED VERSION

To conclude this story, let me briefly explain the word 'weighted' in the title.

The **sharp weighted Nash inequality** for \( p > 0 \) generalizes Carlen/Loss/Nash:

\[
\| f \|_2^{2 + n/(4+2p)} \leq C_{n,p} \| \nabla f \|_2^2 \cdot \| |x|^p f \|_1^{n/(4+2p)}.
\]

Legendre transforming, as before, the equivalent **dual weighted Nash inequality** is:

\[
L_{n,p} \| g \|_2^{2n+4/n+4} \geq \inf_h \left\{ \frac{1}{2} \| (-\Delta)^{-1/2} (g - h) \|_2^2 + \| |x|^{-p} h \|_{\infty} \right\}.
\]

In contrast to the unweighted case, neither sharp constant was known. When \( p \) is an even integer, however, the method we just described can be applied and yields a **new result**: The sharp values of \( C_{n,p} \) and \( L_{n,p} \).

Weights different from \( |x|^p \) are possible, but only in this case can we easily find a formula for the sharp constants.
THANK YOU FOR LISTENING!