Progress in Syntomic Cohomology

Goal Give overview of new theory of syntomic cohomology and its rel^ns to

- p-adic Hodge thy
- K-theory
- prismatic cohomology

Based on:

- Bhatt-M-Scholze 2 "THH & Integral p-adic Hodge thy"
  (Syn. cohom defined as graded pieces of a motivic filtration on topological cyclic homology)
- Bhatt-Scholze "Prisms & Prismatic cohomology"

also: Antieau-Mathew-M-Nikolaus
Kelly-M

Lüders-M

§1 Classical Syntomic Cohomology

(Fontaine, Messing, Kato)

Rmk: Not needed for new approach, we will be quick | imprecise.

Def^n (Mazur) A morphism of schemes $Y \to Z$ is syntomic if it is
- flat

and
- a local complete intersection

Big syntomic site := all schemes (maybe over fixed base) with covers := syntomic surjections

- Mod p^n structure sheaf $\mathcal{O}_n$:= sheafification of $\operatorname{Spec} A \mapsto A/p^n A$
- Mod p^n crystalline structure sheaf $\mathcal{O}_n^{crys}$:=

sheafification of

$$\operatorname{Spec} A \longmapsto \text{absolute crystalline cohomology } \underbrace{H^0_{crys}\left(A/pA \,/\, \mathbb{Z}/p^n\mathbb{Z}\right)}$$

$\cong$ pd envelope of $W_n(A/pA)$ along ideal $\left\{(a_0, \dots, a_{n-1}) : a_i \in A/pA,\ a_0^p = 0\right\}$

$$\longrightarrow A/p^n A$$

$$(a_0, \dots, a_{n-1}) \longmapsto \tilde{a}_0^{p^n} + p\tilde{a}_1^{p^{n-1}} + \dots + p^{n-1}\tilde{a}_{n-1}^{p}$$

- pd ideal sheaf

$$J_n := \ker\left(\mathcal{O}_n^{crys} \longrightarrow \mathcal{O}_n\right)$$

and its divided powers

$$J_n^{[j]} \quad (j \geq 0)$$

Summary : $\mathcal{O}_n^{crys}$ crys. structure sheaf

+ pd filtration

+ $\phi \circlearrowright \mathcal{O}_n^{crys}$

Def^n (Fontaine–Messing)

- $\mathcal{S}_n(j) := \ker\left(\phi - p^j : J_n^{[j]} \to \mathcal{O}_n^{crys}\right)$
- Weight j, rational syntomic cohom. of a nice scheme X is

$$H^*_{syn}(X, \mathcal{S}(j)_{\mathbb{Q}_p}) := \left(\varprojlim_n H^*_{syn}(X, \mathcal{S}_n(j))\right)\left[\tfrac{1}{p}\right]$$

Eg) proper smooth over $\mathcal{O}_K$ or $\mathcal{O}_{\bar{K}}$

ie) "$\phi = p^j$ eigenspace of absolute cryst. cohomology along pd filtration"

Rmk: Can do better if $j < p-1$. Then

$$\phi\left(J_n^{[j]}\right) \subseteq p^j \mathcal{O}_n^{crys},$$

and can replace $\mathcal{S}_n(j)$ by finer invariant $\ker\left(\phi/p^j - 1\right)$.

Thm (Fontaine, Kato, Messing) For X proper smooth / $\mathcal{O}_K :=$ ring of integers of p-adic field, then for $n \leq j < p-1$,

$$H^n_{syn}(X \times_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}, S(j)) \cong H^n_{et}(X_{\bar{K}}, \mathbb{Z}/p(j))$$

So:

$$\begin{matrix}\text{cryst} \\ \text{cohom.}\end{matrix} \xleftrightarrow[\text{def}^n]{\text{by}} \begin{matrix}\text{syntomic} \\ \text{cohom.}\end{matrix} \xleftrightarrow{\text{Thm}} \begin{matrix}p\text{-adic} \\ \text{etale} \\ \text{cohom.}\end{matrix}$$

$\rightsquigarrow$ p-adic Hodge theory

Weaknesses of classical Thy:

(1) Only works well if the weight j is $< p-1$

(2) Local calculations done with

$$\mathbb{F}_p[X_1^{1/p^r}, \ldots, X_d^{1/p^r}] / \text{reg seq.}$$

and then let $r \to \infty$. Also work mod p^n, then $\varprojlim_n$, then $\frac{1}{p}$.

all these $\lim\limits_{\leftarrow}$ and $\lim\limits_{\rightarrow}$ are problematic on sites.

Fix

(1) replace crystalline & pd constructions by prismatic analogues

(2) Quasisyntomic site

Plan : §2 Quasisyntomic site
§3 The prism of a qrsp ring
§4 Globalise and syntomic cohom
§5 applications

§2 Quasisyntomic site.

Motivated by reinterpretation of syntomic:

Prop^n (Avramov) A flat, finite type map $R \to S$ between Noetherian ring is syntomic (ie lci) iff the cotangent complex $\mathbb{L}_{S/R}$ has Tor amplitude $\subseteq [-1,0]$.

Generalization to p-adic, non finite type case:

Defn: A ring R is quasisyntomic if

- R is p-adic complete and has bounded p^∞-torsion
- $\mathbb{L}_{R/\mathbb{Z}_p}$ has p-complete Tor amplitude $\subseteq [-1,0]$

ie) $\mathbb{L}_{R/\mathbb{Z}_p} \otimes^{\mathbb{L}}_R M \in D^{[-1,0]}$ for all R/pR-modules M.

Big quasisyntomic := quasisyntomic ring
site QSyn

covers are $f: R \to S$ s.t

- f is p-completely faithfully flat
  ie) $S \otimes^{\mathbb{L}}_R R/pR$ is a faithfully flat R/pR-module
- and $\mathbb{L}_{S/R}$ has p-complete Tor amplitude $\subseteq [-1,0]$

Rmk: We restrict to affines for simplicity, but no problem extending to p-adic formal schemes.

Examples

- QSyn contains all p-adically complete, regular Noetherian rings R. Also $R/$ reg seq.
  $\Rightarrow$ enough to study smooth, semistable, ncd, ...
- Most important covers in QSyn are (p-adic completions) of
  - étale covers
  - $R \longrightarrow R[X]/X^p - r$
  - filtered colimits of these
    eg) add all p-power roots of all elements

Unlike classical context, have nice basis for our site :

Def^n $S \in \mathrm{QSyn}$ is *quasiregular semi-perfectoid* if there exists a perfectoid ring R and a surjection $R \twoheadrightarrow S$.

- If R is an $\mathbb{F}_p$-alg, this means that R is perfect
- If p is a non zero divisor in R, this means
  - R p-adic complete
  - R/pR is semiperfect
  - $\exists\, u \in R^\times$ s.t. there exist $(up)^{1/p}, (up)^{1/p^2}, \ldots \in R$
  - $f \in R[\frac{1}{p}]$ s.t. $f^p \in R$ then $f \in R$.

Example: Perfectoid ring / regular seq $\in \mathrm{QRSP}$

Lemma $\mathrm{QRSP} \subseteq \mathrm{QSyn}$ forms a basis for the site

Proof

$$\widehat{\mathbb{Z}_p[X_i : i \in I]} \xrightarrow{\text{cover}} \underbrace{\widehat{\mathbb{Z}_p[p^{1/p^\infty}][X_i^{1/p^\infty} : i \in I]}}_{\text{perfectoid}}$$

in QSyn

$$\mathrm{QSyn} \ni R \xrightarrow[\text{cover in QSyn}]{} S \in \mathrm{QRSP}$$

$\square$

Corol: To define sheaves on QSyn, it is enough to define them on QRSP.

Summary of Lecture 1:

QSyn^{op} = p-adic version of classical syntomic site, but with no finiteness conditions (affine only for simplicity).

$\Big\|$ basis

QRSP^{op} = quotients of perfectoid rings

Today: Calculations here.

§3: Local calculations: QRSP rings

Goal : for each $S \in \mathrm{QRSP}$, we will define

- a prism (Δ_S, I)
  $\circlearrowright \phi$
- Nygaard filtration $N^{\geq j}\Delta_S$, $j \geq 0$
- Breuil-Kisin twists $\Delta_S\{j\}$, $j \geq 0$
- divided Frobenius map $\phi_j : N^{\geq j}\Delta_S\{j\} \to \Delta_S\{j\}$

These refine

- $H^0_{crys}(S/pS \,|\, \mathbb{Z}_p)$, (p)
- its pd filtration
- trivial twists
- $\phi/p^j : \mathrm{Fil}^j_{pd} H^0_{crys} \to H^0_{crys}$

§3.1 Prisms

Quick review of some of Bhatt-Scholze's thy of prisms

Defn A δ-ring is a ring A together with a map $\delta : A \to A$ "satisfying the "obvious" relations so that

$$\phi(a) := a^p + p\delta(a)$$

is a ring endomorphism

ie) if p is a non-zero divisor then

$$\left\{ \begin{matrix} \delta \text{ maps} \\ \text{on } A \end{matrix} \right\} \xleftrightarrow{1\text{-to-}1} \left\{ \begin{matrix} \text{lifts to } A \text{ of} \\ \text{the abs. Frob. on } A/pA \end{matrix} \right\}$$

Defn A prism is a pair (A, I) where A is a δ-ring and $I \subseteq A$ is an ideal s.t

- I is locally generated by single non zero divisor ($V(I)$ is Cartier divisor)
- $p \in \langle I, \phi(I) \rangle$
- A is (derived) (p, I)-adic complete.

ie) "A is 1-parameter deformation of A/I, equipped with Frob lift"

Base Examples

(1) $\left(\mathbb{Z}_p[[u]], I = \langle u-p \rangle \right)$

$\phi : u \mapsto u^p$

So $\mathbb{Z}_p[[u]]/I = \mathbb{Z}_p$.

(not so important for these talks)

(2) S perfect $\mathbb{F}_p$-algebra

$\rightsquigarrow \left(W(S), I = \langle p \rangle \right)$

ϕ usual Witt vector Frob

So $W(S)/I = S$.

(3) R perfectoid ring. Its tilt is

$$R^\flat := \left(R/pR \right)^{perf} = \varprojlim_{\phi} R/pR$$

perfect $\mathbb{F}_p$-alg

associated prism:

$$\left(\underset{\substack{\| \\ A_{inf}(R)}}{W(R^\flat)}, I = \ker\left(W(R^\flat) \xrightarrow[\text{Fontaine}]{} R \right) \right)$$

So $A_{inf}(R)/I = R$

Example: If R is p-torsion free and contains a compatible seq of p-power roots of unity, then I has a preferred generator

$$d := 1 + [\varepsilon] + \dots + [\varepsilon]^{p-1} \in A_{inf}(R)$$

where:

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots) \in R^\flat$$

also put $\mu = [\varepsilon] - 1 \in A_{inf}(R)$, so that

$$\phi(\mu) = d\mu$$

(useful later) for ϕ eigenspaces.

"Universal" Examples

(4) (A, I) prism, $t \in A$ sufficiently regular (t mod I is non zero divisor on A/I).

For simplicity assume I principle $= \langle d \rangle$.

Prop Then $\exists$ initial prism "envelope"

$$\left(A\left\{\frac{t}{d}\right\}, J \right)$$

over (A,I) s.t. $t \in J$.

Idea Formally add $\frac{t}{d}, \delta\left(\frac{t}{d}\right), \delta^2\left(\frac{t}{d}\right), \dots$ to A + obvious rel^{n}s $\square$

(5) Continuing (4), assume $I = (p)$. Then

$$A\left\{\frac{t}{p}\right\} \otimes_{A,\varphi} A \cong A\left\{\frac{t^p}{p}\right\}$$

$\cong$ (p-adic compl. of) pd envelope of $A \to A/tA$

direct verification

Moral Prismatic envelopes are mixed char generalisation & refinement of pd envelopes.

§3.2 The prism of a QRSP ring

Generalisation of prismatic envelope:

Thm (Bhatt-Scholze): Let $S \in$ QRSP. Then the category of prisms (A,I) equipped

the category of prisms (A, I) equipped
with a map $S \longrightarrow A/I$ has an
initial object

$$(\Delta_S, I)$$

"prismatic period ring"

Proof

We treat special case $S = R/tR$ where
R is perfectoid and $t \in R$ is non-zero-divisor.

$\rightsquigarrow$ associated prism (from Ex 3)

$$(A_{\inf}(R), \langle d \rangle)$$

Lift $t \in R$ to some $\tilde{t} \in A_{\inf}(R)$.

$\rightsquigarrow$ Envelope $A_{\inf}(R)\left\{\frac{\tilde{t}}{d}\right\}$

Claim: This is the desired initial object.

proof: For any prism (A, I), have

$$\text{maps } (A_{\inf}(R), \langle d \rangle) \to (A, I) \xleftrightarrow{1-1} \text{maps } R \to A/I$$

(use deformation thy: $\hat{\mathbb{L}}_{A_{\inf}(R)/\mathbb{Z}_p} \simeq 0$)

Therefore:

maps $\left(A_{\inf}(R)\left\{\frac{\tilde{t}}{d}\right\}, \langle d\rangle\right) \to (A, I)$ $\longleftrightarrow$ maps $R \to A/I$ s.t. $t \mapsto 0$

$\longleftrightarrow$ maps $S \to A/I$

$\square$

Properties / Structure of the Prism Δ_S

Fix $S \in QRSP$, and a perfectoid ring $R \to S$.

(1) **Thm** (B-S) Δ_S is $\simeq$ the derived prismatic cohomology of S over $(A_{\inf}(R), I)$.

$\Rightarrow$ Control Δ_S via smooth algs $/R$, for which we have main results of prismatic cohomology thy.

Example applications

(i) Hodge-Tate filtration: the S-algebra Δ_S/I admits an exhaustive, increasing filtr. by ideals s.t.

$$gr^i = \Gamma^i_\circ(J/J^2)$$

where $S = R/J$.

(ii) $S \mapsto \Delta_S$ is a sheaf on QRSP, with no higher cohomology
($\Rightarrow$ can use for Čech calculations)

(iii) $\exists$ comparison map $\Delta_S \to H^0_{crys}(S/\mathbb{Z}_p)$
(base change in prismatic cohom + prismatic-crystalline comparison)

I think:

$$\Delta_S \underset{A_{inf}(R)}{\otimes} A_{crys}(R) \xrightarrow{\cong} H^0_{crys}(S/\mathbb{Z}_p)$$

imagine $R = \mathcal{O}_{\mathbb{C}_p}$

(2) Nygaard Filtration by ideals

$$N^{\geq i}\Delta_S := \{ f \in \Delta_S : \phi(f) \in I^i \}$$

Fact: (use (i)) the map

$$\Delta_S \xrightarrow{\phi} \Delta_S \longrightarrow \Delta_S/I$$

$$\cup\!\!\!\!\mid$$

$$S$$

induces an isom

$$\Delta_S/_{N^{\geq 1}\Delta_S} \xrightarrow{\cong} S$$

cf)
$$\begin{tikzcd} \Delta_S \arrow[rr] \arrow[dr, "\substack{\text{mod}\\ N^{\geq 1}\Delta_S}"'] & & H^0_{crys}(S/_{p}S/\mathbb{Z}_p) \arrow[dl, "\text{mod } \mathrm{Fil}^1_{pd} H_{crys}"] \\ & S & \end{tikzcd}$$

But $N^{\geq j}\Delta_S$ is better than than Fil^j_{pd} if $j \geq p-1$.

Last time:

QRSP $\ni S \longmapsto$ prism (Δ_S, I) with ϕ acting on Δ_S

+ its Nygaard filtr.

... $\longrightarrow$ (+ Breuil-Kisin twists)

We will briefly explain their role below. (+ divided Frobenius)

§4 Syntomic cohomology

§4.1 The definitions

We define (new) weight j syntomic sheaf $\mathbb{S}(j)$ on QSyn^{op} as follows:

$$\mathrm{QRSP} \ni S \xmapsto[\text{of rings}]{\text{sheaf}} \Delta_S \supseteq I = \langle d \rangle$$

$$N^{\geq j}\Delta_S \subseteq \Delta_S$$

eigenspaces

$$\mathrm{QRSP} \ni S \xmapsto[\text{of } \mathbb{Z}_p\text{-modules}]{\text{sheaf}} \Delta_S^{\phi = d^j} = \left\{ f \in \Delta_S : \phi(f) = d^j f \right\} = \ker\left(\tfrac{\phi}{d^j} - 1 : N^{\geq j}\Delta_S \to \Delta_S\right)$$

$\bigcap$ basis

extend along basis

$$\mathrm{QSyn} \ni R \xmapsto[\text{sheaf of } \mathbb{Z}_p\text{-modules}]{} \mathbb{S}(j)(R)$$

(known to be p-torsion free)

Def^n: Weight j (new) syntomic cohomology of $R \in \mathrm{QSyn}$ is

$$\mathbb{Z}_p(j)(R) := R\Gamma_{\mathrm{QSyn}}(R, \mathcal{S}(j)) \in D^{\geq 0}(\mathbb{Z}_p)$$

(for $\mathbb{Z}/p^n\mathbb{Z}$-coeffs, just replace $\mathcal{S}(j)$ by $\mathcal{S}(j)/p^n$)

Rmks

(i) The def^n above has big gap: $S \mapsto \Delta_S^{\phi = d^j}$ is not even a functor, since d depends (non uniquely) on S.

Solution: $\exists$ rank 1, invertible Δ_S-modules $\Delta_S\{j\}$ $\forall j \in \mathbb{Z}$, called Breuil-Kisin twists, together with ϕ-semilinear map "divided Frob"

$$\underbrace{\phi_j : N^{\geq j}\Delta_S\{j\} \longrightarrow \Delta_S\{j\}}$$

Functorial

in S!

$$\overbrace{\frac{\phi}{d^{j}} : N^{\geq j}\Delta_S \longrightarrow \Delta_S}^{\text{//}S \text{ for } S \text{ fixed}}$$

Then replace non-functorial

$$S \longmapsto \ker\left(\frac{\phi}{d^{j}} - 1 : N^{\geq j}\Delta_S \to \Delta_S\right)$$

by functorial

$$S \longmapsto \ker\left(\phi_j - 1 : N^{\geq j}\Delta_S\{j\} \xrightarrow{(*)} \Delta_S\{j\}\right)$$

$$\left\ulcorner \Delta_S\{1\} = TP_2(S) \right\lrcorner$$

(ii) Rel^{n}s to other def^{n}s: Def^n in BMS2 was different: we instead set

$$\mathbb{Z}_p(j)(S) := \text{the complex } (*),$$

for $S \in$ QRSP, as a sheaf-valued complex on QRSP

$\rightsquigarrow$ sheaf-valued complex $\mathbb{Z}_p(j)(-)$ on QSyn

This agrees with previous def^n thanks to

Thm (Odd vanishing conj — Bhatt-Scholze)
Locally on QRSP, $\phi_j - 1$ is surj.

analogue of Fontaine-Messing's "tedious calculations"

Corol: Exact seq of sheaves on QRSP

$$\mathcal{S}(j) \to N^{\geq j}\Delta\{j\} \xrightarrow{\phi_j - 1} \Delta\{j\}$$

Extend to QSyn & take colim gives a distinguished triangle $\forall R \in$ QSyn

$$\mathbb{Z}_p(j)(R) \to N^{\geq j}\Delta_R\{j\} \xrightarrow{\phi_j - 1} \Delta_R\{j\}$$

twist of absolute prismatic cohom of R

Nygaard filtr. on ——— " ———

(iii) K-thy approach: the notation $\mathcal{S}(j)$ does not appear in the papers, as we know that for any $S \in$ QRSP,

$$\mathbb{S}(j)(S) \cong K_{2j}(S;\mathbb{Z}_p) \cong TC_{2j}(S)$$

p-adic K-group of S

topological cyclic homology of S.

$$\Rightarrow \mathbb{Z}_p(j)(R) = R\Gamma(R_{QSyn}, \text{sheafification on QSyn of } S \mapsto K_{2j}(S;\mathbb{Z}_p))$$

§4.2 Comparisons / Properties

Low weights

$R \in QSyn$

- $\mathbb{Z}_p(0)(R) = R\Gamma_{et}(R, \mathbb{Z}_p)$

  $\left(:= R\lim_{\overleftarrow{n}} R\Gamma_{et}(R, \mathbb{Z}/p^n)\right)$

Proof: $\mathbb{S}(0)(S) \cong K_0(S;\mathbb{Z}_p) \underset{\text{local}}{\cong} \mathbb{Z}_p$

- $\mathbb{Z}_p(1)(R) \simeq R\Gamma_{et}(R, \mathbb{G}_m)^{\wedge}[-1]$

ie) $\mathbb{Z}_p(1)(R)/p^n \simeq R\Gamma_{fppf}(R, \mu_{p^n}^{\otimes d})$

Proof: For $S \in \mathrm{QRSP}$ containing all p-power roots of unity, then

$$\mathcal{S}(1)(S) = \Delta_S^{\phi=d} \xleftarrow[q\text{-}\log]{\cong} T_p(S^\times)$$

for this isom, see Anschütz-Le Bras

Comparisons

- If R is smooth / field of char p, then

Thm: $\mathbb{Z}_p(j)(R) \simeq R\Gamma_{et}(R, W\Omega^j_{\log})[-j]$

usual p-adic étale motivic cohom of R
- Milne
- Geisser-Levine

- Thm (Bhatt-Clausen-Mathew in progress)

R p-adic completion of a smooth $\mathcal{O}_K$-alg, where K is a p-adic field. Then

$\mathbb{Z}_p(j)(R) \simeq$ Sato + Schneider construction, namely $\varprojlim_n$ of $R\Gamma_{et}(R, -)$ of the homotopy fibre of Kato's residue map

$$i^* R\psi_* \mu_{p^n}^{\otimes j} \longrightarrow W_n \Omega^{j-1}_{log}[-j]$$

where $\psi : \mathrm{Spec}\, R[\frac{1}{p}]_{et} \to \mathrm{Spec}\, R_{et}$
$\simeq \mathrm{Spec}\, R/\pi R_{et}$

Vague idea:
Both sides are graded pieces of "motivic" filtrations on $K^{et}(R; \mathbb{Z}_p)$. But only one such filtr. can exist!

- More generally, have étale comparison map

$$\mathbb{Z}_p(j)(R) \longrightarrow \varprojlim_n R\Gamma_{et}(R[\tfrac{1}{p}], \mu_{p^n}^{\otimes j})$$

Should be an isom in degrees $< j$ if R is regular, but this is wide open.

• Bounds and high degree $\geq j$

Thm (Antieau–Mathew–M–Nikolaus)

$\mathbb{Z}_p(j)(R)$ is supported in degrees $\leq j+1$, given in top degree by

$$H^{j+1}(\mathbb{Z}_p(j)(R))/p \cong \operatorname{coker}\left(\Omega^j_{R/pR} \xrightarrow{C^{-1}-1} \Omega^j_{R/pR}/d\Omega^{j-1}_{R/pR}\right)$$

same for R and R/I, if I henselian.

"Artin–Schreier"

appears in Kato's conjectures.

Thm (Lüders–M.)

For R local,

p-adic compl. of improved Milnor K-thy