

# On the canonical, fpqc and finite topologies: classical questions, new answers

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In the first lecture, we stated and explained the following theorems.  
 $R$  regular ring,  $S \supset R$  finite extension.

### Theorem 1

$R \rightarrow S$  splits (in  $Mod_R$ ).

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### Theorem 1

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### Theorem 2

There exists a ring map  $S \rightarrow T$  such that the composition  $R \rightarrow T$  is faithfully flat.

In the second lecture, we sketched a proof of Theorem 2 (which implies Theorem 1) based on perfectoid techniques. This simplified proof uses only two ingredients: the perfectoid flatness lemma and a result ensuring the non-degeneracy of the construction.

The aim of this third lecture is twofold:

- 1) *explain the geometric meaning of these statements,*
- 2) *disclose the role of regularity: for what kind of rings  $R$  such statements hold?*

[Talk is based on joint work with Luisa Fiorot (Ann. Scuola Norm. Sup. Pisa)].

# Base change and descent

$R \xrightarrow{\alpha} S$  homomorphism of commutative rings.

Base change functor:  $\alpha^* = - \otimes_R S : \text{Mod}_R \rightarrow \text{Mod}_S$ .

- $\alpha^*$  exact:  $\alpha$  flat
- $\alpha^*$  faithful exact:  $\alpha$  faithfully flat
- $\alpha^*$  faithful:  $\alpha$  pure

(faithfully flat = flat + pure).

Fact:  $\alpha$  pure  $\Leftrightarrow \alpha$  universally injective (i.e. remains injective after any base-change).

# Base change and descent

Criteria:

- If  $\alpha$  splits in  $\text{Mod}_R$  (i.e.  $R$  is a direct summand of the  $R$ -module  $S$ ), then  $\alpha$  is pure.

Proposition (Lazard, Fedder)

Converse is true if either  
 $S/R$  is finitely presented,  
or  $R$  is a complete Noetherian local ring.

- If there exists an  $S$ -module  $M$  which is faithfully flat over  $R$ , then  $\alpha$  is pure (Bourbaki).

Question (Ferrand)

Converse?

(solved in our paper...)

# Base change and descent

“Reverse base-change”: descent.

**Descent data:**  $S$ -module  $N$  + isomorphism

$$N \otimes_R S \xrightarrow{\phi} S \otimes_R N$$

satisfying the usual cocycle relations.

They form a category  $DD(\alpha)$  (objects:  $(N, \phi)$ ), and  $\alpha^*$  factors as:

$$Mod_R \xrightarrow{C_\alpha} DD(\alpha) \xrightarrow{\text{forget } \phi} Mod_S.$$

Grothendieck's descent:

$\alpha$  *faithfully flat*  $\Rightarrow C_\alpha$  *is an equivalence*.



# Base change and descent

Grothendieck's descent:

$\alpha$  *faithfully flat*  $\Rightarrow C_\alpha$  *is an equivalence*.

Olivier's descent (70'): **flatness is irrelevant!**

$\alpha$  *pure*  $\Leftrightarrow C_\alpha$  *is an equivalence*  $\Leftrightarrow C_\alpha$  *is fully faithful*.

(This remarkable result was long overlooked, partly because it was published only as an announcement, in a CRAS note.

In our paper, we reconstruct the proof in detail, starting from a key lemma stated in that note as a hint.)

No need to comment about the geometric meaning of flatness in Algebraic Geometry...

But what is the geometric meaning of purity?

This is best expressed in terms of Grothendieck topologies, as follows...

# Topological interpretation of purity

$\mathcal{C}$ : category with initial object and fibered products, e.g.  $Aff_k$  (affine schemes over a commutative ring  $k$ ).

- Grothendieck pretopology: class of "covering families" ( $Y_i \rightarrow X$ ) in  $\mathcal{C}$ , containing all isomorphisms, stable by base change and by composition.
- Grothendieck topology: moreover, any family refined by a covering family is a covering family.

Ex: • fpqc pretopology on  $Aff_k$ : covering maps  $Y \rightarrow X$  are faithfully flat maps.

- fpqc topology on  $Aff_k$ :  $Y \rightarrow X$  is a covering map iff there exists  $Z \rightarrow Y$  such that the composition  $Z \rightarrow X$  is faithfully flat.
- effective descent topology on  $Aff_k$ : covering families: those for which the corresponding  $C_\alpha$ 's are equivalences.

# Topological interpretation of purity

- Canonical topology on  $\mathcal{C}$ : finest Grothendieck topology s.t. all representable presheaves are sheaves (for all covering families  $(Y_i \rightarrow X)$ ,  $F(X) \rightarrow \prod F(Y_i) \rightrightarrows \prod F(Y_i \times_X Y_j)$  is exact).

Ex. • canonical topology on  $Aff_k$ :

Basic fact:

$\text{Spec } S \rightarrow \text{Spec } R$  is a canonical covering iff  $R \rightarrow S$  is pure.

By Olivier's result ('70): on  $Aff_k$ , **Canonical topology = Effective descent topology**.

(On the other hand, the fpqc topology is strictly coarser than the canonical topology (Raynaud-Gruson '71)).

# Topological interpretation of purity

Basic fact:

$\text{Spec } S \rightarrow \text{Spec } R$  is a canonical covering iff  $R \rightarrow S$  is pure.

Issue:  $\alpha$  pure  $\Rightarrow \forall T, T \xrightarrow{\alpha_T} S \otimes_R T \rightrightarrows S \otimes_R S \otimes_R T$  is exact.

Prove the analog in  $\text{Mod}_R$ :  $M \xrightarrow{\alpha_M} S \otimes_R M \rightrightarrows S \otimes_R S \otimes_R M$ , with double map  $(\eta_1 = \alpha_{S \otimes_R M}, \eta_2 = 1_S \otimes_R \alpha_M)$ . Let  $K \xrightarrow{\beta} S \otimes_R M$  be such that  $\eta_1 \beta = \eta_2 \beta$  and look at

$$\begin{array}{ccc}
 & & K \\
 & \swarrow & \searrow \beta \\
 M & \xrightarrow{\alpha_M} & S \otimes_R M \\
 \alpha_M \downarrow & \swarrow \beta & \downarrow 1_S \otimes_R \alpha_M \\
 S \otimes_R M & \xrightarrow{\alpha_{S \otimes_R M}} & S \otimes_R S \otimes_R M \\
 \pi \downarrow & & \downarrow 1_S \otimes \pi \\
 \text{Coker } \alpha_M & \xrightarrow{\alpha_{\text{Coker } \alpha_M}} & S \otimes_R \text{Coker } \alpha_M
 \end{array}$$

# Topological interpretation of purity

Remarks.

- this topological interpretation of purity allows to globalize to non-affine schemes. However, for a finite covering  $Y \rightarrow X$  (i.e. a finite surjective map),  $\mathcal{O}_X \rightarrow \mathcal{O}_Y$  need not split if  $X$  is not affine.
- For a long time, only subcanonical topologies were considered to be relevant to Algebraic/Arithmetic Geometry.

Voevodsky changed this viewpoint: some finer topologies are relevant because one can compute their points (local rings).

Another later motivation: (cohomological) descent properties.

- the  $v$ -topology ( $Y \rightarrow X$  is a covering map if valuation rings of  $X$  lift to valuation rings of  $Y$ ):  $v$ -descent plays a fundamental role in the geometric  $p$ -adic Langlands program (Fargues-Scholze).

- the arc-topology: defined similarly but for rank 1 valuations rings (Bhatt-Mathew): arc-descent plays a fundamental role in prismatic theory (Bhatt-Scholze).

Back to: finite coverings = finite surjective morphisms.

*Question:* behaviour of finite coverings w.r.t. the canonical or fpqc topology?

- Examples of non-canonical finite coverings  $Y \rightarrow X$  (with  $X$  affine Noetherian)?
  - The normalization of any non-normal excellent scheme is an example (e.g. two crossing lines:  $k[x, y]/(xy) \rightarrow k[x] \times k[y]$  is not pure, since its reduction mod  $(x - y)$  sends  $x \neq 0$  to 0).
  - Ex. with  $X$  normal:  $Y = \mathbb{A}_k^4 \rightarrow X = \mathbb{A}_k^4/(\mathbb{Z}/4)$ , where  $\text{char } k = 2$  and  $\mathbb{Z}/4$  acts cyclically on the variables.

# Finite coverings

The point is that here  $X$  is not Cohen-Macaulay (M.J. Bertin '67), whereas a finite canonical covering  $\text{Spec } S \rightarrow \text{Spec } R$  descends the Cohen-Macaulay property:

Indeed, in the following diagram, the vertical and the top horizontal arrows are injective since  $\alpha$  is pure and  $S$  is Cohen-Macaulay

$$\begin{array}{ccc} \frac{S}{(x_1, \dots, x_i)} & \xrightarrow{\cdot X_{i+1}} & \frac{S}{(x_1, \dots, x_i)} \\ \uparrow & & \uparrow \\ \frac{R}{(x_1, \dots, x_i)} & \xrightarrow{\cdot X_{i+1}} & \frac{R}{(x_1, \dots, x_i)} \end{array}$$



Remark. No such example in char. 0. If  $\text{char } k = 0$  and  $X$  is normal, any finite covering  $Y \rightarrow X$  is canonical (one may assume  $R \subset S$  is an extension of integral domains, then  $\text{tr}_{S/R}/[S : R]$  gives a splitting).

Going further, is there any example with  $X$  regular? **No:**

## Theorem 1

Any finite covering of a regular scheme is canonical (hence an effective descent morphism).

Due to the above interpretation of purity, this amounts to a geometric translation of the direct summand conjecture (Hochster '69):

*any finite extension  $S$  of a regular ring  $R$  splits in  $\text{Mod}_R$ ,*  
proved by him in the presence of a base field, and by me in '16 in general, using perfectoid methods.

- Examples of canonical finite coverings  $Y \rightarrow X$  which are not coverings for the fpqc topology?

- Frobenius-type examples:  $M$  finite  $R$ -module,  $b : \text{Sym}^2 M \rightarrow R$ , associative:  $\ell b(m, n) = b(\ell, m)n$ . Then  $\text{Ann}_R M$  kills  $b(m, m)$ , hence  $b(m, m) \in J := \text{Ann}_R(\text{Ann}_R M)$ .

$S := R \oplus M$ ,  $(r, m)(r', m') = (rr' + b(m, m'), rm' + r'm)$ .

## Proposition

If  $b(m, m) \notin J^2$ , then the finite canonical covering  $\text{Spec } S \rightarrow \text{Spec } R$  is not a covering for the fpqc topology.

e.g.  $R = \mathbb{Z}/4$ ,  $M = \mathbb{Z}/2$ ,  $J = (2)$ ,  $b(1, 1) = 2$ .

- Can one find a normal example? Yes: the quadric cone.

$k$ : field of char.  $\neq 2$ ,  $S := k[x, y]$ ,  $\mathbb{Z}/2$ -action  $(x, y) \mapsto (-x, -y)$ ,

$$S = k[x, y]^+ \oplus k[x, y]^-.$$

Thus  $R := k[x, y]^+ = k[x^2, xy, y^2] \rightarrow S$  is pure.

## Proposition

$\text{Spec } S \rightarrow \text{Spec } R$  is not a covering for the fpqc topology.

Moreover, no  $S$ -module is faithfully flat over  $R$ .

(Negative answer to Ferrand's question, using recent work of Bhatt-Iyengar-Ma).

Going further, is there any example with  $X$  regular? **No:**

## Theorem 2

Any finite covering of a regular scheme is a covering for the fpqc topology.

Remarks. 1)  $Y \rightarrow X$  is a covering for the fpqc *pre*topology iff  $Y$  is Cohen-Macaulay.

2) Thm 2 is much stronger than Thm 1: e.g. unlike Thm 1, it is quite non-trivial in char. 0 (reduction to char.  $p$  and ultraproducts).

3) Some finite coverings of  $\mathbb{A}_k^3$  are not coverings for the fppf topology (in general,  $Z$  is not expected to be of finite type, nor even Noetherian).

Thm. 2 is a geometric translation of the existence of "big" Cohen-Macaulay algebras, proved by Hochster-Huneke in the presence of a base field, by me in general ('16).

Reduce to  $S$  complete local, finite extension of regular  $R$  (Cohen):  
a (not necessarily Noetherian)  $S$ -algebra  $T$  is (big) Cohen-Macaulay if every secant sequence in  $S$  becomes regular in  $T$ . This is equivalent to the faithful flatness of  $T$  over  $R$ .

## Question

*Is the canonical topology on  $\mathbf{Aff}_k$  generated by the fpqc topology and by morphisms  $Y \rightarrow X$  such that  $\mathcal{O}_X \rightarrow \mathcal{O}_Y$  splits as  $\mathcal{O}_X$ -modules?*

True if the base  $X$  is Noetherian.

- In char.  $p$ , the Frobenius map  $F_X : X \rightarrow X$  is faithfully flat iff  $X$  is regular (Kunz).

Enhancement (from the fpqc pretopology to the fpqc topology):

## Proposition

$F_X$  is a covering for the fpqc topology iff  $X$  is regular.



Remark. Kunz' theorem can also be stated as:

A noetherian ring  $R$  of char.  $p$  is regular iff there exists a perfect, faithfully flat  $R$ -algebra.

Generalization in mixed char. (Bhatt-Iyengar-Ma '18):

A Noetherian  $p$ -adically complete ring  $R$  is regular iff there exists a perfectoid, faithfully flat  $R$ -algebra.

Remarks. 1)  $F_X$  is often a finite covering (e.g. for schemes of finite type over a perfect field).

2) Whether  $F_X$  is a canonical covering (*F-purity*: Frobenius splitting) was an influential question in the study of singularities in char.  $p$  in the 80's, 90's ("*F-singularity theory*").

Related notion: *strongly F-regular* (Noetherian domain)  $S$ :

$\forall s \in S, \exists e, \exists S^{1/p^e} \rightarrow S$ , s.t.  $s^{1/p^e} \mapsto 1$ .

*F-pure* and *strongly F-regular* singularities are char.  $p$  analogs of *log-canonical* and *log-terminal* singularities in the MMP.

These have now analogs in mixed char. (Ma, Schwede), based on Thm. 2.

Ma and Schwede remark that for a resolution of singularities  $g : Z \rightarrow Y = \text{Spec } S$  in char. 0, Grauert-Riemenschneider + local duality implies  $\mathbb{H}_Y^i(Rg_*\mathcal{O}_Z) = 0$  for  $i < \dim Y$ , hence  $Rg_*\mathcal{O}_Z$  looks like a Cohen-Macaulay  $S$ -algebra... except that it is an algebra object in the derived category.

This supports their strategy to replace  $Rg_*\mathcal{O}_Z$  by a Cohen-Macaulay  $S$ -algebra  $T$  to study singularities in mixed char.

- Back to: finite coverings vs. canonical topology.

A Noetherian affine scheme  $X = \text{Spec } R$  is a **splinter** if any finite covering of  $X$  is canonical

- equivalently: if any finite extension  $R \rightarrow S$  splits in  $\text{Mod}_R$  (whence the name).

- Any splinter is normal.
- Thm. 1 says that any **regular**  $X$  is a splinter.

Are there any other? **Yes.**

- In char. 0, any normal  $X$  is a splinter (by the  $\text{tr}/\text{deg}$  trick)
  - In char  $p$ , any strongly  $F$ -regular  $X$  is a splinter (converse true in the Gorenstein case, and conjecturally in general)
- e.g. the quadric cone  $\text{Spec } k[x^2, xy, y^2]$  in char.  $\neq 2$  is a splinter.
- **Splinters in positive or mixed char. are Cohen-Macaulay**  
(a straightforward consequence of the strong Hochster-Huneke-Bhatt theorem about  $\widehat{R}^+$ ).
  - **$X$  is a splinter if there exists a canonical covering  $Y$  which is regular** - a main supply of splinters (the proof uses a “weakly functorial” version of Thm. 2)

- What if we replace the canonical topology by the fpqc topology ("fpqc analogs of splinters")?

*Conjecture: only regular  $X$  are fpqc analogs of splinters.*

Evidence: - true in char.  $p$  in the  $F$ -finite case (a straightforward consequence of our enhancement of Kunz' theorem),

- regularity descends along maps which are coverings for the fpqc topology, hence the "main supply of splinters" cannot provide any non-regular fpqc analog of splinters.

*p.s.* No need to comment about the geometric meaning of flatness or faithful flatness in Algebraic Geometry??

In EGA, we find many properties of schemes and morphisms which descend along faithfully flat morphisms (i.e. w.r.t. the fpqc pretopology).

But which properties descend w.r.t. the fpqc topology (not only w.r.t. the fpqc pretopology).

For instance (Ma):

**regularity descends w.r.t. the fpqc topology**

(the proof uses methods involved in the perfectoid version of Kunz' theorem).

While this issue could have been addressed in the 60's, it brings us back, after a long detour, to fundamental problems about base-change and descent we started with.

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