Generators for the level 2 twist subgroup of the mapping class group of a non-orientable surface and its abelianization

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$N_g = \sharp_g \mathbb{R}P^2$: a closed conn. non-ori. surface of genus $g \geq 1$.

$\mathcal{M}(N_g) := \mathrm{Diff}(N_g)$/isotopy: the *mapping class group* of N_g, where $\mathrm{Diff}(N_g) := \{f : N_g \to N_g \text{ diffeo.}\}$.

Put $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

$$\Gamma_2(N_g) := \ker(\mathcal{M}(N_g) \to \mathrm{Aut}\ H_1(N_g; \mathbb{Z}_2))$$

: the *level 2 mapping class group* of N_g.

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Theorem (Hirose-Sato (2014))

For $g \geq 4$,

- $\Gamma_2(N_g)$ *is generated by* $\binom{g}{3} + \binom{g}{2}$ *elements.*
- $H_1(\Gamma_2(N_g); \mathbb{Z}) \cong \mathbb{Z}_2^{\binom{g}{3}+\binom{g}{2}}$.

$\rightsquigarrow$ They used the mod 2 Johnson homomorphism to determine the abelianization of $\Gamma_2(N_g)$!!

Definition

c: *a simple closed curve on* N_g.

- c: *one-sided* $\overset{\text{def}}{\Longleftrightarrow}$ *a neighborhood of* c *in* N_g *is a Möbius band.*
- c: *two-sided* $\overset{\text{def}}{\Longleftrightarrow}$ *a neighborhood of* c *in* N_g *is an annulus.*

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For a two-sided simple closed curve c on N_g, we can define the Dehn twist t_c!!

Remark

We also need to take an orientation of the neighborhood of c to define t_c.

$\mathcal{T}(N_g) := \langle\{t_c \mid c : \text{a two-sided simple closed curve on } N_g\}\rangle \lhd \mathcal{M}(N_g)$
: the *twist subgroup* of $\mathcal{M}(N_g)$.

Theorem (Lickorish (1965))

$\mathcal{T}(N_g) \subset \mathcal{M}(N_g)$: an index 2 subgroup.

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Theorem (Lickorish (1965))

$\mathcal{T}(N_g) \subset \mathcal{M}(N_g)$*: an index 2 subgroup.*

$\mathcal{T}_2(N_g) := \Gamma_2(N_g) \cap \mathcal{T}(N_g)$: the *level 2 twist subgroup* of $\mathcal{M}(N_g)$.

Remark

- $\mathcal{T}_2(N_2) = \mathcal{T}_2(N_1) = \{1\}$.
- $\mathcal{T}_2(N_3) \cong \ker(SL(2;\mathbb{Z}) \to SL(2;\mathbb{Z}_2))$.

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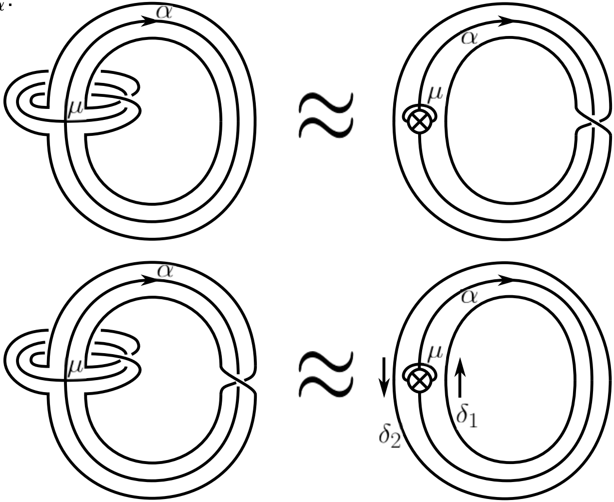
Today's talk

- A finite generating set for $\mathcal{T}_2(N_g)$,
- The first homology group of $\mathcal{T}_2(N_g)$.

Crosscap pushing map

μ: a one-sided s.c.c. on N_g, α: a s.c.c. on N_g w/ $|\mu \cap \alpha| = 1$,

$Y_{\mu,\alpha}$:



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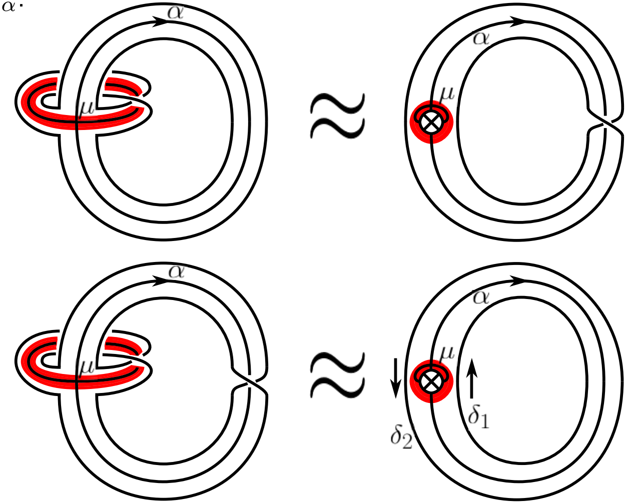

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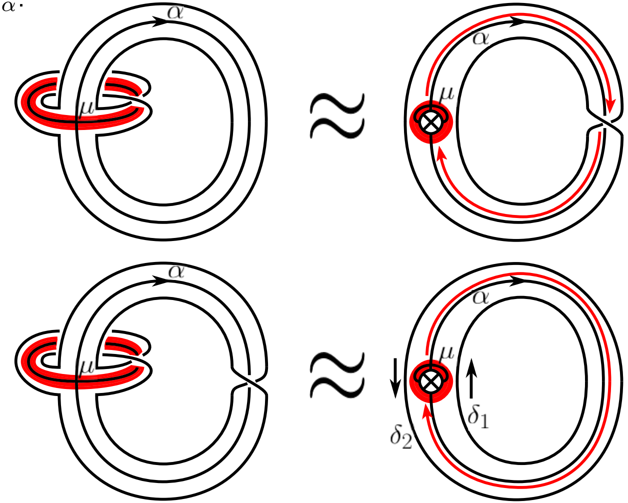
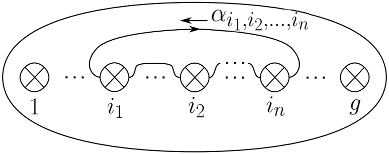


: Y-homeo.

$= t_{\delta_1} t_{\delta_2}^{-1}$

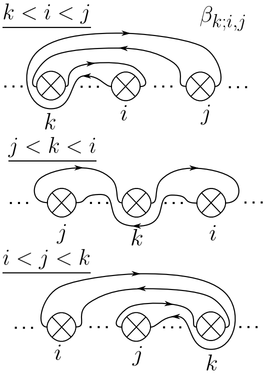
$\alpha_{i_1,i_2,\ldots,i_n}$: the s.c.c. on N_g for distinct $i_1, i_2, \ldots, i_n \in \{1, \ldots, g\}$,
$\beta_{k;i,j}$: the s.c.c. on N_g for $k < i < j$, $j < k < i$, or $i < j < k$.

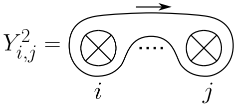
$T_{i,j,k,l} := t_{\alpha_{i,j,k,l}}$,
$Y_{i,j} := Y_{\alpha_i,\alpha_{i,j}}$: the Y-homeomorphism,
$a_{k;i,j} := Y_{\alpha_k,\alpha_{i,j,k}}$,
$b_{k;i,j} := Y_{\alpha_k,\beta_{k;i,j}}$.



Remark

- $T^2_{i,j,k,l} \in \mathcal{T}_2(N_g)$.
- $a_{k;i,j},\ b_{k;i,j} \in \mathcal{T}_2(N_g)$.
- $Y_{i,j} \in \Gamma_2(N_g)$, but $Y_{i,j} \notin \mathcal{T}_2(N_g)$.
- $Y^2_{i,j} \in \mathcal{T}_2(N_g)$.





Theorem (R. Kobayashi-O.)

For $g \geq 3$, $\mathcal{T}_2(N_g)$ *is generated by the following elements:*

(i) $a_{k;i,i+1}, b_{k;i,i+1}, a_{k;k-1,k+1}, b_{k;k-1,k+1}$ *(*$1 \leq k \leq g$, $1 \leq i \leq g-1$, $i \neq k-1, k$*),*

(ii) $Y_{1,j}^2$ *(*$2 \leq j \leq g$*),*

(iii) $T_{1,j,k,l}^2$ *(when* $g \geq 4$, $2 \leq j < k < l \leq g$*).*

Outline of the proof

$$\begin{aligned}\Gamma_2(N_g)/\mathcal{T}_2(N_g) &= \Gamma_2(N_g)/(\Gamma_2(N_g) \cap \mathcal{T}(N_g)) \cong (\Gamma_2(N_g)\mathcal{T}(N_g))/\mathcal{T}(N_g)\\ &= \mathcal{M}(N_g)/\mathcal{T}(N_g)\\ &\cong \mathbb{Z}_2[Y_{1,2}].\end{aligned}$$

We use the Reidemeister-Schreier method for $\mathcal{T}_2(N_g) < \Gamma_2(N_g)$!! □

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Theorem (R. Kobayashi-O.)

$$H_1(\mathcal{T}_2(N_g);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}_2 & (g=3), \\ \mathbb{Z}_2^{\binom{g}{3}+\binom{g}{2}-1} & (g \geq 5). \end{cases}$$

Remark

- $|\{\text{generators of } \mathcal{T}_2(N_3) \text{ in the thm.}\}| = \dim_{\mathbb{Z}_2} H_1(\mathcal{T}_2(N_3);\mathbb{Z})$.
- For $g \geq 5$,
  $|\{\text{generators of } \mathcal{T}_2(N_g) \text{ in the thm.}\}| - \dim_{\mathbb{Z}_2} H_1(\mathcal{T}_2(N_g);\mathbb{Z})$
  $= \frac{1}{6}(g^3 + 6g^2 - 7g - 12) - (\binom{g}{3} + \binom{g}{2} - 1)$
  $= g^2 - g - 1.$

Remark

- $|\{\text{generators of } \mathcal{T}_2(N_3) \text{ in the thm.}\}| = \dim_{\mathbb{Z}_2} H_1(\mathcal{T}_2(N_3); \mathbb{Z})$.
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Key theorem for the abelianization:

Theorem (R. Kobayashi-O.)

- *For $g = 3$ or $g \geq 5$,*
  $\mathcal{T}_2(N_g)$ is normally generated by $a_{1;2,3}$ in $\mathcal{M}(N_g)$.
- *$\mathcal{T}_2(N_4)$ is normally generated by $a_{1;2,3}$ and $T^2_{1,2,3,4}$ in $\mathcal{M}(N_4)$.*

The abelianization of $\mathcal{T}_2(N_g)$ for $g \geq 5$

Put $H_1(\mathcal{T}_2(N_g)) := H_1(\mathcal{T}_2(N_g); \mathbb{Z})$.
We have the exact sequence

$$1 \longrightarrow \mathcal{T}_2(N_g) \longrightarrow \Gamma_2(N_g) \longrightarrow \mathbb{Z}_2[Y_{1,2}] \longrightarrow 0.$$

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where

$$H_1(\mathcal{T}_2(N_g))_{\mathbb{Z}_2} := H_1(\mathcal{T}_2(N_g))/\langle f \cdot m - m \mid m \in H_1(\mathcal{T}_2(N_g)),\ f \in \mathbb{Z}_2 \rangle.$$

: the co-invariant part, where

$\mathbb{Z}_2 = \Gamma_2(N_g)/\mathcal{T}_2(N_g) \curvearrowright H_1(\mathcal{T}_2(N_g)) = \mathcal{T}_2(N_g)^{\mathrm{ab}}$: conjugations.

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Proposition (by using the normal generating set for $\mathcal{T}_2(N_g)$)

For $g \geq 5$, the action $\mathbb{Z}_2 \curvearrowright H_1(\mathcal{T}_2(N_g))$ is trivial.

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An observation for the abelianization of $\mathcal{T}_2(N_4)$

Remark (by a private communication with B. Szepietowski)

- The conjugate action $\mathbb{Z}_2 \curvearrowright H_1(\mathcal{T}_2(N_4))$ is not trivial.
- $[T^2_{1,2,3,4}] \in H_1(\mathcal{T}_2(N_4))$ has infinite order.

Proposition

$\mathcal{G}$: the subgroup of $\mathcal{T}_2(N_g)$ which is normally generated by $a_{1;2,3}$ in $\mathcal{M}(N_g)$.
For $g \geq 4$, $\mathcal{G}$ is generated by involutions.

Theorem (R. Kobayashi-O. (again))

- *For $g = 3$ or $g \geq 5$, $\mathcal{T}_2(N_g)$ is normally generated by $a_{1;2,3}$ in $\mathcal{M}(N_g)$.*
- *$\mathcal{T}_2(N_4)$ is normally generated by $a_{1;2,3}$ and $T^2_{1,2,3,4}$ in $\mathcal{M}(N_4)$.*

$\rightsquigarrow$ $\mathcal{T}_2(N_4)$ is not normally generated by $a_{1;2,3}$ in $\mathcal{M}(N_4)$.

Hirose-Sato defined the mod 2 Johnson homomorphism $\tau_1 : \Gamma_2({N_g}^*) \to A^*$ for some $\mathbb{Z}_2$-vector space A^*.

Theorem (R. Kobayashi-O. (again))

$\mathcal{T}_2(N_4)$ is generated by the following elements:

(i) $a_{1;2,3}$, $a_{1;3,4}$, $a_{2;1,3}$, $a_{2;3,4}$, $a_{3;1,2}$, $a_{3;2,4}$, $a_{4;1,2}$, $a_{4;2,3}$, $b_{1;2,3}$, $b_{1;3,4}$, $b_{2;1,3}$, $b_{2;3,4}$, $b_{3;1,2}$, $b_{3;2,4}$, $b_{4;1,2}$, $b_{4;2,1}$, $Y_{1,2}^2$, $Y_{1,3}^2$, $Y_{1,4}^2$,

(ii) $T_{1,2,3,4}^2$.

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Theorem (R. Kobayashi-O. (again))

$\mathcal{T}_2(N_4)$ is generated by the following 9 elements:

(i) $a_{1;2,3}$, $a_{1;3,4}$, $a_{2;1,3}$, $a_{3;1,2}$, $a_{3;2,4}$, $Y^2_{1,2}$, $Y^2_{1,3}$, $Y^2_{1,4}$,

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(ii) $T^2_{1,2,3,4}$.

Observations: for $g = 4$,

- $[T^2_{1,2,3,4}] \in H_1(\mathcal{T}_2(N_4))$ has infinite order,
- $[a_{k;i,j}] \in H_1(\mathcal{T}_2(N_4))$ has order $n \leq 2$ ($\because \mathcal{G}$ is generated by involutions),
- $Y^2_{i,j} \in \mathcal{G}$ ($\Longrightarrow [Y^2_{i,j}] \in H_1(\mathcal{T}_2(N_4))$ also has order $n \leq 2$),
- $\bar{\tau}_1(a_{k;i,j}) \neq 0$ and $\bar{\tau}_1(Y^2_{i,j}) = 0$ in A,

$\rightsquigarrow 8 \geq \exists d \geq 1$ s.t. $H_1(\mathcal{T}_2(N_4)) \cong \mathbb{Z}_2^d \oplus \mathbb{Z}[T^2_{1,2,3,4}]$.

Thank you for your attention !