Torelli group versus invariants of homology spheres and beyond

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based on jw/w Takuya SAKASAI and Masaaki SUZUKI

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Contents

Homology 3-spheres and the Torelli group (1)

$\mathfrak{M}(3) = \{$closed oriented 3-manifold$\}/$ori. pres. diffeo.

$\cup$

$\mathfrak{H}(3) = \{$closed oriented homology 3-sphere$\}/$ori. pres. diffeo.

Heegaard decomposition:

$\mathfrak{M}(3) \ni^{\forall} [M],\ M = H_g \cup_\varphi -H_g \quad (H_g$: handlebody, $\varphi \in \mathcal{M}_g)$

$\mathcal{M}_g$: mapping class group

$S^3 = H_g \cup_{\iota_g} -H_g \quad (\iota_g \in \mathcal{M}_g : 90^\circ$-rotation on each handle$)$

$$\Rightarrow\ \mathcal{M}_g \ni [\varphi] \longmapsto [M_\varphi = H_g \cup_\varphi -H_g] \in \mathfrak{M}(3)$$

Homology 3-spheres and the Torelli group (2)

Theorem (Reidemeister-Singer)

$$\left(\coprod_g \mathcal{M}_g\right) / \textit{R.S. stabilization} = \mathfrak{M}(3)$$

restriction to the Torelli group :

$$\mathcal{I}_g = \mathrm{Ker}(\mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z}))$$

Proposition

$$\lim_{g\to\infty} \mathcal{I}_g / \sim = \mathfrak{H}(3)$$

$$(\mathcal{I}_g \ni \varphi \longmapsto W_\varphi = H_g \cup_{\iota_g \varphi} -H_g \in \mathfrak{H}(3))$$

Homology 3-spheres and the Torelli group (3)

two filtrations of $\mathcal{I}_g$:

$$\mathcal{I}_g = \mathcal{M}_g(1) \supset \mathcal{M}_g(2) \supset \cdots \text{ (Johnson filtration)}$$
$$\mathcal{I}_g = \mathcal{I}_g(1) \supset \mathcal{I}_g(2) = [\mathcal{I}_g(1), \mathcal{I}_g(1)] \supset \cdots \text{ (lower central series)}$$

$$\mathcal{I}_g(k) \subset \mathcal{M}_g(k) \text{ for any } k$$

$$\mathcal{M}_g(2) = \mathrm{Ker}(\tau_1 : \mathcal{I}_g \overset{\text{first Johnson hom.}}{\rightarrow} \wedge^3 H/H) \ (H = H_1(\Sigma_g;\mathbb{Z}))$$

$$\mathcal{M}_g(k+1) = \mathrm{Ker}(\tau_k : \mathcal{M}_g(k) \overset{\text{Johnson hom.}}{\rightarrow} \mathfrak{h}_g(k))$$

Theorem (Johnson)

$$\mathcal{I}_g(2) \overset{\textit{finite index}}{\subset} \mathcal{M}_g(2) = \mathcal{K}_g = \langle \textit{Dehn twists on BSCC} \rangle$$
$$\mathcal{I}_g(1)/\mathcal{I}_g(2) = H_1(\mathcal{I}_g) \cong \wedge^3 H/H \oplus 2\textit{-torsion}$$

Casson invariant and the first MMM class (1)

Casson invariant (1985):

$\lambda : \mathfrak{H}(3) \to \mathbb{Z}$

$$
\begin{aligned}
&\text{(i) } \lambda \equiv \text{ Rohlin homomorphism} : \mathfrak{H}(3) \to \mathbb{Z}/2 \ (\text{mod}\, 2) \\
&\text{(ii) } \lambda = \frac{1}{2} \text{ “alg. number” of } \{\text{irred. rep.} : \pi_1 W \to \mathrm{SU}(2)\}/\text{conj.} \\
&\text{(iii) } \lambda(-W) = -\lambda(W), \text{ additive w.r.t. connected sum} \\
&\text{(iv) } W \supset K \text{ (knot)} \Rightarrow \lambda(W_{1/n}(K)) = \lambda(W) + n\frac{1}{2}\bar{\Delta}''_K(1)
\end{aligned}
$$

Extensions by Walker (to rational homology 3-spheres) and

Lescop (to all 3-manifolds)

Casson invariant and the first MMM class (2)

Consider the mapping

$$\lambda^* : \mathcal{I}_g \to \mathbb{Z} \quad \text{defined by } \lambda^*(\varphi) = \lambda(W_\varphi)$$

NOT a homomorphism, but its restriction to $\mathcal{K}_g$

$$\lambda^* : \mathcal{K}_g \to \mathbb{Z}$$

can be shown to be a homomorphism!

What is it?

Answer: secondary class associated to the fact: the first

MMM-class vanishes in the Torelli group $\quad e_1 = 0 \in H^2(\mathcal{I}_g; \mathbb{Q})$

Casson invariant and the first MMM class (3)

$e_1 \in H^2(\mathcal{M}_g;\mathbb{Z})$

geometric meaning: signature of surface bundles over surfaces

$\Rightarrow\ e_1 = 0 \in H^2(\mathcal{I}_g;\mathbb{Q})$ because signature of any fiber bundle

$F \to E \to B$ vanishes if $\pi_1 B$ acts on $H_*(F;\mathbb{Q})$ trivially

(Chern-Hirzebruch-Serre)

There are two canonical cocycles representing e_1:

pull back of $\quad -3c_1 \in Z^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q})$

image of $\quad \wedge^2 (\wedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \to \wedge^2 \tilde{k} \in Z^2(\mathcal{M}_g;\mathbb{Q})$

under $\quad (\tilde{k}, \rho_0) : \mathcal{M}_g \to \wedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}} \rtimes \mathrm{Sp}(2g,\mathbb{Z})$

Casson invariant and the first MMM class (4)

$$\Rightarrow 3c_1 + \wedge^2 \tilde{k} = d^{\exists} u_1, \quad u_1 \in C^1(\mathcal{M}_g; \mathbb{Q})$$

$$\begin{cases} c_1|_{\mathcal{I}_g} & = 0 \\ \wedge^2 \tilde{k}|_{\mathcal{K}_g} & = 0 \end{cases} \Rightarrow d(u_1|_{\mathcal{K}_g}) = 0$$

Theorem (M.)

(i) $H^1(\mathcal{K}_g; \mathbb{Q})^{\mathcal{M}_g} \cong \mathbb{Q}$ $(g \geq 2)$ *generated by* $d_1 := [u_1|_{\mathcal{K}_g}]$

$$\text{(ii) } \lambda^* = \frac{1}{24} d_1 + \bar{\tau}_2 : \ \mathcal{K}_g \to \mathbb{Q}$$

$$\bar{\tau}_2 = \mathcal{K}_g \overset{\tau_2}{\to} \mathfrak{h}_g(2) \overset{\text{certain quotient}}{\longrightarrow} \mathbb{Q}$$

$\lambda^* = \frac{1}{24} d_1$ on $\mathcal{M}_g(3) \Rightarrow d_1$ is the core of the Casson invariant

Difference between two filtrations of the Torelli group (1)

(recall) two filtrations of $\mathcal{I}_g$:

$$\mathcal{I}_g = \mathcal{M}_g(1) \supset \mathcal{M}_g(2) = \mathcal{K}_g \supset \mathcal{M}_g(3) \cdots \text{ (Johnson filtration)}$$
$$\mathcal{I}_g = \mathcal{I}_g(1) \supset \mathcal{I}_g(2) = [\mathcal{I}_g(1), \mathcal{I}_g(1)] \supset \mathcal{I}_g(3) \cdots \text{ (lower central series)}$$

$\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ for any k, Johnson showed

$\mathcal{M}_g(2)/\mathcal{I}_g(2) \otimes \mathbb{Q} = 0$ and asked $[\mathcal{M}_g(k) : \mathcal{I}_g(k)] < \infty$?

Theorem (M. 1988)

The index of $\mathcal{I}_g(3) = [[\mathcal{I}_g, \mathcal{I}_g], \mathcal{I}_g]$ in $\mathcal{M}_g(3)$ is infinite

This was proved by showing that

$d_1 \neq 0$ on $\mathcal{M}_g(3)$ whereas $d_1 = 0$ on $\mathcal{I}_g(3)$, alternatively:

$$\text{under} \quad \tau_1 : \mathcal{I}_g \to \wedge^3 H/H \overset{\text{over } \mathbb{Q}}{\cong} (\mathcal{I}_g/\mathcal{I}_g(2)) \otimes \mathbb{Q}$$

$$\tau_1^* : H^2(\wedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \to H^2(\mathcal{I}_g;\mathbb{Q})$$

is trivial because the image is e_1 which vanishes on $\mathcal{I}_g$

Hain determined τ_1^* on H^2 completely and by Hodge theory :

Theorem (Hain 1997)

$$\bigoplus_{k=1}^{\infty} \mathfrak{t}_g(k) \cong \textit{Free Lie } \langle \wedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}} \rangle / \textit{quad. relation} \quad (g \geq 6)$$

$$\mathbb{Q}+? \to \bigoplus_{k=1}^{\infty} \mathfrak{t}_g(k) \twoheadrightarrow \bigoplus_{k=1}^{\infty} \mathfrak{m}_g(k) \quad (\textit{Johnson image})$$

$$\mathfrak{t}_g(k) := (\mathcal{I}_g(k)/\mathcal{I}_g(k+1)) \otimes \mathbb{Q},\ \mathfrak{m}_g(k) := (\mathcal{M}_g(k)/\mathcal{M}_g(k+1)) \otimes \mathbb{Q}$$

Finite type invariants and the Torelli group (1)

$\mathbb{Q}$: a "manifestation" of the Casson invariant in $\mathcal{I}_g \Rightarrow$

natural to ask whether Ohtsuki's (finite type) invariants:

$$\lambda_k : \mathfrak{H}(3) \to \mathbb{Q} \quad (\lambda_1 = \lambda, k = 1, 2, \dots)$$

appear in the kernel $\mathbb{Q} + ?$ or not and more generally

it is an important and difficult problem to identify

$$\mathfrak{i}_g = \bigoplus_{k=1}^{\infty} \mathfrak{i}_g(k) := \mathrm{Ker}\left(\bigoplus_{k=1}^{\infty} \mathfrak{t}_g(k) \twoheadrightarrow \bigoplus_{k=1}^{\infty} \mathfrak{m}_g(k)\right) = \mathbb{Q} + ?$$

Ohtsuki filtration based on the LMO-invariant

$$\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)_{(3)} \supset \mathbb{Q}\mathfrak{H}(3)_{(6)} \supset \mathbb{Q}\mathfrak{H}(3)_{(9)} \supset \cdots$$

Finite type invariants and the Torelli group (2)

$\mathcal{A}(\emptyset) =$ commutative algebra generated by vertex oriented connected trivalent graphs/AS+IHX

degree $=$ half the number of vertices

Theorem (Garoufalidis-Ohtsuki+Le-Murakami-Ohtsuki)

There exists an isomorphism

$$\mathrm{Gr}_m \mathcal{A}(\emptyset) \cong \mathbb{Q}\mathfrak{H}(3)_{(3m)}/\mathbb{Q}\mathfrak{H}(3)_{(3m+1)}$$

Theorem (Garoufalidis-Levine)

There exists a mapping

$$\mathfrak{t}_g(2m) \to \mathrm{Gr}_m \mathcal{A}^{\mathrm{conn}}(\emptyset)$$

which is surjective for $g \geq 5m+1$

In particular

$$\mathfrak{t}_g(2m)^{\mathrm{Sp}} \otimes \mathbb{Q}$$

gives rise to invariants for

$$\mathbb{Q}\mathfrak{H}(3)_{(3m)}/\mathbb{Q}\mathfrak{H}(3)_{(3m+1)}$$

The case $m = 1$:

$$\mathrm{Gr}_1\mathcal{A}(\emptyset) \cong \mathbb{Q}\mathfrak{H}(3)_{(3)}/\mathbb{Q}\mathfrak{H}(3)_{(4)} \cong \mathbb{Q}$$

$$\mathfrak{t}_g(2)^{\mathrm{Sp}} \otimes \mathbb{Q} \cong \mathrm{Gr}_1\mathcal{A}^{\mathrm{conn}}(\emptyset) \cong \mathbb{Q}$$

given by

theta graph, Casson invariant λ and $d_1 : \mathcal{K}_g \to \mathbb{Q}$

Results (1)

recall : $\quad 0 \to \mathfrak{i}_g \to \mathfrak{t}_g \to \mathfrak{m}_g \to 0$

$$\mathfrak{i}_g = \bigoplus_{k=1}^{\infty} \mathfrak{i}_g(k), \ \mathfrak{t}_g = \bigoplus_{k=1}^{\infty} \mathfrak{t}_g(k), \ \mathfrak{m}_g = \bigoplus_{k=1}^{\infty} \mathfrak{m}_g(k)$$

$$\mathfrak{t}_g(k) = (\mathcal{I}_g(k)/\mathcal{I}_g(k+1)) \otimes \mathbb{Q}, \quad \mathfrak{m}_g(k) = (\mathcal{M}_g(k)/\mathcal{M}_g(k+1)) \otimes \mathbb{Q}$$

$$\mathfrak{t}_g(k) = \mathfrak{i}_g(k) \oplus \mathfrak{m}_g(k), \quad \mathfrak{i}_g(1) = 0, \ \mathfrak{i}_g(2) = \mathbb{Q}$$

Problem

Determine $\mathfrak{i}_g(k)$ *for* $k = 3, 4, \ldots$

Theorem (M. 1999)

$\mathfrak{i}_g(3) = 0$ *and hence* $\quad \mathfrak{t}_g(3) \cong \mathfrak{m}_g(3)$

Results (2)

Theorem (Sakasai-Suzuki-M.)

$$\mathfrak{i}_g(4) = 0 \quad \textit{and hence} \quad \mathfrak{t}_g(4) \cong \mathfrak{m}_g(4)$$

$$\mathfrak{i}_g(5) = 0 \quad \textit{and hence} \quad \mathfrak{t}_g(5) \cong \mathfrak{m}_g(5)$$

Corollary

Any finite type invariant of degree 2*, including Ohtsuki's* λ_2*,*

can be expressed by:

d_1 *and (lifts of) Johnson homomorphisms* $\tilde{\tau}_2 \sim (\tau_2, \tau_3)$

Proposition (Levine)

$$\lambda_k : \mathcal{I}_g(k+1)/\mathcal{I}_g(2k+1) \to \mathbb{Q}$$

is a homomorphism

Results (3)

sketch of proof

$$0 \to \mathfrak{i}_g \to \mathfrak{t}_g \to \mathfrak{m}_g \to 0$$

$$H_2(\mathfrak{t}_g) \overset{\text{Hain}}{=} H_2(\mathfrak{t}_g)_2 \to H_2(\mathfrak{m}_g) \to H_1(\mathfrak{i}_g)_{\mathfrak{m}_g} \to H_1(\mathfrak{t}_g) \overset{\cong}{\to} H_1(\mathfrak{m}_g)$$

$\Rightarrow$

Proposition

$$0 \to H_2(\mathfrak{t}_g)_2 \to H_2(\mathfrak{m}_g)_2 \to (H_1(\mathfrak{i}_g)_{\mathfrak{m}_g})_2 = \mathbb{Z} \to 0$$

and for any $w \geq 3$

$$H_2(\mathfrak{m}_g)_w \cong (H_1(\mathfrak{i}_g)_{\mathfrak{m}_g})_w$$

Results (4)

Corollary

$$\mathfrak{i}_g(3) \cong H_2(\mathfrak{m}_g)_3 = 0, \quad \mathfrak{i}_g(4) \cong H_2(\mathfrak{m}_g)_4$$

and for any $w \geq 4$*, we have*

$$0 \to \bigoplus_{k=3}^{w-1} [\mathfrak{i}_g(k), \mathfrak{t}_g(w-k)] \to \mathfrak{i}_g(w) \to H_2(\mathfrak{m}_g)_w \to 0 \quad (\textit{exact})$$

In general, we have the following

Proposition

Assume $\mathfrak{i}_g(k) = 0$ *for* $k = 3, 4, \ldots, m-1$ $(m \geq 4)$*, then we have*

$$\mathfrak{i}_g(m) \cong H_2(\mathfrak{m}_g)_m$$

Results (5)

proof of $H_2(\mathfrak{m}_g)_4 = 0$:

2-cycles Z_2 and 2-boundaries B_2 of weight 4 of $\mathfrak{m}_g$

exact:

$$B_2 := \mathrm{Im}\left(\wedge^2\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(2) \to (\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(3)) \oplus \wedge^2\mathfrak{m}_g(2)\right) \to$$
$$Z_2 := \mathrm{Ker}\left((\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(3)) \oplus \wedge^2\mathfrak{m}_g(2) \to \mathfrak{m}_g(4)\right) \twoheadrightarrow H_2(\mathfrak{m}_g)_4$$

$$\wedge^2\,\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(2) \ni (u \wedge v) \otimes w \overset{\text{boundary}}{\longmapsto}$$
$$(u \otimes [v,w] - v \otimes [u,w], -[u,v] \wedge w) \in (\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(3)) \oplus \wedge^2\mathfrak{m}_g(2)$$

Results (6)

Sp-irreducible decompositions:

$$
\begin{aligned}
H_2(\mathfrak{m}_g)_4 \cong \mathrm{Coker}\,(\wedge^2[1^3] \otimes [2^2] \to \\
&[\mathbf{42^2}][\mathbf{421^2}][\mathbf{32^21}][\mathbf{41^4}][\mathbf{321^3}][\mathbf{31^5}] \\
&[\mathbf{41^2}]\mathbf{2}[\mathbf{321}][\mathbf{31^3}][\mathbf{2^21^2}][\mathbf{21^4}] \quad [\mathbf{4}][\mathbf{31}][\mathbf{2^2}][\mathbf{21^2}] \qquad \oplus \\
&[\mathbf{431}][\mathbf{32^21}] \quad [\mathbf{42}][\mathbf{321}][\mathbf{31^3}][\mathbf{2^3}] \quad [\mathbf{31}][\mathbf{21^2}] \quad [\mathbf{2}])
\end{aligned}
$$

we checked that all the 2-cycles (24-types of Young diagrams) are boundaries proof of $H_2(\mathfrak{m}_g)_5 = 0$:

2-cycles Z_2 and 2-boundaries B_2 of weight 5 of $\mathfrak{m}_g$

exact:

$$\begin{aligned} B_2 :=& \mathrm{Im}((\wedge^2\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(3)) \oplus (\mathfrak{m}_g(1) \otimes \wedge^2\mathfrak{m}_g(2)) \to \\ & \qquad\qquad (\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(4)) \oplus (\mathfrak{m}_g(2) \otimes \mathfrak{m}_g(3))) \to \\ Z_2 :=& \mathrm{Ker}\,((\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(4)) \oplus (\mathfrak{m}_g(2) \otimes \mathfrak{m}_g(3)) \to \mathfrak{m}_g(5)) \twoheadrightarrow H_2(\mathfrak{m}_g)_5 \end{aligned}$$

$$\wedge^2\,\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(3) \ni (u \wedge v) \otimes w \longmapsto$$

$$u \otimes [v,w] - v \otimes [u,w] - [u,v] \otimes w \in (\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(4)) \oplus (\mathfrak{m}_g(2) \otimes \mathfrak{m}_g(3)$$

$$\mathfrak{m}_g(1) \otimes \wedge^2\mathfrak{m}_g(2) \ni u \otimes (v \wedge w) \longmapsto$$

$$u \otimes [v,w] - v \otimes [u,w] + w \otimes [u,v] \in (\mathfrak{m}_g(1) \otimes \mathfrak{m}_g(4)) \oplus (\mathfrak{m}_g(2) \otimes \mathfrak{m}_g(3)$$

Sp-irreducible decompositions:

$H_2(\mathfrak{m}_g)_5 \cong \mathrm{Coker}\,((\wedge^2[1^3] \otimes [31^2]) \oplus ([1^3] \otimes \wedge^2[2^2]) \to$
$\mathbf{2[531]2[521^2][432]2[431^2]2[42^21]3[421^3][41^5]2[3^3]2[3^221][32^3]}$
$\mathbf{3[32^21^2][321^4][31^6][2^31^3]} \quad \mathbf{2[52][51^2]2[43]6[421]5[41^3]3[3^21]4[32^2]}$
$\mathbf{7[321^2]4[31^4]3[2^31][2^21^3][21^5]} \quad \mathbf{4[41]4[32]10[31^2]4[2^21]5[21^3]}$
$\mathbf{3[3]4[21]2[1^3]})$

we checked that all the 2-cycles (34-types of Young diagrams)

are boundaries

we are now computing $\mathbf{H_2(\mathfrak{m}_g)_6}$

already proved $H_2(\mathfrak{m}_g)_6[0][2][1^2]\cdots[1^6]\cdots = 0$ for 27 among 67 types

Extending the above picture in a broader context (1)

$$H^2(\mathcal{M}_g;\mathbb{Q}) \ni e_1 \mapsto 0 \in H^2(\mathcal{I}_g;\mathbb{Q}) \quad \Rightarrow \quad \lambda : \mathfrak{H}(3) \to \mathbb{Z}$$

More precisely

$$\mathcal{K}_g \to \mathcal{M}_g \to \mathcal{M}_g/\mathcal{K}_g \overset{\mathbb{Q}}{\cong} U_{\mathbb{Q}} \rtimes \mathrm{Sp}(2g,\mathbb{Z}) \quad (U = \wedge^3 H/H)$$

$$H^1(\mathcal{M}_g;\mathbb{Q}) = 0 \to H^1(\mathcal{K}_g;\mathbb{Q})^{\mathcal{M}_g} \cong \mathbb{Q} \to$$
$$H^2(U_{\mathbb{Q}} \rtimes \mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q}) \cong \mathbb{Q}^2 \to H^2(\mathcal{M}_g;\mathbb{Q}) \cong \mathbb{Q}$$

the difference of two natural cocycles for e_1 $\Rightarrow$

$$H^1(\mathcal{K}_g;\mathbb{Q})^{\mathcal{M}_g} \cong \mathbb{Q} \quad \Rightarrow \quad \text{Casson invariant}$$

Extending the above picture in a broader context (2)

extending $\mathcal{M}_g \Rightarrow \mathcal{H}_{g,1}$ and $e_1 \Rightarrow \tilde{t}_{2k+1}$, ultimate goal:

$$H^2(\mathcal{H}^{\mathrm{top}}_{g,1};\mathbb{Q}) \ni \tilde{t}_{2k+1} \mapsto 0 \in H^2(\mathcal{H}^{\mathrm{smooth}}_{g,1};\mathbb{Q}) \;\Rightarrow\; \nu_k : \boldsymbol{\Theta}^3 \to \mathbb{Q}$$

Garoufalidis-Levine (based on Goussarov and Habiro)

$\mathcal{H}^{\mathrm{smooth}}_{g,1} = \{$homology cylinder over $\Sigma_{g,1}\}/$smooth H-cobordism

$\mathcal{H}^{\mathrm{smooth}}_{0,1} = \boldsymbol{\Theta}^3 = \mathfrak{H}(3)/$smooth H-cobordism $\overset{\text{central}}{\subset}$ $\mathcal{H}^{\mathrm{smooth}}_{g,1}$

$$\boldsymbol{\Theta}^3 \to \mathcal{H}^{\mathrm{smooth}}_{g,1} \to \overline{\mathcal{H}}_{g,1} = \mathcal{H}^{\mathrm{smooth}}_{g,1}/\boldsymbol{\Theta}^3 \quad \text{(central extension)}$$

Extending the above picture in a broader context (3)

exact sequence:

$$0 \to H^1(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^1(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q}) \to H^1(\boldsymbol{\Theta}^3;\mathbb{Q})$$
$$\cong \mathrm{Hom}(\boldsymbol{\Theta}^3,\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}} \to H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})$$

Theorem (Furuta, Fintushel-Stern)

$\boldsymbol{\Theta}^3$ *has infinite rank*

$\Rightarrow \boldsymbol{\Theta}^3/$torsion $\subset \mathbb{Q}^\infty$ (because $\boldsymbol{\Theta}^3$ is countable)

$\Rightarrow \mathrm{Hom}(\boldsymbol{\Theta}^3,\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}}$ (direct product of countably many $\mathbb{Q}$)

so there exist (uncountably) many homomorphisms

$$\boldsymbol{\Theta}^3 \to \mathbb{Q}$$

but explicitly known one(s): Frøyshov and Ozsváth-Szabó

Problem

How is the huge group $H^1(\mathbf{\Theta}^3;\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}}$ *divided into*

$$\mathrm{Coker}\left(H^1(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^1(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})\right) \quad \textit{and}$$

$$\mathrm{Ker}\left(H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})\right)\ ?$$

Coker is non-trivial $\Leftrightarrow$

$^\exists$ homomorphism $\mathbf{\Theta}^3 \to \mathbb{Q}$ $(\neq 0)$ which extends to $\mathcal{H}_{g,1}^{\mathrm{smooth}} \to \mathbb{Q}$

many works on $\mathcal{H}_{g,1}$ by

Sakasai, Habiro, Massuyeau, Cha-Friedl-Kim,...

Extending the above picture in a broader context (5)

Mal'cev completion of $\pi_1\Sigma_{g,1}$: $\cdots \to N_d \to \cdots \to N_1 = H_{\mathbb{Q}}$

Theorem (Garoufalidis-Levine)

$\exists$ $\tilde{\rho}_\infty : \mathcal{H}_{g,1}^{\text{smooth}} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d$ (*symplectic auto. groups*)

each factor $\tilde{\rho}_d : \mathcal{H}_{g,1}^{\text{smooth}} \to \mathrm{Aut}_0\, N_d$ *is surjective over* $\mathbb{Z}$

candidates for Ker: constructed a homomorphism

$$\tilde{\rho} : \overline{\mathcal{H}}_{g,1} \to \left(\wedge^3 H_{\mathbb{Q}} \oplus \prod_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}\right) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$$

and defined

$$(\wedge^2 S^{2k+1} H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \ni 1 \mapsto \tilde{\mathfrak{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$$

replacing $\overline{\mathcal{H}}_{g,1}$ with more geometric object

(2008, after a comment by Orr):

$\mathcal{H}_{g,1}^{\mathrm{top}} = \{$homology cylinder over $\Sigma_{g,1}\}/$topological H-cobordism

Theorem (Freedman)

Any homology 3-sphere bounds a contractible topological 4-mfd

It follows that $\mathcal{H}_{g,1}^{\mathrm{smooth}} \to \mathcal{H}_{g,1}^{\mathrm{top}}$ factors through $\overline{\mathcal{H}}_{g,1}$

$$\boldsymbol{\Theta}^3 \to \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \overline{\mathcal{H}}_{g,1} \to \mathcal{H}_{g,1}^{\mathrm{top}}$$

and the homomorphisms $\tilde{\rho}_\infty, \tilde{\rho}$ are actually defined on $\mathcal{H}_{g,1}^{\mathrm{top}}$

$$\Rightarrow \ \tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q})$$

Extending the above picture in a broader context (7)

how about Coker ?

$\mathfrak{h}_{g,1} =$ symplectic derivation Lie algebra of $\mathcal{L}(H_{\mathbb{Q}})$

extremely rich and mysterious structure

Theorem (Massuyeau-Sakasai)

(i) $\mathcal{H}_{g,1} \overset{\textit{homo.}}{\to} \hat{H}_1(\mathfrak{h}^+_{g,1}) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ *with dense image*

(ii) $H_1(\mathcal{H}_{g,1};\mathbb{Q}) \supset \mathbb{Q}$ (*sharp contrast: $\mathcal{M}_g$ is perfect* $(g \geq 3)$)

$\Rightarrow \hat{H}^1_c(\hat{\mathfrak{h}}_{g,1}) \subset H^1(\mathcal{H}^{\mathrm{smooth}}_{g,1};\mathbb{Q})$ but this part comes from $H^1(\mathcal{H}^{\mathrm{top}}_{g,1};\mathbb{Q})$ so that it vanishes in the Coker

At present, there is no information about

$$\mathrm{Coker}\left(H^1(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^1(\mathcal{H}^{\mathrm{smooth}}_{g,1};\mathbb{Q})\right)$$

Extending the above picture in a broader context (8)

back to Ker

Theorem (Sakasai-Suzuki-M.)

$$\exists\ \tilde{\rho}^*_\infty : H^*_c(\hat{\mathfrak{h}}^+_{\infty,1})^{\mathrm{Sp}} \otimes H^*(\mathrm{Sp}(2\infty,\mathbb{Z})) \to H^*(\mathcal{H}^{\mathrm{top}}_{g,1};\mathbb{Q})$$

$$\Rightarrow\ H^2_c(\hat{\mathfrak{h}}_{\infty,1}) \to H^2(\mathcal{H}^{\mathrm{top}}_{g,1};\mathbb{Q}) \to H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$$

$H^2_c(\hat{\mathfrak{h}}_{\infty,1}) \ni \mathfrak{t}_{2k+1}$ (Lie algebra version) $\mapsto \tilde{\mathfrak{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$

Conant-Kassabov-Vogtmann defined more classes on the LHS,

but, at present, only $\mathfrak{t}_3, \mathfrak{t}_5, \mathfrak{t}_7$ are known to be non-trivial...

Prospect (1)

only known homomorphism(s) (Frøyshov and Ozsváth-Szabó)

$$\mathbf{\Theta}^3 \to \mathbb{Z}$$

candidate: Neumann-Siebenmann, Fukumoto-Furuta-Ue, Saveliev

$$\nu := \sum_{i=0}^{7} (-1)^{\frac{i(i+1)}{2}} \mathrm{rank}\, HF^i \text{ (instanton Floer homology)}$$

recall:

Theorem (Taubes)

$\sum_{i=0}^{7} (-1)^i \mathrm{rank}\, HF^i = 2\lambda$ *(Casson invariant)*

Theorem (Manolescu)

The Rohlin homomorphism $\mathbf{\Theta}^3 \to \mathbb{Z}/2$ *does not split*

Prospect (2)

geometric meaning of the classes $\tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q})$:

Intersection numbers of higher and higher Massey products

(using works of Kitano, Garoufalidis-Levine)

Conjecture

The homomorphism $H^1(\mathbf{\Theta}^3;\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}} \to H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$ *induced by*

$$0 \to \mathbf{\Theta}^3 \to \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \overline{\mathcal{H}}_{g,1} \to 1$$

is highly non-trivial (possibly injective) and its image contains the classes $\tilde{\mathbf{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \ \Rightarrow$

$$\tilde{\mathbf{t}}_{2k+1} \neq 0 \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q}) \ \text{ and}$$
$$\tilde{\mathbf{t}}_{2k+1} = 0 \in H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})$$

Prospect (3)

If Conjecture is true $\Rightarrow$ obtain homomorphisms

$$\nu_k : \mathbf{\Theta}^3 \to \mathbb{Q} \quad (k = 1, 2, \ldots)$$

homology cobordism invariants