A normal generating set for the Torelli group of a compact non-orientable surface

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Contents

Orientable surface

Σ_g^b : a genus g compact orientable surface with b boundary components.
The mapping class group of Σ_g^b is defined as

$$\mathcal{M}(\Sigma_g^b) = \{f : \Sigma_g^b \xrightarrow{\text{diffeo.}} \Sigma_g^b \mid \text{ori.-pres.}, f|_{\partial\Sigma_g^b} = \text{id}\}/\text{isotopy}.$$

The Torelli group of Σ_g^b is defined as

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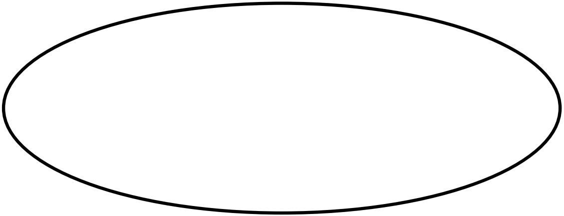

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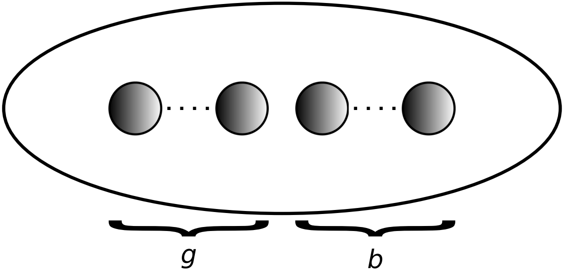

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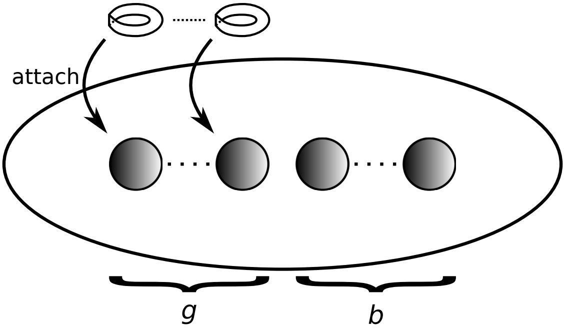

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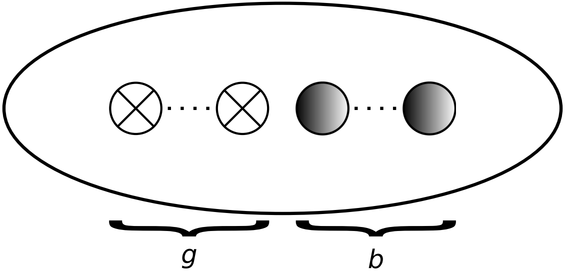

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- A generating set for $\mathcal{I}(\Sigma_g^0)$ was found by Powell (1978).
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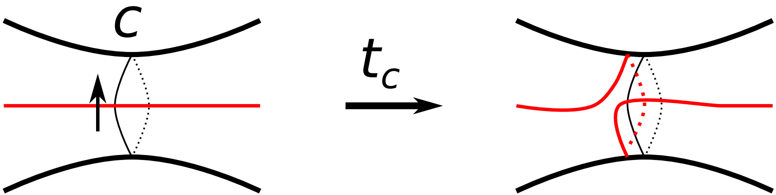
Dehn twist

For a two sided simple closed curve c, the Dehn twist t_c is defined as

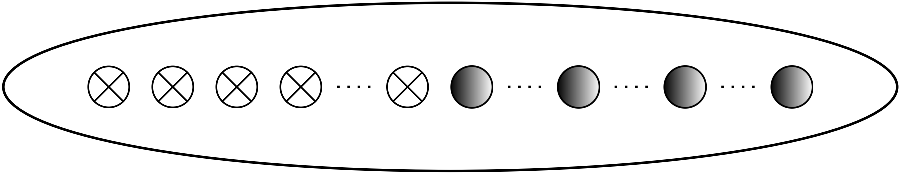


Main result

Theorem (Hirose-K. ($b = 0$), K. ($b \geq 1$))

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_α, $t_\beta t_{\beta'}^{-1}$,
- t_{δ_i}, t_{ρ_i} ($1 \leq i \leq b-1$),
- $t_{\sigma_{ij}}$, $t_{\bar{\sigma}_{ij}}$ ($1 \leq i < j \leq b-1$) and
- t_γ (only if $g = 4$).



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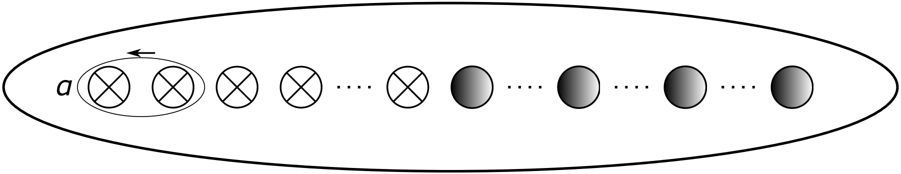

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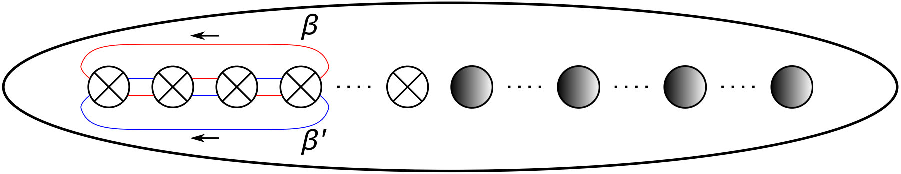

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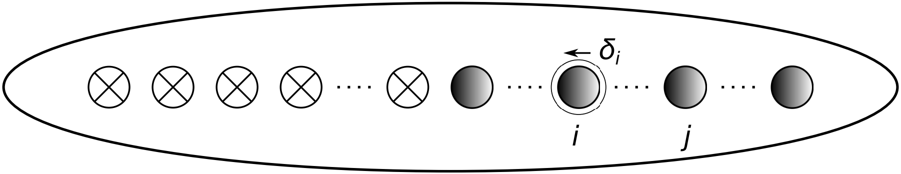

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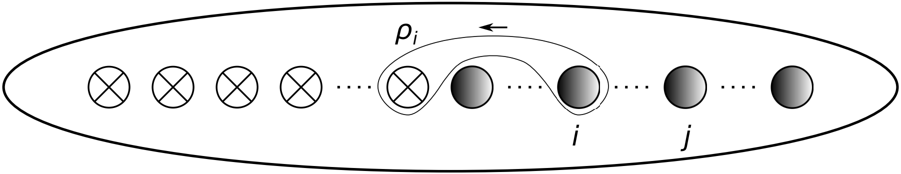

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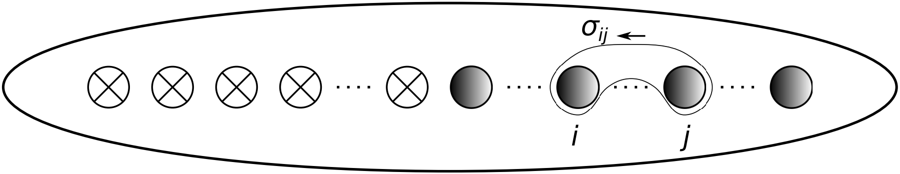

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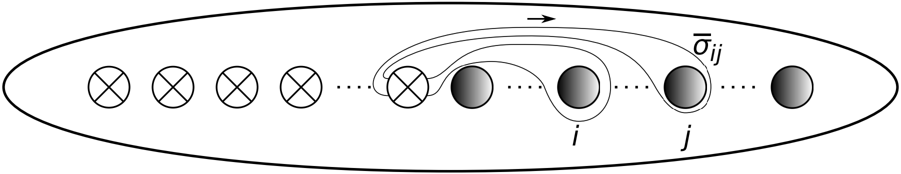

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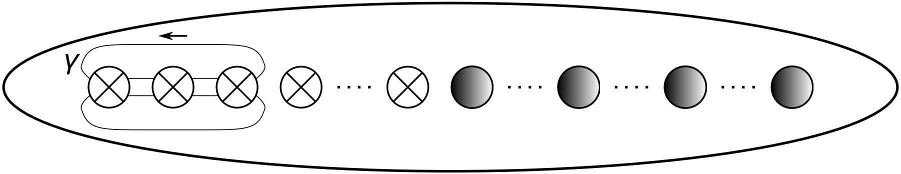

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The case of a closed surface

Theorem (Hirose-K.)

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The level 2 mapping class group of N_g^b is defined as

$$\Gamma_2(N_g^b) = \ker(\mathcal{M}(N_g^b) \to \mathrm{Aut}(H_1(N_g^b; \mathbb{Z}/2\mathbb{Z}))).$$

The level-2 principal congruence subgroup of $GL(n; \mathbb{Z})$ is defined as

$$\Gamma_2(n) = \ker(GL(n; \mathbb{Z}) \to GL(n; \mathbb{Z}/2\mathbb{Z})).$$

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Lemma

We have the short exact sequence

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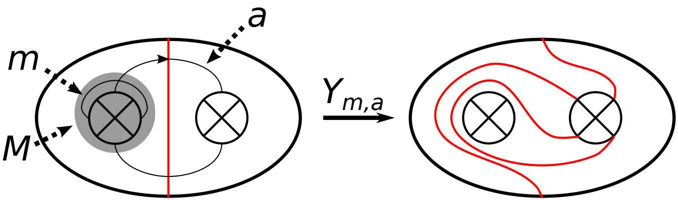
In general, if there is a short exact sequence

$$1 \to G \to \langle X \mid Y \rangle \xrightarrow{\phi} \langle \phi(X) \mid Z \rangle \to 1,$$

then G is normally generated by $\{\tilde{z} \mid \phi(\tilde{z}) \in Z\}$.

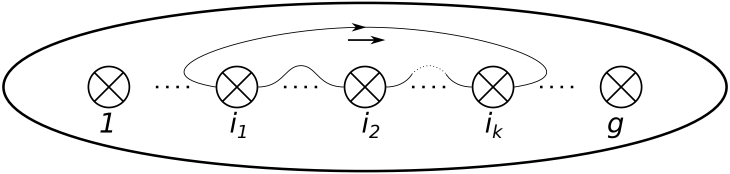
Crosscap slide

m : a one sided simple closed curve,
a : a two sided oriented simple closed curve,
(m and a are intersect transversely at only one point)
M : a regular neighborhood of m.
The crosscap slide $Y_{m,a}$ is defined as



Generating sets for $\Gamma_2(N_g^0)$

For $1 \leq i_1 < i_2 < \cdots < i_k \leq g$, $\alpha_{i_1,\ldots,i_k}$ is defined as



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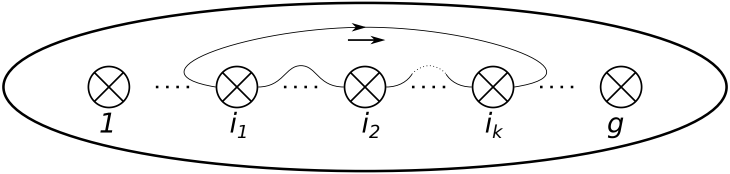

Theorem (Szepietowski (2013))

For $g \geq 4$, $\Gamma_2(N_g^0)$ is finitely generated by

1. $Y_{\alpha_i,\alpha_{i,j}}$ *for* $1 \leq i \leq g-1$, $1 \leq j \leq g$ *and* $i \neq j$,
2. $t^2_{\alpha_{i,j,k,l}}$ *for* $1 \leq i < j < k < l \leq g$.

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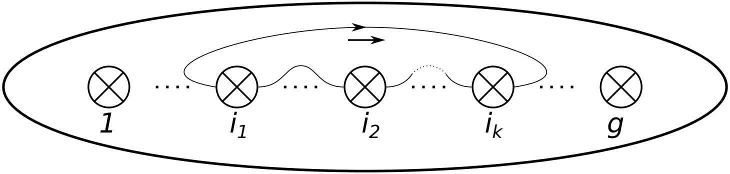

Theorem (Hirose-Sato (2014))

For $g \geq 4$, $\Gamma_2(N_g^0)$ is minimally generated by

1. $Y_{\alpha_i,\alpha_{i,j}}$ *for* $1 \leq i \leq g-1$, $1 \leq j \leq g$ *and* $i \neq j$,
2. $t^2_{\alpha_{1,j,k,l}}$ *for* $1 < j < k < l \leq g$.

Presentations for $\Gamma_2(n)$

$$E_{ij} = \begin{array}{c} \\ \\ i \\ \\ \\ \\ \\ \end{array} \!\!\! \begin{array}{c} \begin{array}{ccccccc} & & & & j & & \end{array} \\ \begin{pmatrix} 1 & & & & \vdots & & \\ & \ddots & & & \vdots & & \\ \cdots & \cdots & 1 & \cdots & 2 & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Presentations for $\Gamma_2(n)$

$$F_i = \begin{array}{cc} & \begin{array}{ccccccc} & & & i & & & \end{array} \\ i & \begin{pmatrix} 1 & & & \vdots & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \vdots & & & \\ \cdots & \cdots & \cdots & -1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Presentations for $\Gamma_2(n)$

Theorem (cf. Fullarton (2014), K. (2015))

For $n \geq 1$, $\Gamma_2(n)$ has a finite presentation with the generators E_{ij} and F_i, for $1 \leq i, j \leq n$ with $i \neq j$, and with the relators

1. F_i^2,
2. $(E_{ij}F_i)^2$, $(E_{ij}F_j)^2$, $(F_iF_j)^2$ *(when $n \geq 2$),*
3.
    1. $[E_{ij}, E_{ik}]$, $[E_{ij}, E_{kj}]$, $[E_{ij}, F_k]$, $[E_{ij}, E_{ki}]E_{kj}^2$ *(when $n \geq 3$),*
    2. $(E_{ji}E_{ij}^{-1}E_{kj}^{-1}E_{jk}E_{ik}E_{ki}^{-1})^2$ *for $i < j < k$ (when $n \geq 3$),*
4. $[E_{ij}, E_{kl}]$ *(when $n \geq 4$),*

where $1 \leq i, j, k, l \leq n$ are all different, $[X, Y] = X^{-1}Y^{-1}XY$.

$$Y_{ij} = \begin{cases} E_{ij}F_i & 1 \leq i, j \leq g-1, \\ F_i & 1 \leq i, j \leq g-1, j = g. \end{cases}$$

Then we have $\Gamma_2(N_g^0) \ni Y_{\alpha_i, \alpha_{i,j}} \mapsto Y_{ij} \in \Gamma_2(g-1)$.

$$Y_{ij} = \begin{array}{cc} & \begin{array}{ccccccc} & & & i & & j & \end{array} \\ i & \begin{pmatrix} 1 & & & \vdots & & \vdots & \\ & \ddots & & \vdots & & \vdots & \\ & & 1 & \vdots & & \vdots & \\ \cdots & \cdots & \cdots & -1 & \cdots & 2 & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

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Proposition

For $g-1 \geq 1$, $\Gamma_2(g-1)$ has a finite presentation with the generators Y_{ij} for $1 \leq i \leq g-1$ and $1 \leq j \leq g$ with $i \neq j$, and with the relators

1. Y_{ij}^2 *for* $1 \leq i \leq g-1$ *and* $1 \leq j \leq g$,
2. $[Y_{ik}, Y_{jk}]$ *for* $1 \leq i,j \leq g-1$ *and* $1 \leq k \leq g$,
3. $[Y_{ij}, Y_{ik}Y_{jk}]$ *for* $1 \leq i,j \leq g-1$ *and* $1 \leq k \leq g$,
4. $[Y_{ij}, Y_{kl}]$ *for* $1 \leq i,k \leq g-1$ *and* $1 \leq j,l \leq g$,
5. $(Y_{ij}Y_{ik}Y_{il})^2$ *for* $1 \leq i \leq g-1$ *and* $1 \leq j,k,l \leq g$,
6. $(Y_{ji}Y_{ij}Y_{kj}Y_{jk}Y_{ik}Y_{ki})^2$ *for* $1 \leq i,j,k \leq g-1$,

where $[X,Y] = X^{-1}Y^{-1}XY$ and i,j,k,l are all different.

Remark

For $g \geq 4$, $\Gamma_2(N_g^0)$ is generated by

1. $Y_{\alpha_i,\alpha_{i,j}}$ for $1 \leq i \leq g-1$, $1 \leq j \leq g$ and $i \neq j$,
2. $t^2_{\alpha_{1,j,k,l}}$ for $1 < j < k < l \leq g$.

$\Gamma_2(g-1)$ is generated by Y_{ij} and T_{1jkl}, and has the relators

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3. $[Y_{ij}, Y_{ik}Y_{jk}]$ for $1 \leq i,j \leq g-1$ and $1 \leq k \leq g$,
4. $[Y_{ij}, Y_{kl}]$ for $1 \leq i,k \leq g-1$ and $1 \leq j,l \leq g$,
5. $(Y_{ij}Y_{ik}Y_{il})^2$ for $1 \leq i \leq g-1$ and $1 \leq j,k,l \leq g$,
6. $(Y_{ji}Y_{ij}Y_{kj}Y_{jk}Y_{ik}Y_{ki})^2$ for $1 \leq i,j,k \leq g-1$,
7. $T_{1jkl} \cdot$ (a product of Y_{ij}'s).

$(t^2_{\alpha_{i,j,k,l}} \mapsto T_{ijkl})$

Remark

For $g \geq 4$, $\Gamma_2(N_g^0)$ is generated by

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7. $T_{1jkl} \cdot$ (a product of Y_{ij}'s).

$$1 \to \mathcal{I}(N_g^0) \to \Gamma_2(N_g^0) \to \Gamma_2(g-1) \to 1$$

Let $Y_{\alpha_i,\alpha_{i,j}} = Y_{i;j}$ and $t^2_{\alpha_{i,j,k,l}} = T_{i,j,k,l}$.

Corollary

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by followings in $\Gamma_2(N_g^0)$,

1. $Y_{i;j}^2$ *for* $1 \leq i \leq g-1$ *and* $1 \leq j \leq g$,
2. $[Y_{i;k}, Y_{j;k}]$ *for* $1 \leq i,j \leq g-1$ *and* $1 \leq k \leq g$,
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We checked that these are products of conjugations of t_α, $t_\beta t_{\beta'}^{-1}$ and t_γ.

We have

Theorem (Hirose-K. (2016))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by

- t_α, $t_\beta t_{\beta'}^{-1}$ *and*
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The case of a surface with boundary

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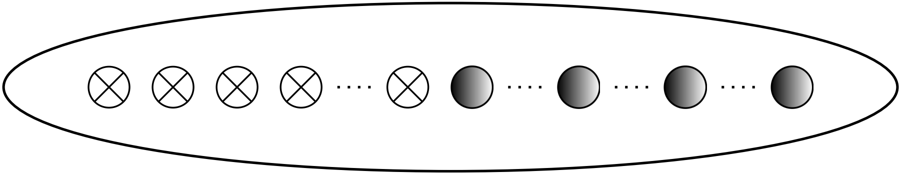

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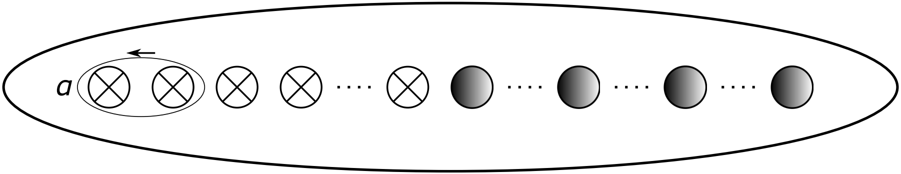

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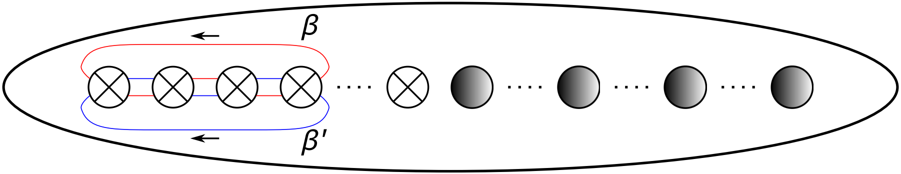

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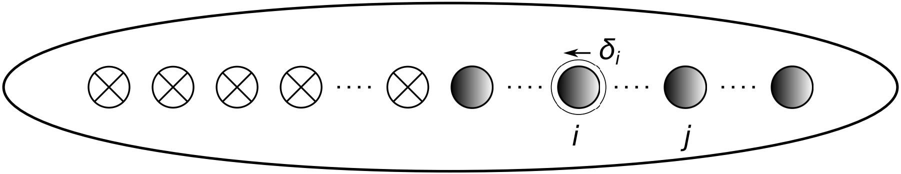

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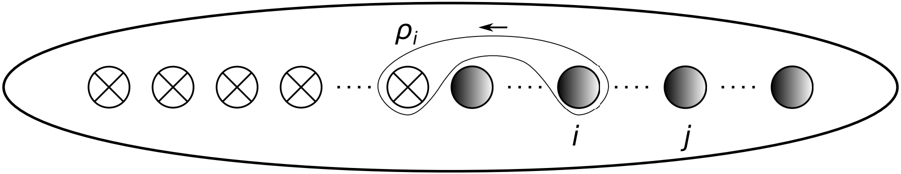

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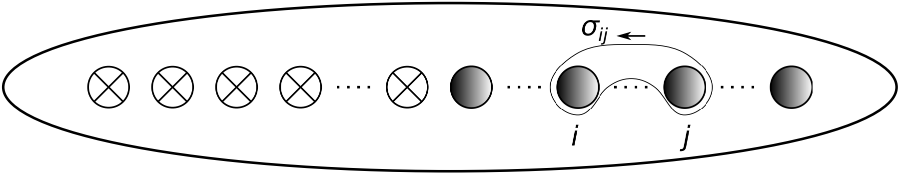

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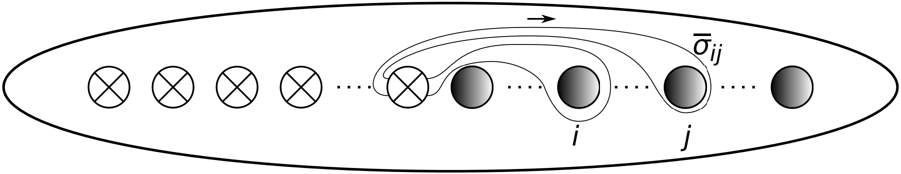

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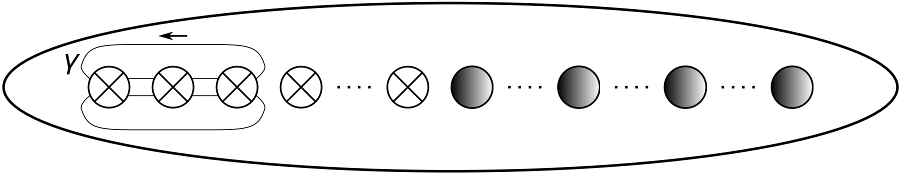

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Capping homomorphisms

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\})/\text{isotopy}$$

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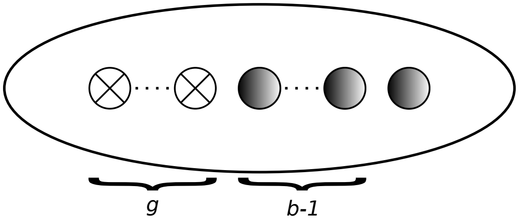

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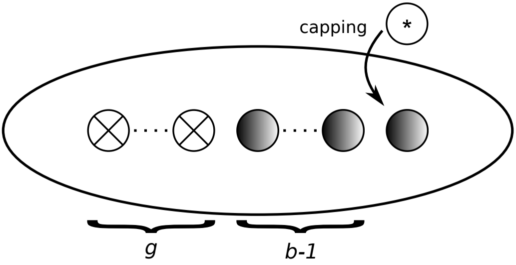

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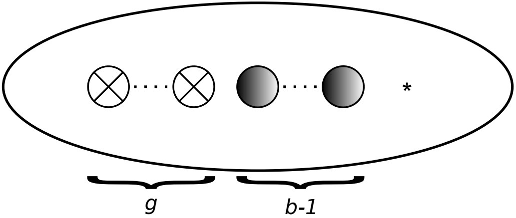

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Remark

- $\ker \mathcal{C}_g^b$ is generated by t_{δ_b}.
- $\ker \mathcal{C}_g^b|_{\mathcal{I}(N_g^b)}$ is generated by t_{δ_b}.
- $\mathcal{C}_g^b$ and $\mathcal{C}_g^b|_{\mathcal{I}(N_g^b)}$ are not surjective.

Pushing and Forgetful homomorphisms

- $\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$: the pushing homomorphism.
- $\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$: the forgetful homomorphism.

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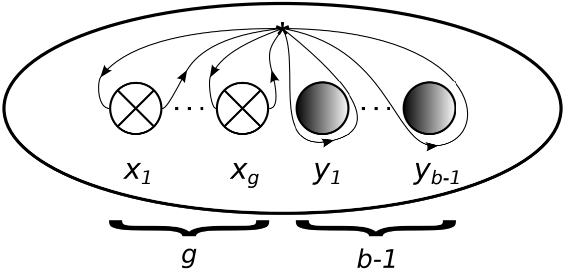

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Remark

We have short exact sequences

$$\begin{array}{ccccccc} \pi_1(N_g^{b-1}, *) & \xrightarrow{\mathcal{P}_g^{b-1}} & \mathcal{M}(N_g^{b-1}, *) & \xrightarrow{\mathcal{F}_g^{b-1}} & \mathcal{M}(N_g^{b-1}) & \longrightarrow & 1, \\ \pi_1(N_g^{b-1}, *) & \longrightarrow & \mathcal{I}(N_g^{b-1}, *) & \longrightarrow & \mathcal{I}(N_g^{b-1}) & \longrightarrow & 1. \end{array}$$

$$\mathcal{I}(N_g^{b-1}, *) = \ker(\mathcal{M}(N_g^{b-1}, *) \to \mathrm{Aut}(H_1(N_g^b; \mathbb{Z})))$$

By the short exact sequence

$$1 \to \ker \mathcal{C}_g^b|_{\mathcal{I}(N_g^b)} \to \mathcal{I}(N_g^b) \to \mathcal{C}_g^b(\mathcal{I}(N_g^b)) \to 1,$$

$\mathcal{I}(N_g^b)$ is normally generated by

- lifts of normal generators of $\mathcal{C}_g^b(\mathcal{I}(N_g^b))$ in $\mathcal{C}_g^b(\mathcal{M}(N_g^b))$ and
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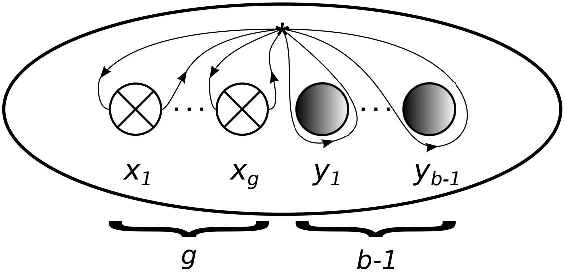
x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.
$p : \pi_1(N_g^{b-1}, *) \to \pi_1(N_g^0, *)$: the projection (w/ $p(x_i) = x_i$, $p(y_j) = 1$).
For $x \in \pi_1(N_g^{b-1}, *)$ we can denote $p(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_t}^{\varepsilon_t}$ ($\varepsilon_k = \pm 1$).



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$$\begin{aligned} O_i(x) &= \sharp\{i_{2k-1}\mid i_{2k-1}=i\}, \\ E_i(x) &= \sharp\{i_{2k}\mid i_{2k}=i\}, \\ \Gamma_g^{b-1} &= \{x\in\pi_1(N_g^{b-1},*)\mid O_i(x)=E_i(x), 1\le i\le g\} \end{aligned}$$

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Example

$x = x_1 y_2 x_2 x_3^{-1} y_5 y_1^{-2} x_1 x_2^{-1} y_4^3 x_3^{-1} \Longrightarrow p(x) = x_1 x_2 x_3^{-1} x_1 x_2^{-1} x_3^{-1}$.
$O_1(x) = E_1(x) = 1$, $O_2(x) = E_2(x) = 1$, $O_3(x) = E_3(x) = 1$,
$O_i(x) = E_i(x) = 0$ $(i \geq 4)$.
$\rightsquigarrow x \in \Gamma_g^{b-1}$.

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Example

$x = x_1 y_2 x_2 x_3^{-1} y_5 y_1^{-2} x_1 x_2^{-1} y_4^3 x_3^{-1} \Longrightarrow p(x) = x_1 x_2 x_3^{-1} x_1 x_2^{-1} x_3^{-1}$.
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$\rightsquigarrow x \in \Gamma_g^{b-1}$.

Proposition

1. $\mathcal{F}_g^{b-1}(\mathcal{C}_g^b(\mathcal{I}(N_g^b))) = \mathcal{I}(N_g^{b-1})$.
2. $\ker(\mathcal{C}_g^b(\mathcal{I}(N_g^b)) \to \mathcal{I}(N_g^{b-1})) = \mathcal{P}_g^{b-1}(\Gamma_g^{b-1})$.
3. $\mathcal{P}_g^{b-1}(\Gamma_g^{b-1})$ *is the normal closure of* $\mathcal{P}_g^{b-1}(x_g^2)$, $\mathcal{P}_g^{b-1}(y_j)$ *and* $\mathcal{P}_g^{b-1}(x_g y_j x_g^{-1})$ *(*$1 \leq j \leq b-1$*) in* $\mathcal{C}_g^b(\mathcal{M}(N_g^b))$.

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$$\begin{array}{ccccc}
\mathcal{M}(N_g^b) & \overset{\mathcal{C}_g^b}{\to} & \mathcal{M}(N_g^{b-1}, *) & \overset{\mathcal{F}_g^{b-1}}{\to} & \mathcal{M}(N_g^{b-1}) \\
\cup & & \cup & & \cup \\
\mathcal{I}(N_g^b) & \twoheadrightarrow & \mathcal{C}_g^b(\mathcal{I}(N_g^b)) & \twoheadrightarrow & \mathcal{I}(N_g^{b-1})
\end{array}$$

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$$
\begin{array}{ccccccc}
\pi_1(N_g^{b-1}, *) & \overset{\mathcal{P}_g^{b-1}}{\rightarrow} & \mathcal{I}(N_g^{b-1}, *) & \overset{\mathcal{F}_g^{b-1}}{\rightarrow} & \mathcal{I}(N_g^{b-1}) & \rightarrow & 1 \\
\cup & & \cup & & & & \\
\pi_1^+(N_g^{b-1}, *) & \overset{\mathcal{P}_g^{b-1}}{\rightarrow} & \mathcal{C}_g^b(\mathcal{I}(N_g^b)) & \overset{\mathcal{F}_g^{b-1}}{\rightarrow} & \mathcal{I}(N_g^{b-1}) & \rightarrow & 1 \\
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Proposition

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3. *$\mathcal{P}_g^{b-1}(\Gamma_g^{b-1})$ is the normal closure of $\mathcal{P}_g^{b-1}(x_g^2)$, $\mathcal{P}_g^{b-1}(y_j)$ and $\mathcal{P}_g^{b-1}(x_g y_j x_g^{-1})$ $(1 \leq j \leq b-1)$ in $\mathcal{C}_g^b(\mathcal{M}(N_g^b))$.*

Proposition

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$$\begin{array}{ccccccc}
\pi_1(N_g^{b-1},*) & \overset{\mathcal{P}_g^{b-1}}{\to} & \mathcal{M}(N_g^{b-1},*) & \overset{\mathcal{F}_g^{b-1}}{\to} & \mathcal{M}(N_g^{b-1}) & \to & 1 \\
\cup & & \cup & & \cup & & \\
\Gamma_g^{b-1} & \to & \mathcal{C}_g^b(\mathcal{I}(N_g^b)) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1
\end{array}$$

Proposition

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Corollary

$\mathcal{C}_g^b(\mathcal{I}(N_g^b))$ *is normally generated by* $\mathcal{P}_g^{b-1}(x_g^2)$, $\mathcal{P}_g^{b-1}(y_j)$, $\mathcal{P}_g^{b-1}(x_g y_j x_g^{-1})$ *(*$1 \leq j \leq b-1$*) and lifts by* $\mathcal{F}_g^{b-1}$ *of normal generators of* $\mathcal{I}(N_g^{b-1})$*, in* $\mathcal{C}_g^b(\mathcal{M}(N_g^b))$.

$$1 \to \ker \mathcal{C}_g^b|_{\mathcal{I}(N_g^b)} \to \mathcal{I}(N_g^b) \to \mathcal{C}_g^b(\mathcal{I}(N_g^b)) \to 1$$

$\mathcal{I}(N_g^b)$ is normally generated by

- t_{δ_b} and
- lifts by $\mathcal{C}_g^b$ of normal generators of $\mathcal{C}_g^b(\mathcal{I}(N_g^b))$.

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  - lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

$$1 \to \ker \mathcal{C}_g^b|_{\mathcal{I}(N_g^b)} \to \mathcal{I}(N_g^b) \to \mathcal{C}_g^b(\mathcal{I}(N_g^b)) \to 1$$

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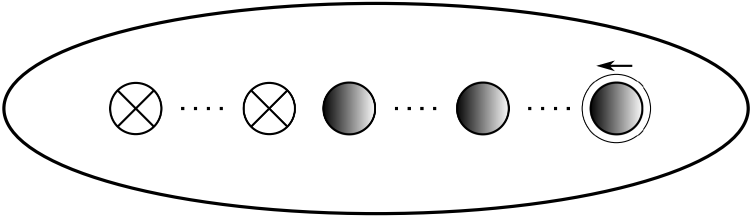

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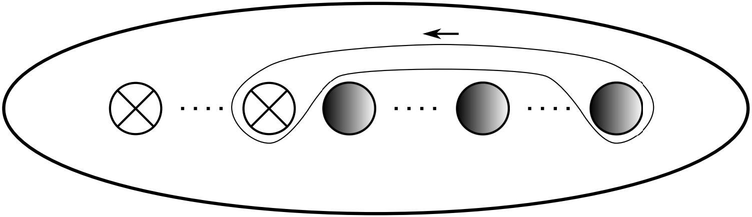

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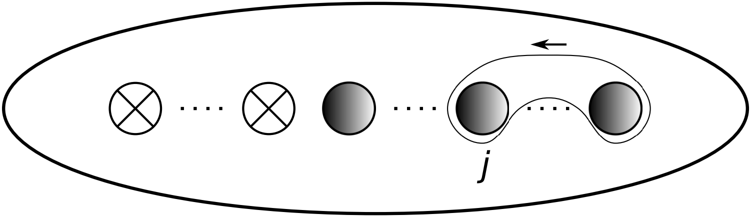

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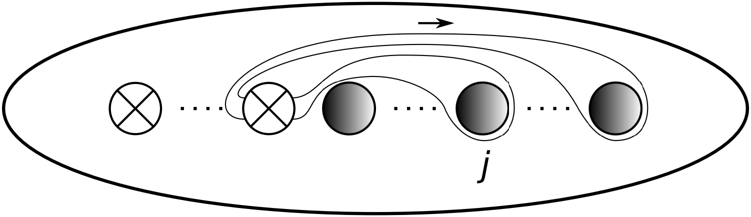

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$\mathcal{I}(N_g^b)$ is normally generated by

- t_{δ_b}, t_{ρ_b}, $t_{\sigma_{jb}}$, $t_{\bar{\sigma}_{jb}}$ and
- lifts by $\mathcal{F}_g^{b-1} \circ \mathcal{C}_g^b$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

Theorem (Hirose-K.)

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by t_α, $t_\beta t_{\beta\prime}^{-1}$ (and t_γ).

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$\vdots$

Theorem (K.)

For $g \geq 4$ and $b \geq 1$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_α, $t_\beta t_{\beta'}^{-1}$,
- t_{δ_i}, t_{ρ_i} *($1 \leq i \leq b$),*
- $t_{\sigma_{ij}}$, $t_{\bar{\sigma}_{ij}}$ *($1 \leq i < j \leq b$) and*
- t_γ *(only if $g = 4$).*

Theorem (Hirose-K.)

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by t_α, $t_\beta t_{\beta\prime}^{-1}$ (and t_γ).

$\mathcal{I}(N_g^1)$ is normally generated by t_α, $t_\beta t_{\beta\prime}^{-1}$, t_{δ_1}, t_{ρ_1} (and t_γ).
$\mathcal{I}(N_g^2)$ is normally generated by t_α, $t_\beta t_{\beta\prime}^{-1}$, t_{δ_1}, t_{ρ_1}, t_{δ_2}, t_{ρ_2}, $t_{\sigma_{12}}$, $t_{\bar{\sigma}_{12}}$ (and t_γ).

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Thank you for your attention!