Johnson homomorphisms up to degree 6

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Mapping class groups

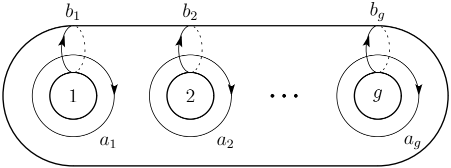
- Σ_g: a closed oriented connected surface of genus g
- $\mathcal{M}_g := \mathrm{Diff}_+\Sigma_g/(\text{isotopy}) = \pi_0\mathrm{Diff}_+\Sigma_g$
  : the mapping class group of Σ_g
- $H_\mathbb{Z} := H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$
- Intersection form on $H_\mathbb{Z}$:

$$\mu : H_\mathbb{Z} \otimes H_\mathbb{Z} \longrightarrow \mathbb{Z} \qquad \begin{pmatrix} \text{non-degenerate} \\ \text{skew-symmetric} \end{pmatrix}$$

- Poincaré duality:

$$H_\mathbb{Z} := H_1(\Sigma_g; \mathbb{Z}) = H_1(\Sigma_g; \mathbb{Z})^* = H^1(\Sigma_g; \mathbb{Z}) = H_\mathbb{Z}^*.$$

- Fix a symplectic basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of $H_{\mathbb{Z}}$ w.r.t. μ:



- symplectic element (class):

$$
\begin{aligned}
\omega_0 &= \sum_{i=1}^{g} (a_i \otimes b_i - b_i \otimes a_i) \in H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \\
&= \sum_{i=1}^{g} a_i \wedge b_i \in \wedge^2 H_{\mathbb{Z}}.
\end{aligned}
$$

- $\mathrm{Sp}(H_{\mathbb{Z}}) \cong \mathrm{Sp}(2g, \mathbb{Z})$: symplectic group,

$$\mathrm{Sp}(H_{\mathbb{Z}}) \curvearrowright H_{\mathbb{Z}} \qquad \mu\text{-preserving } (\omega_0\text{-preserving}) \text{ action.}$$

- $\mathcal{M}_g$ acts on $H_{\mathbb{Z}}$ with preserving μ. This gives

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathcal{M}_g \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1 \qquad \text{(exact)}$$

where $\mathcal{I}_g$ is called the Torelli group.

We also consider

- $\Sigma_{g,1}$: a compact oriented connected surface of genus g
  w/ one boundary component

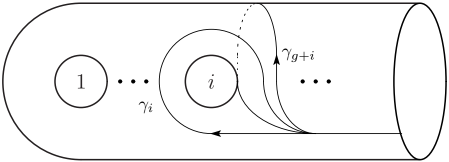
- $\mathcal{M}_{g,1} := \mathrm{Diff}(\Sigma_{g,1} \,\mathrm{rel}\, \partial\Sigma_{g,1})/(\mathrm{isotopy})$
  : the mapping class group of $\Sigma_{g,1}$

- $H_1(\Sigma_{g,1}, \mathbb{Z}) = H_{\mathbb{Z}} \cong \mathbb{Z}^{2g}$

- Corresponding Torelli group:

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1 \qquad \text{(exact)}$$

- $\pi_1\Sigma_{g,1} = \langle \gamma_1, \gamma_2, \ldots, \gamma_{2g} \rangle = F_{2g}$, where



$\zeta := \prod_{i=1}^{g} [\gamma_i, \gamma_{g+i}]$ is the boundary loop.

- $\pi_1\Sigma_{g,1} \twoheadrightarrow \pi_1\Sigma_g = \langle \gamma_1, \gamma_2, \ldots, \gamma_{2g} \rangle / \langle \zeta \rangle$

- $\mathcal{M}_{g,1}$ acts naturally on $\pi_1\Sigma_{g,1}$:

$$\sigma : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}\,(\pi_1\Sigma_{g,1}),$$
$$\overline{\sigma} : \mathcal{M}_{g} \longrightarrow \mathrm{Out}\,(\pi_1\Sigma_{g}) := \mathrm{Aut}\,(\pi_1\Sigma_{g})/\mathrm{Inn}\,(\pi_1\Sigma_{g})$$

Theorem [Dehn, Nielsen, Baer, Epstein, Zieschang et al.]

The homomorphisms σ and $\overline{\sigma}$ are injective and

$$\mathrm{Im}\,\sigma = \{\varphi \in \mathrm{Aut}\,(\pi_1\Sigma_{g,1}) \mid \varphi(\zeta) = \zeta\},$$
$$\mathrm{Im}\,\overline{\sigma} = \mathrm{Out}_+(\pi_1\Sigma_g) : \text{ (orientation-preserving).}$$

Johnson homomorphisms

In the following, we mainly focus on the $\mathcal{M}_{g,1}$-case.

- $\mathcal{I}_{g,1}$ measures the gap between $\mathcal{M}_{g,1}$ and $\mathrm{Sp}(2g,\mathbb{Z})$.
- It is known that

$$H_1(\mathcal{M}_{g,1}) = \mathcal{M}_{g,1}/[\mathcal{M}_{g,1},\mathcal{M}_{g,1}] = 0 \quad \text{for } g \geq 3.$$

  $\rightsquigarrow$ It is not easy to make an "approximation" of $\mathcal{M}_{g,1}$ without looking the structure of $\mathcal{I}_{g,1}$.

- The structure of $\mathcal{I}_{g,1}$ is more complicated than that of $\mathcal{M}_{g,1}$.

In a series of papers, Dennis Johnson showed:

Theorem [Johnson]

1. $\mathcal{I}_{g,1}$ is finitely generated for $g \geq 3$.
2. (The first Johnson homomorphism)
   There exists an $\mathcal{M}_{g,1}$-equivariant homomorphism
   $$\tau_{g,1}(1) : \mathcal{I}_{g,1} \longrightarrow\!\!\!\!\!\rightarrow \wedge^3 H_{\mathbb{Z}}.$$
   Dehn twists along BSCC form a generating system of $\mathrm{Ker}\,\tau_{g,1}(1)$.
3. $\tau_{g,1}(1)$ gives the abelianization $H_1(\mathcal{I}_{g,1}) = \mathcal{I}_{g,1}/[\mathcal{I}_{g,1}, \mathcal{I}_{g,1}]$ modulo 2-torsions.
   (The torsion part is given by Birman-Craggs homormophisms.)

- Putman gave another proof for the above facts.

Morita's generalization

- $\pi := \pi_1(\Sigma_{g,1}) = \langle \gamma_1, \gamma_2, \dots, \gamma_{2g} \rangle$.
- $\pi = \Gamma_1(\pi) \supset \Gamma_2(\pi) \supset \Gamma_3(\pi) \supset \cdots$

  : The lower central series of π defined by

$$\Gamma_{i+1}(\pi) = [\Gamma, \Gamma_i(\pi)] \qquad \text{for } i \geq 1.$$

- $\mathcal{L}(H_{\mathbb{Z}}) = \bigoplus_{i=1}^{\infty} \mathcal{L}_i(H_{\mathbb{Z}})$: the free Lie algebra generated by $H_{\mathbb{Z}}$

$$a \in \mathcal{L}_1(H_{\mathbb{Z}}) = H_{\mathbb{Z}},$$
$$[a, b] \in \mathcal{L}_2(H_{\mathbb{Z}}) \cong \wedge^2 H_{\mathbb{Z}},$$
$$[a, [b, c]] \in \mathcal{L}_3(H_{\mathbb{Z}}) \cong (H_{\mathbb{Z}} \otimes (\wedge^2 H_{\mathbb{Z}})) / \wedge^3 H_{\mathbb{Z}},$$
$$\vdots$$

Fact

There exists an $\mathcal{M}_{g,1}$-equivariant isomorphism

$$\begin{array}{ccc} \Gamma_i(\pi)/\Gamma_{i+1}(\pi) & \stackrel{\cong}{\longrightarrow} & \mathcal{L}_i(H_{\mathbb{Z}}) \\ \uplus & & \uplus \\ [\alpha_1, [\alpha_2, \cdots, \alpha_i]] \cdots] & \longmapsto & [\overline{\alpha_1}, [\overline{\alpha_2}, \cdots, \overline{\alpha_i}]] \cdots] \end{array}$$

where $\pi \ni \alpha_j \longmapsto \overline{\alpha_j} \in H_{\mathbb{Z}}$.

- Iterating expansion

$$[X, Y] \longmapsto X \otimes Y - Y \otimes X$$

gives an (degree preserving) embedding $\mathcal{L}(H_{\mathbb{Z}}) \hookrightarrow \bigoplus_{i=1}^{\infty} H_{\mathbb{Z}}^{\otimes i}$.

- $\mathcal{M}_{g,1} \subset \mathrm{Aut}(\pi) \curvearrowright \Gamma_i(\pi)$ for $i \geq 1$.

$\rightsquigarrow \ \mathcal{M}_{g,1} \curvearrowright \pi/\Gamma_i(\pi) \qquad (\pi/\Gamma_2(\pi) = H_{\mathbb{Z}})$

Definition (Johnson filtration)

$$\mathcal{M}_{g,1}[0] = \mathcal{M}_{g,1} \supset \mathcal{M}_{g,1}[1] = \mathcal{I}_{g,1} \supset \mathcal{M}_{g,1}[2] \supset \mathcal{M}_{g,1}[3] \supset \cdots,$$

where

$$\mathcal{M}_{g,1}[k] := \mathrm{Ker}\left(\sigma_k : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}(\pi/\Gamma_{k+1}(\pi))\right).$$

Definition (The k-th Johnson homomorphism)

We have an $\mathcal{M}_{g,1}$-equivariant homomorphism defined by

$$\begin{array}{rccc} \tau_{g,1}(k): & \mathcal{M}_{g,1}[k] & \longrightarrow & \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_{k+1}(H_{\mathbb{Z}})) \\ & \cup & & \cup \\ & f & \longmapsto & (\overline{\gamma} \mapsto [f(\gamma)\gamma^{-1}]) \end{array}$$

where $[f(\gamma)\gamma^{-1}] \in \Gamma_{k+1}(\pi)/\Gamma_{k+2}(\pi) = \mathcal{L}_{k+1}(H_{\mathbb{Z}})$.

- By definition,

$$\mathrm{Ker}\,\tau_{g,1}(k) = \mathcal{M}_{g,1}[k+1],$$
$$\mathrm{Im}\,\tau_{g,1}(k) = \mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1].$$

- $\mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_{k+1}(H_{\mathbb{Z}})) = H_{\mathbb{Z}}^* \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}}) \overset{\mathrm{PD}}{=\!=} H_{\mathbb{Z}} \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$.

Theorem [Morita]

1. The image of $\tau_k : \mathcal{M}_{g,1}[k] \to H_{\mathbb{Z}} \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$ is included in

$$\mathfrak{h}_{g,1}(k) := \mathrm{Ker}\left(H_{\mathbb{Z}} \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}}) \xrightarrow{[\cdot,\cdot]} \mathcal{L}_{k+2}(H_{\mathbb{Z}}) \right).$$

2. The direct sums

$$\mathrm{Im}\,\tau_{g,1} := \bigoplus_{k=1}^{\infty} \mathrm{Im}\,\tau_{g,1}(k) \quad \text{and} \quad \mathfrak{h}_{g,1}^{+} := \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$$

have natural positively graded Lie algebra structures and

$$\tau_{g,1} := \bigoplus_{k=1}^{\infty} \tau_{g,1}(k) : \mathrm{Im}\,\tau_{g,1} \longrightarrow \mathfrak{h}_{g,1}^{+}$$

is a Lie algebra embedding.

Problem

Determine:

(I) the Lie subalgebra $\operatorname{Im}\tau_{g,1} = \bigoplus_{k=1}^{\infty} \operatorname{Im}\tau_{g,1}(k)$ of $\mathfrak{h}_{g,1}^{+}$.

(II) the abelianization

$$H_1(\mathfrak{h}_{g,1}^{+}) = \mathfrak{h}_{g,1}^{+}/[\mathfrak{h}_{g,1}^{+}, \mathfrak{h}_{g,1}^{+}] = \bigoplus_{k=1}^{\infty} H_1(\mathfrak{h}_{g,1}^{+})_k.$$

of $\mathfrak{h}_{g,1}^{+}$, where

$$\begin{cases} H_1(\mathfrak{h}_{g,1}^{+})_1 = \mathfrak{h}_{g,1}(1) \\ H_1(\mathfrak{h}_{g,1}^{+})_k = \mathfrak{h}_{g,1}(k) \Bigg/ \sum\limits_{\substack{i+j=k \\ i,j\geq 1}} [\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(j)] \qquad (k \geq 2). \end{cases}$$

Remarks

- In the following, we consider the rational ($\mathbb{Q}$-)version:

$$H := H_1(\Sigma_g;\mathbb{Q}) = H_{\mathbb{Z}} \otimes \mathbb{Q}$$
$$\tau_{g,1} \otimes \mathbb{Q} : \operatorname{Im} \tau_{g,1} \otimes \mathbb{Q} \longrightarrow \mathfrak{h}^+_{g,1} \otimes \mathbb{Q}$$

  For simplicity, we omit "$\otimes\mathbb{Q}$".

- By using the Maguns expansion (and its generalization),

  Kitano, Kawazumi, Massuyeau

  gave other ways to define $\tau_{g,1}$.

- Kawazumi-Kuno gave a geometric description of $\tau_{g,1}$ by using the completed Goldman Lie algebra.

Related theory

- $\mathrm{Aut}\, F_n$: Nielsen, Magnus, Andreadakis, T.Satoh
- Link theory: Milnor, Habegger-Lin, Orr, Habegger-Masbaum, Meilhan-Yasuhara
- Number theory: Ihara, Oda, Nakamura, Hain, Matsumoto, Asada, Kaneko, Takao

In this workshop, we shall see the relationship among them!

Important tools

(I) Representation theory of $\mathrm{Sp}(2g,\mathbb{Q})$

- The actions of $\mathcal{M}_{g,1}$ on $\mathrm{Im}\,\tau_{g,1}$ and $\mathfrak{h}^+_{g,1}$ descend to those of $\mathrm{Sp}(2g,\mathbb{Z}) = \mathcal{M}_{g,1}/\mathcal{I}_{g,1} = \mathcal{M}_{g,1}[0]/\mathcal{M}_{g,1}[1]$.

  $\rightsquigarrow$ We have an $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant embedding

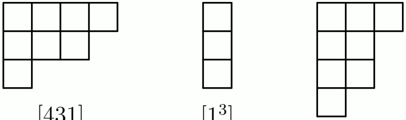
$$\tau_{g,1} : \mathrm{Im}\,\tau_{g,1} \longrightarrow \mathfrak{h}^+_{g,1}.$$

- $\mathrm{Im}\,\tau_{g,1}(k)$ and $\mathfrak{h}_{g,1}(k)$ are finite dimensional $\mathrm{Sp}(2g,\mathbb{Q})$-module.

- As pointed out by Asada-Nakamura, $\tau_{g,1}$ is in fact an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant embedding.

Fact (Representations of $\mathrm{Sp}(2g, \mathbb{Q})$)

$$\left\{ \begin{array}{l} \text{Finite dimensional irreducible} \\ \text{polynomial representations} \\ \text{of } \mathrm{Sp}(2g, \mathbb{Q}) \end{array} \right\} \overset{\cong}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Young diagrams} \\ \text{w/ } \sharp(\text{rows}) \leq g \end{array} \right\}$$



$[431]$ $[1^3]$ $[32^21]$

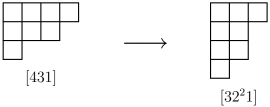
<u>Example</u>

$$\begin{aligned} \mathbb{Q} &= [0] \quad \text{(trivial representation)}, \\ H &= [1] \quad \text{(fundamental representation)}, \\ S^k H &= [k], \\ \wedge^{2k} H &= [1^{2k}] + [1^{2k-2}] + \cdots + [0], \\ \wedge^{2k+1} H &= [1^{2k+1}] + [1^{2k-1}] + \cdots + [1]. \end{aligned}$$

Irreducible representation V_λ for the Young diagram λ.

Example For $\lambda = [431]$,

1. Take the transpose $\lambda' = [32^2 1]$:



2. V_λ is the minimum $\mathrm{Sp}(2g, \mathbb{Q})$-module containing

$$v_\lambda := (a_1 \wedge a_2 \wedge a_3) \otimes (a_1 \wedge a_2) \otimes (a_1 \wedge a_2) \otimes a_1$$

in

$$(\wedge^3 H) \otimes (\wedge^2 H) \otimes (\wedge^2 H) \otimes (\wedge^1 H).$$

v_λ is called the *highest weight vector* of V_λ.

Irreducible decomposition of $H^{\otimes k}$

Fact

Any irreducible subrepresentation V_λ in $H^{\otimes k}$ can be detected by a combination of

1. contractions $\mu_{i,j} : H^{\otimes n} \longrightarrow H^{\otimes (n-2)}$,
2. projections $\wedge^n : H^{\otimes n} \longrightarrow \wedge^n H$

as a quotient representation of $H^{\otimes k}$.

(Just detect the highest weight vector v_λ.)

Example $2[21] \subset H^{\otimes 3}$ are detected by

$$\wedge_{1,2} : H^{\otimes 3} \to (\wedge^2 H) \otimes H \quad (x_1 \otimes x_2 \otimes x_3 \mapsto (x_1 \wedge x_2) \otimes x_3),$$
$$\wedge_{1,3} : H^{\otimes 3} \to (\wedge^2 H) \otimes H \quad (x_1 \otimes x_2 \otimes x_3 \mapsto (x_1 \wedge x_3) \otimes x_2).$$

In fact, two linearly independent $v_{[21]} = (a_1 \wedge a_2) \otimes a_1$ are captured by these maps:

$$\begin{array}{ll} \wedge_{1,2}(a_1 \otimes a_2 \otimes a_1) = v_{[21]}, & \wedge_{1,3}(a_1 \otimes a_2 \otimes a_1) = 0, \\ \wedge_{1,2}(a_1 \otimes a_1 \otimes a_2) = 0, & \wedge_{1,3}(a_1 \otimes a_1 \otimes a_2) = v_{[21]}. \end{array}$$

Namely,

$$\wedge_{1,2} \oplus \wedge_{1,3} : H^{\otimes 3} \longrightarrow 2[21] \subset ((\wedge^2 H) \otimes H) \oplus ((\wedge^2 H) \otimes H).$$

In our setting $\mathfrak{h}_{g,1}^{+} = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$,

- $\mathfrak{h}_{g,1}(k)$ is a finite dimensional $\mathrm{Sp}(2g, \mathbb{Q})$-module.
  $\Longrightarrow$ $\mathfrak{h}_{g,1}(k)$ has the irreducible decomposition.

- $\mathfrak{h}_{g,1}(k) \subset H \otimes \mathcal{L}_{k+1}(H) \subset H^{\otimes(k+2)}$ ($\mathrm{Sp}(2g, \mathbb{Q})$-submodule).
  $\Longrightarrow$ The irreducible decomposition of $\mathfrak{h}_{g,1}(k)$ is obtained by combinations of contractions and projections in $H^{\otimes(k+2)}$.

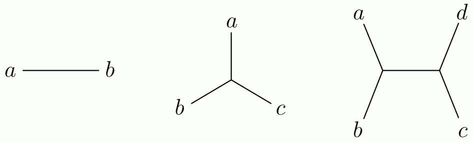
- We may assume that g is sufficiently large ($g \geq 3k$).
  $\Longrightarrow$ The irreducible decomposition stabilizes.

(II) Graphical description of the Lie algebra $\mathfrak{h}_{g,1}^{+}$

Fact

Let

$$\mathcal{A}^t(H) := \mathbb{Q}\left\{ \begin{array}{l} H\text{-colord tree-shaped} \\ \text{Jacobi diagram} \end{array} \right\} \Big/ \left(\begin{array}{l} \text{AS, IHX,} \\ \text{multi-linear} \end{array} \right).$$



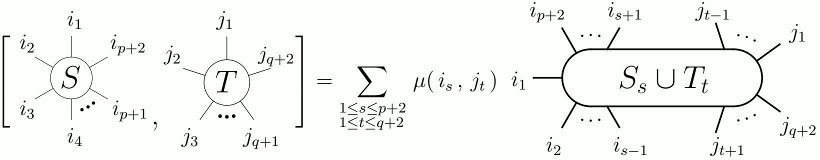
$(a, b, c, d \in H)$

$\mathcal{A}^t_k(H)$: subspace generated by diagrams w/ k trivalent vertices.

$$\mathcal{A}^t_k(H) \cong \mathfrak{h}_{g,1}(k).$$

Formula

Brackets in $\mathcal{A}^t(H)$:



where $S_s \cup T_t$ is obtained by welding S and T at the legs s and t.

Then we have

$$\mathcal{A}^t_0(H) \cong \mathfrak{sp}(2g,\mathbb{Q}), \qquad \bigoplus_{k=1}^{\infty} \mathcal{A}^t_k(H) \cong \mathfrak{h}^+_{g,1}$$

as Lie algebras.

- $\mathcal{A}^t(H)$ appears in the theory of finite type invariants (clasper surgery) for 3-manifolds.

(III) Hain's theory

Hain determined the infinitesimal presentation of $\mathcal{I}_g$ by using the Hodge theory (Mixed Hodge Structures). From this,

Theorem [Hain]

1. The Lie subalgebra $\operatorname{Im}\tau_{g,1}$ is generated by its degree 1 part $\operatorname{Im}\tau_{g,1}(1) = \mathfrak{h}_{g,1}(1) = \wedge^3 H$.

2. There exists an ideal $\mathfrak{j}_{g,1} = \bigoplus_{k=1}^{\infty} \mathfrak{j}_{g,1}(k)$ in $\mathfrak{h}_{g,1}^{+}$ such that

$$\mathfrak{j}_{g,1}(k) \cap \operatorname{Im}\tau_{g,1}(k) = \{0\} \qquad \text{for all } k \geq 3.$$

Precisely speaking,

$$\begin{aligned}\mathfrak{j}_{g,1}(k) &:= \operatorname{Ker}(\mathfrak{h}_{g,1}(k) \twoheadrightarrow \mathfrak{h}_{g,*}(k)) \\ &= \operatorname{Ker}\left(H \otimes (\mathcal{L}_{k+1}(H)/\langle\omega_0\rangle_{k+1}) \xrightarrow{[\cdot\,,\cdot]} (\mathcal{L}_{k+2}(H)/\langle\omega_0\rangle_{k+2}) \right).\end{aligned}$$

Remarks

- Our problem (I) is equivalent to:

Problem

(I') Determine the Lie subalgebra of $\mathfrak{h}_{g,1}^+$ generated by its degree 1 part $\mathfrak{h}_{g,1}(1) = \operatorname{Im}\tau_{g,1}(1) = \wedge^3 H$.

- $\operatorname{Im}\tau_{g,1}(k) \subset \operatorname{Ker}\left(\mathfrak{h}_{g,1}(k) \to H_1(\mathfrak{h}_{g,1}^+)_k\right)$ for $k \geq 2$.
  (i.e. $H_1(\mathfrak{h}_{g,1}^+)_k \subset \mathfrak{h}_{g,1}(k)/\operatorname{Im}\tau_{g,1}(k)$ as $\mathrm{Sp}(2g,\mathbb{Q})$-module.)

(IV) Trace maps and Enomoto-Satoh's obstruction

Theorem [Morita] (trace map)

For $k \geq 2$, the composition

$$\mathrm{Tr}_{2k-1} : \mathfrak{h}_{g,1}(2k-1) \subset H \otimes \mathcal{L}_{2k}(H) \hookrightarrow H^{\otimes(2k+1)} \xrightarrow{\mu_{1,2}} H^{\otimes(2k-1)} \xrightarrow{\mathrm{proj}} S^{2k-1}H$$

gives

$$S^{2k-1}H = [2k-1] \subset H_1(\mathfrak{h}_{g,1}^{+})_{2k-1}.$$

(i.e. Tr_{2k-1} is a non-trivial homomorphism vanishing on brackets.)

Enomoto-Satoh's obstruction

Theorem [Enomoto-Satoh]

For $k \geq 2$, consider the composition

$$\mathrm{ES}_k : \mathfrak{h}_{g,1}(k) \subset H \otimes \mathcal{L}_{k+1}(H) \hookrightarrow H^{\otimes(k+2)} \xrightarrow{\mu_{1,2}} H^{\otimes k} \xrightarrow{\mathrm{proj}} \left(H^{\otimes k}\right)_{\mathbb{Z}/k\mathbb{Z}},$$

where $\mathbb{Z}/k\mathbb{Z} \curvearrowright H^{\otimes k}$ is given by the cyclic permutation. Then

$$\mathrm{Im}\,\tau_{g,1}(k) \subset \mathrm{Ker}\,\mathrm{ES}_k.$$

$\rightsquigarrow \quad \mathrm{Im}\,\mathrm{ES}_k \subset \mathfrak{h}_{g,1}(k)/\,\mathrm{Im}\,\tau_{g,1}(k).$

We call the map ES_k the ES-obstruction.

(V) Relation with number theory

In 1980's, Oda predicted:

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ should "appear" in $(\mathrm{Coker}\,\tau_g)^{\mathrm{Sp}} \otimes \mathbb{Z}_p$ (p:prime).

Nakamura, Matsumoto: proof and related many works.

"Encounter with the Galois obstruction!"
(The first one appears in $\tau_g(6)$.)

Problem

Describe the Galois image explicitly.

- Earlier foundational works for $g = 0$: Ihara, Deligne.
- More recent works for $g = 1$: Hain-Matsumoto, Nakamura.

(I) Previously known facts on $\mathrm{Im}\,\tau_{g,1} \subset \mathfrak{h}_{g,1}^{+}$ (up to degree 4):

Fact

- $\mathrm{Im}\,\tau_{g,1}(1) = \mathfrak{h}_{g,1}(1) = \wedge^3 H = [1^3] + [1]$ (Johnson),

- $\mathrm{Im}\,\tau_{g,1}(2) = \mathfrak{h}_{g,1}(2) = [2^2] + [1^2] + [0]$ (Hain, Morita),

- $\mathrm{Im}\,\tau_{g,1}(3) = [31^2] + [21] \subsetneqq \mathfrak{h}_{g,1}(3) = [31^2] + [21] + [3]$ (Hain, Asada-Nakamura),

- $\mathrm{Im}\,\tau_{g,1}(4) = [42] + [31^3] + 2[31] + [2^3] + [21^2] + 2[2]$

  $\subsetneqq \mathfrak{h}_{g,1}(4) = [42] + [31^3] + 2[31] + [2^3] + 2[21^2] + 3[2]$ (Morita).

(II) Previously known facts on $H_1(\mathfrak{h}_{g,1}^+)_k$ (up to degree 4):

Fact

- By definition $H_1(\mathfrak{h}_{g,1}^+)_1 = \mathfrak{h}_{g,1}(1) = [1^3] + [1]$.
- Arguments using Trace map gives

$$H_1(\mathfrak{h}_{g,1}^+)_2 = 0, \quad H_1(\mathfrak{h}_{g,1}^+)_3 \cong S^3 H = [3], \quad H_1(\mathfrak{h}_{g,1}^+)_4 = 0.$$

New Results: Degree 5

Theorem 1. [Morita-Suzuki-S.] w/ a correction by Enomoto

- $$\begin{aligned}\operatorname{Im}\tau_{g,1}(5) = &([51^2]+[421]+[3^31]+[321^2]+[2^21^3]) \\ &+(2[41]+2[32]+2[31^2]+2[2^21]+2[21^3]) \\ &+([3]+3[21]+2[1^3])+[1].\end{aligned}$$
- $$\begin{aligned}\mathfrak{h}_{g,1}(5)/\operatorname{Im}\tau_{g,1}(5) = &([5]+[32]+[2^21]+[1^5]) \\ &+(2[21]+2[1^3])+2[1].\end{aligned}$$
  (completely detected by ES-obstruction)
- $H_1(\mathfrak{h}_{g,1}^+)_5 \cong S^5H = [5]$. (only the trace component)

Proof: Computer calculation + ES-obstruction + trace map.

New Results: Degree 6

Theorem 2. [Morita-Suzuki-S.]

- $$\begin{aligned}\mathrm{Im}\,\tau_{g,1}(6) = {} & ([62]+[521]+[51^3]+[4^2]+[431]+2[42^2]+[421^2] \\ & +[41^4]+2[3^21^2]+[32^21]+[321^3]+[2^4]+[2^21^4]) \\ & +(3[51]+3[42]+4[41^2]+3[3^2]+7[321]+3[31^3] \\ & +[2^3]+5[2^21^2]+2[21^4]+[1^6]) \\ & +(4[4]+6[31]+9[2^2]+6[21^2]+4[1^4]) \\ & +(3[2]+6[1^2])+2[0].\end{aligned}$$

Theorem 2 (continue).

- $$\mathfrak{h}_{g,1}(6)/\operatorname{Im}\tau_{g,1}(6) = (2[41^2]+[3^2]+[321]+[31^3]+[2^21^2]) \\ +(2[4]+3[31]+3[2^2]+3[21^2]+2[1^4]) \\ +([2]+5[1^2])+3[0],$$

  in which the ES-obstruction cannot detect $[1^4]+[1^2]+[0]$.

Proof: Theoretical consideration + computer calculations

- $[1^4]+[1^2]$: Two proofs by
  (1) Checking all patterns of brackets.
  (2) Finding a component in the ideal $\mathfrak{j}_{g,1}(6)$ outside of $\operatorname{Im}\mathrm{ES}_6$.
- $[0]$: The Galois obstruction (w/ explicit description).

Abelianization of $H_1(\mathfrak{h}_{g,1}^+)$ (in progress)

Problem (bis)

(II) Determine the abelianization $H_1(\mathfrak{h}_{g,1}^+) = \bigoplus_{k=1}^{\infty} H_1(\mathfrak{h}_{g,1}^+)_k$ of $\mathfrak{h}_{g,1}^+$.

Background of (II): Kontsevich's theorem says:

Theorem [Kontsevich]

There exists an isomorphism

$$PH_n\big(\lim_{g\to\infty} \mathfrak{h}_{g,1}^+\big)_{2k}^{\mathrm{Sp}} \cong H^{2k-n}(\mathrm{Out}(F_{k+1});\mathbb{Q}).$$

$\rightsquigarrow$ If $H_1\big(\lim_{g\to\infty} \mathfrak{h}_{g,1}^+\big)^{\mathrm{Sp}} = 0$, then $H^{2k-3}(\mathrm{Out}(F_k);\mathbb{Q}) = 0$ holds for any $k \geq 2$.

Morita once conjectured that

$$\text{The trace components } \bigoplus_{k=1}^{\infty}[2k+1] \text{ gave } H_1(\mathfrak{h}_{g,1}^+).$$

However, Conant-Kassabov-Vogtmann recently disproved it:

Theorem [Conant-Kassabov-Vogtmann]

There exist much more components other than the trace components $\bigoplus_{k=1}^{\infty}[2k+1]$ in $H_1(\mathfrak{h}_{g,1}^+)$.

They use the Eichler-Shimura isomorphism in the theory of modular forms.

Motivated by their results, we obtained explicit descriptions for (a part of) their new components of $H_1(\mathfrak{h}_{g,1}^+)$:

Theorem 3. [Morita-Suzuki-S.]

1. $H_1(\mathfrak{h}_{g,1}^+)_6 = [31]$. (New component in $H_1(\mathfrak{h}_{g,1}^+)$)

2. For $k \geq 3$, the composition

$$\begin{aligned} H \otimes \mathcal{L}_{2k+1}(H) \hookrightarrow H^{\otimes(2k+2)} &\xrightarrow{\mu_{1,3} \circ \mu_{4,2k+1}} H^{\otimes(2k-2)} \\ &\xrightarrow{\wedge_{1,(2k-2)}} H^{\otimes(2k-4)} \otimes \wedge^2 H \\ &\xrightarrow{\mathrm{proj} \otimes \mathrm{id}} S^{2k-4} H \otimes \wedge^2 H \end{aligned}$$

gives

$$[(2k-3)\,1] \subset H_1(\mathfrak{h}_{g,1}^+)_{2k}.$$

Proof: Combinatorial argument w/o using computer.

Corollary [Morita-Suzuki-S.]

Constructions of explicit Sp-invariant cocycles of $\mathfrak{h}_{g,1}^{+}$ corresponding to homology classes in

$$H_{11}(\mathrm{Out}(F_8);\mathbb{Q}),\quad H_{15}(\mathrm{Out}(F_{10});\mathbb{Q}),\quad H_{17}(\mathrm{Out}(F_{11});\mathbb{Q}),$$
$$H_{19}(\mathrm{Out}(F_{12});\mathbb{Q}),\ \ldots.$$

(Not yet known whether they are non-trivial.)

Example Since

$$([31]\otimes[3]\otimes[5])^{\mathrm{Sp}} \cong \mathbb{Q},$$
$$([31]\otimes[5]\otimes[7])^{\mathrm{Sp}} \cong \mathbb{Q},$$

we obtain Sp-invariant cohomology classes in $H^3(H_1(\mathfrak{h}_{g,1}^{+}))_{14}^{\mathrm{Sp}}$ and $H^3(H_1(\mathfrak{h}_{g,1}^{+}))_{18}^{\mathrm{Sp}}$.

Fin.