The mod 2 Johnson homomorphism and the abelianization of the level 2 mapping class group of a non-orientable surface

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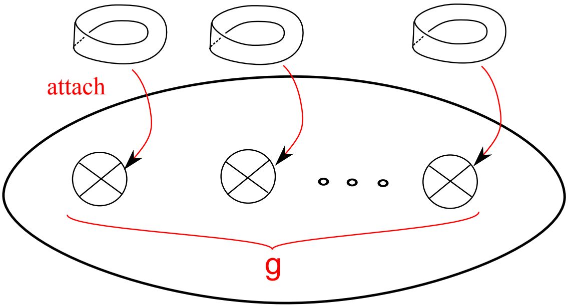
N_g: the non-orientable surface of genus g.



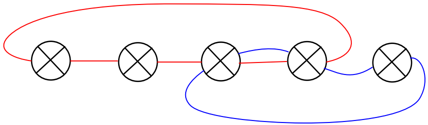
$$\mathcal{M}(N_g) = \frac{\mathrm{Diff}(N_g)}{\text{isotopy}}$$ the mapping class group of N_g

Generators for $\mathcal{M}(N_g)$

c : a simple closed curve on N_g.

c is an A-circle $\Leftrightarrow$ the regular neighborhood of c is an annulus.

c is an M-circle $\Leftrightarrow$ the regular neighborhood of c is a Möbius band.



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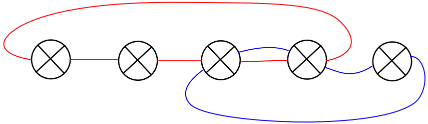

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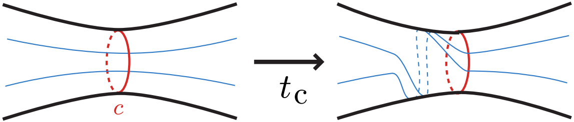

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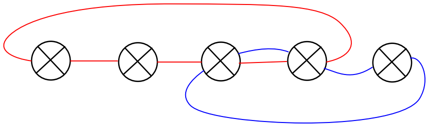

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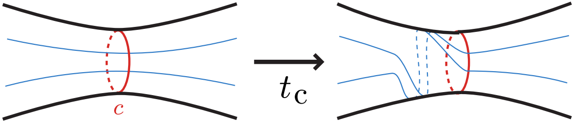

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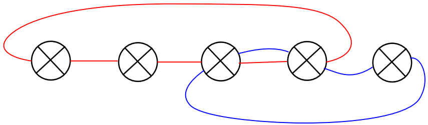

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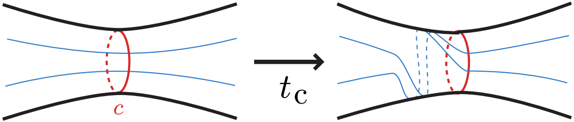

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Y-homeomorphisms are needed.

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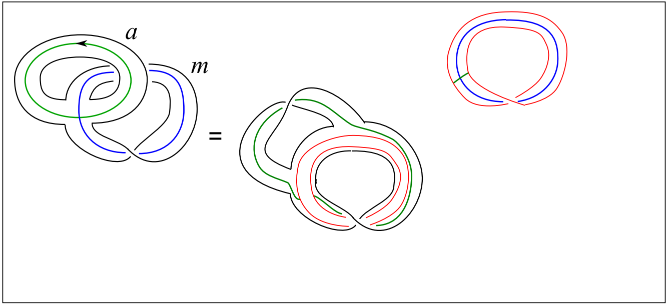

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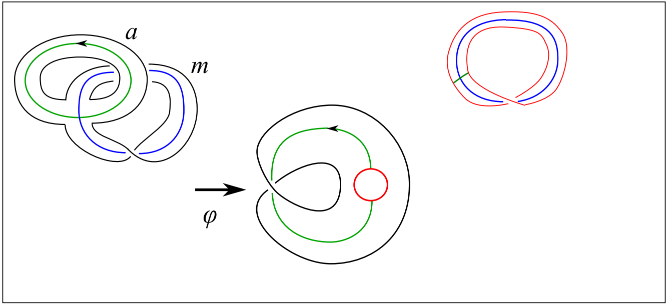

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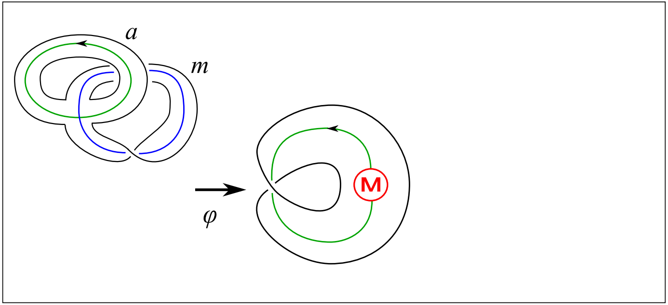

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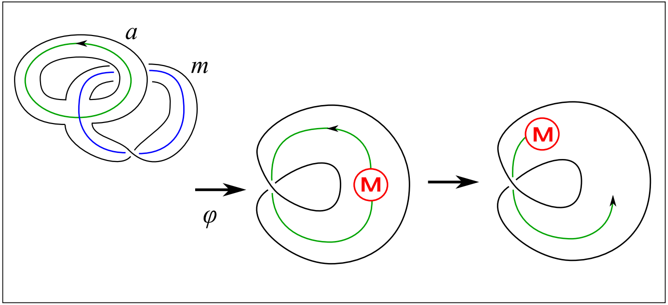

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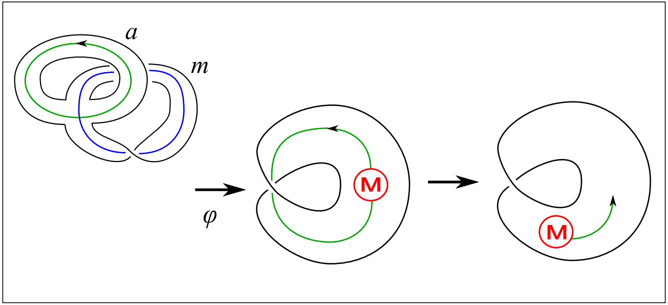

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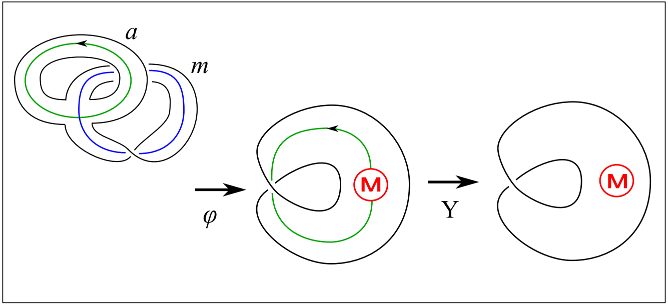

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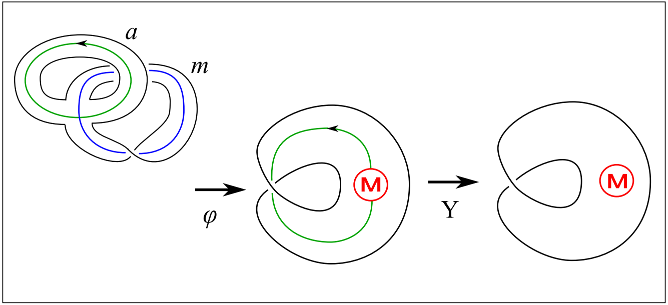

$$Y_{m,a}(x) = \begin{cases} \varphi^{-1} \circ Y \circ \varphi(x) & \text{if } x \text{ is in the neighborhood of } a \cup m, \\ x & \text{otherwise.} \end{cases}$$

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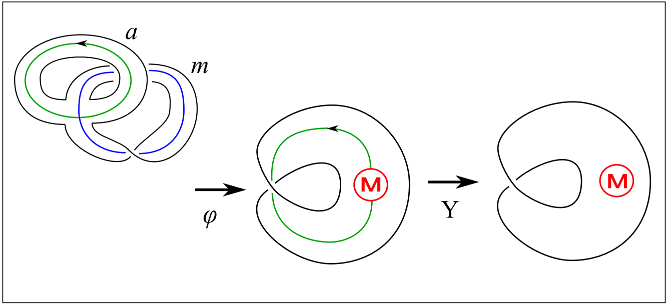

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Lickorish (1963) : $\mathcal{M}(N_g)$ is generated by Dehn-twists and Y-homeomorphisms.

Define $D : \mathcal{M}(N_g) \to \mathbb{Z}_2 = \{+1, -1\}$ by

$$D(f) = det(f_* : H_1(N_g;\mathbb{R}) \to H_1(N_g;\mathbb{R})),$$

then $D(Y\text{-homeomorphism}) = -1$, $D(\text{Dehn twist}) = +1$.
Lickorish (1965) :
$\ker D =$ the subgroup of $\mathcal{M}(N_g)$ generated by all Dehn twists.

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$\cdot : H_1(N_g;\mathbb{Z}_2) \times H_1(N_g;\mathbb{Z}_2) \to \mathbb{Z}_2$: mod 2 intersection form.
$Aut(H_1(N_g;\mathbb{Z}_2), \cdot)$: the group of automorphisms over $H_1(N_g;\mathbb{Z}_2)$ preserving $\cdot$.

Theorem (McCarthy and Pinkall (preprint, 1984))

The natural homomorphism $\rho_2 : \mathcal{M}(N_g) \to Aut(H_1(N_g;\mathbb{Z}_2), \cdot)$ is surjective.

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Level 2 mapping class group: $\Gamma_2(N_g) := \ker \rho_2$

Theorem (Błażej Szepietowski (Geom.Dedicata(2011)))

$\Gamma_2(N_g)$ is generated by Y-homeomorphisms.

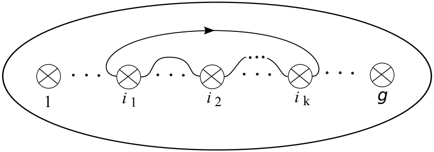
If $g = 3$, $\Gamma_2(N_g) \cong \{A \in GL(2,\mathbb{Z}) \mid A \equiv E_2 \mod 2\}$.

(B. Szepietowski)

Finite system of generators for $\Gamma_2(N_g)$, when $g \geq 4$.

$\{i_1, \ldots, i_k\} \subset \{1, \ldots, g\}$

$\alpha_{\{i_1,\ldots,i_k\}} =$



$Y_{i_1;i_2,\ldots,i_k} := Y_{\alpha_{i_1}, \alpha_{\{i_1,i_2,\ldots,i_k\}}}$

$T_{i_1,\ldots,i_k} := t_{\alpha_{\{i_1,\ldots,i_k\}}}$ = the Dehn twist about $\alpha_{\{i_1,\ldots,i_k\}}$.

Theorem (Błażej Szepietowski (Kodai Math. J. (2013)))

When $g \geq 4$, $\Gamma_2(N_g)$ is generated by the following elements:

1. $Y_{i;j}$ for $i \in \{1, \cdots, g-1\}, j \in \{1, \cdots, g\}$ and $i \neq j$,
2. $T^2_{i,j,k,l}$ for $i < j < k < l$.

$$H_1(\Gamma_2(N_g);\mathbb{Z}) = \Gamma_2(N_g)/[\Gamma_2(N_g),\Gamma_2(N_g)].$$

B. Szepietowski (Geom. Dedicata (2011)) : $\Gamma_2(N_g)$ is generated by involutions. $\Rightarrow H_1(\Gamma_2(N_g);\mathbb{Z})$ is a $\mathbb{Z}/2\mathbb{Z}$-module.

B. Szepietowski (Kodai Math. J. (2013)) $\Rightarrow$

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_1(\Gamma_2(N_g);\mathbb{Z}) \leq (g-1)^2 + \binom{g}{4}.$$

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We have better upper bound !!

Lemma 1

When $g \geq 4$, $\Gamma_2(N_g)$ is generated by the following elements:

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This Lemma $\Rightarrow$

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_1(\Gamma_2(N_g); \mathbb{Z}) \leq \binom{g}{2} + \binom{g}{3}.$$

The lower bound $\Rightarrow$

$$H_1(\Gamma_2(N_g); \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\binom{g}{2} + \binom{g}{3}}.$$

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Corollary

When $g \geq 4$, the set of elements given in the above Lemma is a minimal set of generators for $\Gamma_2(N_g)$.

Proof of Lemma 1

$G =$ the subgroup of $\mathcal{M}(N_g)$ generated by

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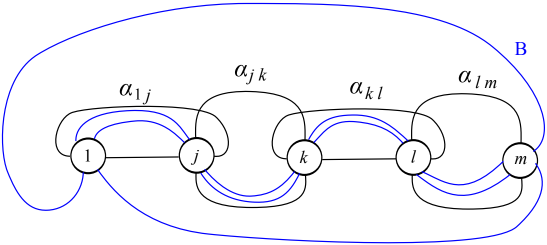
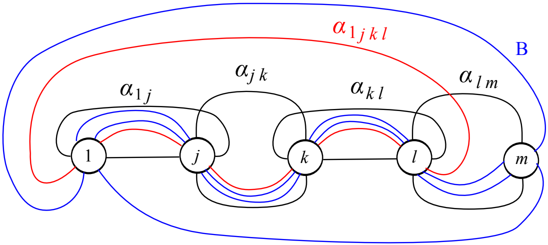
CONVENTION: $\phi_1\phi_2 \in \mathcal{M}(N_g)$ means apply ϕ_1 first, and then ϕ_2.



N = the regular neighborhood of $\alpha_{1,j} \cup \alpha_{j,k} \cup \alpha_{k,l} \cup \alpha_{l,m} \cong \Sigma_{2,1}$,
$B = \partial N$.



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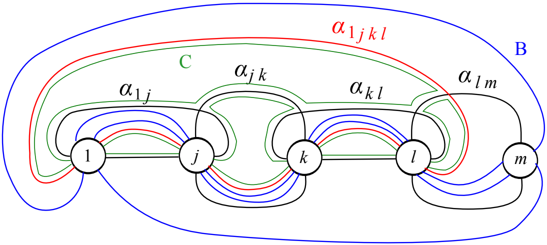
$N' =$ the regular neighborhood of $\alpha_{1,j} \cup \alpha_{j,k} \cup \alpha_{k,l} \cup \alpha_{l,m} \cup \alpha_{1,j,k,l}$
in $N \cong \Sigma_{2,2}$,



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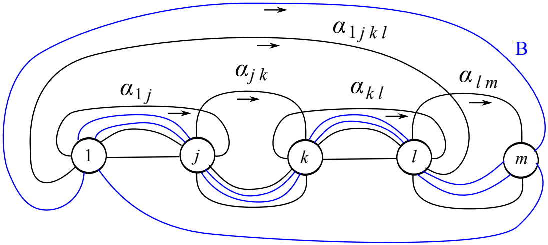
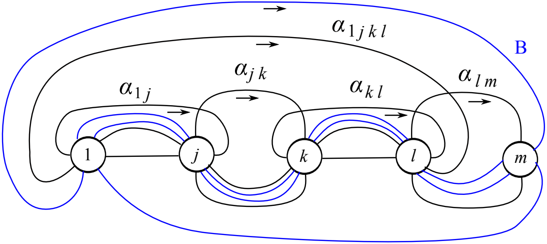
in $N \cong \Sigma_{2,2}$,
$\partial N' = B \cup C$, C bounds a disk in N_g.



For short, $a = T_{\alpha_{1,j}}$, $b = T_{\alpha_{j,k}}$, $c = T_{\alpha_{k,l}}$, $d = T_{\alpha_{l,m}}$, $e = T_{\alpha_{1,j,k,l}}$.
By chain relation, $(abcde)^6 = T_B \cdot T_C = T_B$.



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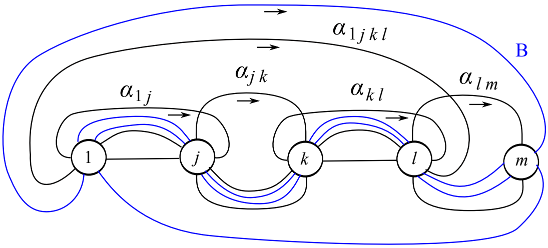
By braid relations,

$$
\begin{aligned}
(abcde)^6 &= abcdeedcba \cdot bcdeedcb \cdot cdeedc \cdot deed \cdot ee \\
&= abcdee\bar{d}\bar{c}\bar{b}\bar{a} \cdot abcdd\bar{c}\bar{b}\bar{a} \cdot abcc\bar{b}\bar{a} \cdot abb\bar{a} \cdot aa \\
&\quad \cdot bcdee\bar{d}\bar{c}\bar{b} \cdot bcdd\bar{c}\bar{b} \cdot bcc\bar{b} \cdot bb \\
&\quad \cdot cdee\bar{d}\bar{c} \cdot cdd\bar{c} \cdot cc \cdot dee\bar{d} \cdot dd \cdot ee.
\end{aligned}
$$



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\begin{aligned}
(abcde)^6 &= abcdeedcba \cdot bcdeedcb \cdot cdeedc \cdot deed \cdot ee \\
&= {T_{j,k,l,m}}^2 \cdot (abcd\bar{c}\bar{b}\bar{a})^2 \cdot (abc\bar{b}\bar{a})^2 \cdot (ab\bar{a})^2 \cdot a^2 \\
&\quad \cdot (bcde\bar{d}\bar{c}\bar{b})^2 \cdot (bcd\bar{c}\bar{b})^2 \cdot (bc\bar{b})^2 \cdot b^2 \\
&\quad \cdot (cde\bar{d}\bar{c})^2 \cdot (cd\bar{c})^2 \cdot c^2 \cdot (de\bar{d})^2 \cdot d^2 \cdot e^2.
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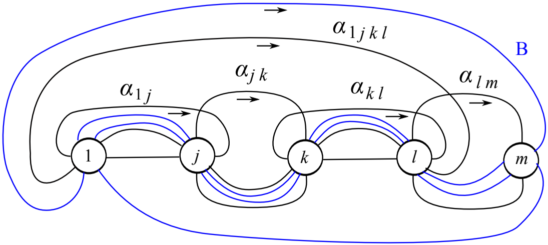
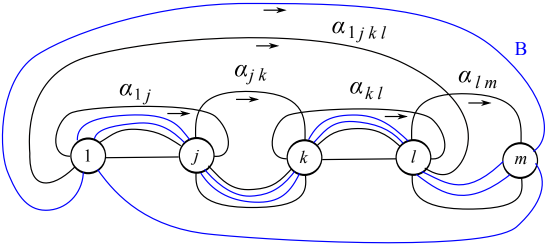


$$T_B = (abcde)^6 = T_{j,k,l,m}{}^2 \cdot (abcd\bar{c}\bar{b}\bar{a})^2 \cdot (abc\bar{b}\bar{a})^2 \cdot (ab\bar{a})^2 \cdot a^2$$
$$\cdot (bcde\bar{d}\bar{c}\bar{b})^2 \cdot (bcd\bar{c}\bar{b})^2 \cdot (bc\bar{b})^2 \cdot b^2$$
$$\cdot (cde\bar{d}\bar{c})^2 \cdot (cd\bar{c})^2 \cdot c^2 \cdot (de\bar{d})^2 \cdot d^2 \cdot e^2.$$

$abcd\bar{c}\bar{b}\bar{a}$ = Dehn twist about $(\alpha_{l,m})\bar{c}\bar{b}\bar{a} = (\alpha_{1,m})Y_{m;j}^{-1}Y_{m;k}^{-1}Y_{m;l}^{-1}$

$\Rightarrow (abcd\bar{c}\bar{b}\bar{a})^2 = Y_{m;l}Y_{m;k}Y_{m;j}T_{1,m}^2Y_{m;j}^{-1}Y_{m;k}^{-1}Y_{m;l}^{-1}$



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$$\cdot (bcded\bar{d}\bar{c}\bar{b})^2 \cdot (bcd\bar{c}\bar{b})^2 \cdot (bc\bar{b})^2 \cdot b^2$$
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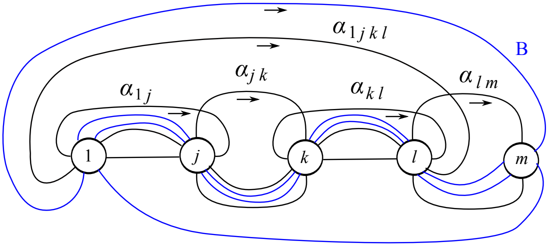
$\Rightarrow (abcd\bar{c}\bar{b}\bar{a})^2 = Y_{m;l}Y_{m;k}Y_{m;j}T_{1,m}^2 Y_{m;j}^{-1}Y_{m;k}^{-1}Y_{m;l}^{-1}$

$T_{1,m}^2 = Y_{1;m}^{-1}Y_{m;1}$ (by B. Szepietowski)

$\Rightarrow (abcd\bar{c}\bar{b}\bar{a})^2 = Y_{m;l}Y_{m;k}Y_{m;j}Y_{1;m}^{-1}Y_{m;1}Y_{m;j}^{-1}Y_{m;k}^{-1}Y_{m;l}^{-1} \in G$



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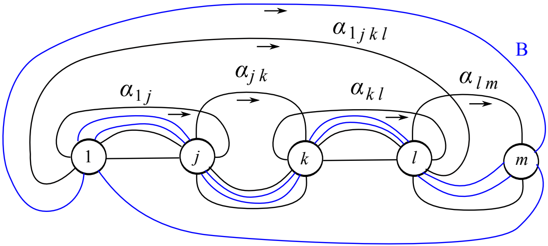
By the same method, $(abc\bar{b}\bar{a})^2$, $(ab\bar{a})^2$, a^2, $(bcde\bar{d}\bar{c}\bar{b})^2$, $(bcd\bar{c}\bar{b})^2$, $(bc\bar{b})^2$, b^2, $(cde\bar{d}\bar{c})^2$, $(cd\bar{c})^2$, c^2, $(de\bar{d})^2$, d^2, e^2 are in G.



$$T_B = (abcde)^6 = {T_{j,k,l,m}}^2 \cdot \text{ an element of } G.$$

$\exists$ simple closed curves β_n in N_g such that

$$T_B = \left(\prod_{n \neq i,j,k,l,m} Y_{\alpha_n, \beta_n} \right)^2 .$$

By using the non-orientable analogy of forgetful exact sequence (or Birman exact sequence), $Y_{\alpha_n,\beta_n} \in G$ (by B. Szepietowski).
$\Rightarrow T^2_{j,k,l,m} \in G$.

In the rest of the talk,

I explain

· the mod d Johnson homomorphism,

· the mod d Johnson filtration,

· the abelianization of the level 2 mapping class group of a nonorientable surface.

Magnus expansion

$F_n = \langle x_1, x_2, \cdots, x_n \rangle$,
R: a commutative ring with a unit element 1,
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$\hat{T}_m = \prod_{p \geq m}^{\infty} H^{\otimes p}$: an ideal consisting of (degree $\geq m$)-part.
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The subset $1 + \hat{T}_1$ is a subgroup of $\hat{T}$ with respect to the multiplication. The homomorphism

$$\theta : F_n \to 1 + \hat{T}_1$$
$$x_i \mapsto 1 + [x_i],$$

is called the standard Magnus expansion.

$\theta_p : F_n \to H^{\otimes p}$: the projection of θ to the degree p part.

Proposition (Magnus)

Let $\Gamma(0) = F_n$, $\Gamma(p+1) = [\Gamma(p), F_n]$.
When $R = \mathbb{Z}$,

$$\Gamma(m) = \bigcap_{p=1}^{m} \operatorname{Ker} \theta_p.$$

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The sequence of subgroups

$$\mathrm{IA}_n(m) = \operatorname{Ker}(\operatorname{Aut} F_n \to \operatorname{Aut}(F_n/\Gamma(m)))$$

is called the Johnson filtration.

Johnson homomorphisms via Magnus expansion

$H = H_1(F_n; R)$.

Theorem (Kawazumi)

For any generalized Magnus expansion θ, the map

$$\begin{aligned}\tau_1 : \operatorname{Aut} F_n &\to \operatorname{Hom}(H, H^{\otimes 2}) \\ \varphi &\mapsto ([\gamma] \mapsto \theta_2(\gamma) - \varphi_* \theta_2(\varphi^{-1}(\gamma))\end{aligned}$$

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$$\begin{aligned} \tau_m : & IA_n(m) \to \mathrm{Hom}(H, H^{\otimes m+1}) \\ & \varphi \mapsto ([\gamma] \mapsto \theta_{m+1}(\gamma) - \theta_{m+1}(\varphi^{-1}(\gamma)), \end{aligned}$$

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Consider the case when $R = \mathbb{Z}/d\mathbb{Z}$, and $H = H_1(F_n; \mathbb{Z}/d\mathbb{Z})$.

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The first one

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If we restrict the crossed homomorphism

$$\begin{aligned} \tau_1 : \operatorname{Aut} F_n &\to \operatorname{Hom}(H, H^{\otimes 2}), \\ \varphi &\mapsto ([\gamma] \mapsto \theta_2(\gamma) - \varphi_* \theta_2(\varphi^{-1}(\gamma))) \end{aligned}$$

to $\operatorname{Aut} F_n[d]$, we have a homomorphism in the same way.

(We will not use it in this talk but)
we also have the m-th mod d Johnson homomorphism

$$\begin{aligned}\tau_m : \mathrm{IA}[d](m) &\to \mathrm{Hom}(H, H^{\otimes m+1}) \\ \varphi &\mapsto ([\gamma] \mapsto \theta_{m+1}(\gamma) - \theta_{m+1}(\varphi^{-1}(\gamma)).\end{aligned}$$

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Note that the mod d Johnson filtration

$$\cdots \subset \mathrm{IA}[d](3) \subset \mathrm{IA}[d](2) \subset \mathrm{IA}[d](1) = \mathrm{Aut}\, F_n[d].$$

are finite index normal subgroups of $\mathrm{Aut}\, F_n$.

Mapping class groups for orientable surfaces

In the following, we assume $g \geq 4$.
Let

$$\mathcal{M}(\Sigma_{g,1}) = \frac{\mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})}{\text{isotopy rel } \partial\Sigma_{g,1}}.$$

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The restriction of the crossed homomorphism

$$\tau_1 : \mathcal{M}(\Sigma_{g,1}) \to \mathrm{Hom}(H, H^{\otimes 2})$$

to the level d mapping class group $\Gamma_d(\Sigma_{g,1})$ is a homomorphism (the mod d Johnson homomorphism).

Remark

Broaddus-Farb-Putman and Perron also constructed the mod d Johnson homomorphism in the level d mapping class group in different ways.

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Proposition

The image of the mod d Johnson homomophism $\tau_1 : \Gamma_d(\Sigma_{g,1}) \to \mathrm{Hom}(H, H^{\otimes 2})$ is

$$\mathrm{Im}\,\tau_1 = \begin{cases} \Lambda^3 H & \text{when } d \text{ is odd} \\ \Lambda^3 H + \langle \frac{d}{2} X^{\otimes 3} \mid X \in H \rangle & \text{when } d \text{ is even} \end{cases}.$$

Since the level d symplectic group generated by transvections, the level d mapping class group is generated by d-powers of Dehn twists and elements in the Torelli group. Proposition is proved using the following lemma.

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Lemma

Let c be a non-separating simple closed curve in $\Sigma_{g,1}$. Then, we have

$$\tau_1(t_c^d) = \begin{cases} \frac{d}{2}[c]^{\otimes 3} \text{ when } d \text{ is even,} \\ 0 \text{ when } d \text{ is odd.} \end{cases}$$

proof of Lemma

Since τ_1 is a crossed homomorphism,
for $\varphi \in \mathcal{M}(\Sigma_{g,1})$ and $\psi \in \Gamma_2(\Sigma_{g,1})$, we have

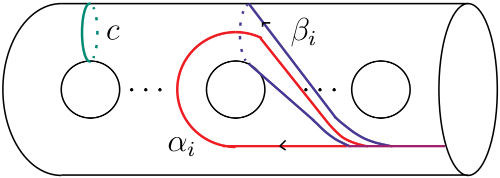
$$\tau_1(\varphi\psi\varphi^{-1}) = \varphi_*\tau_1(\psi) \in H^{\otimes 3}.$$

Hence, it suffices to show that

$$\tau_1(t_c^d) = \frac{d(d-1)}{2}[c]^{\otimes 3}$$

for just one non-separating SCC c.

Choose basis $\{\alpha_i, \beta_i\}_{i=1}^{g}$ and a SCC c as in Figure.



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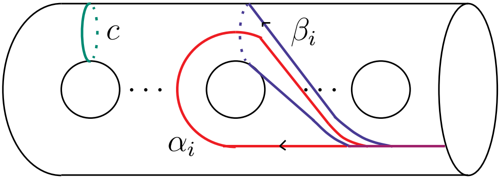

Since $t_c^d(\alpha_1) = \alpha_1 \beta_1^{-d}$, we have

$$\tau_1(t_c^d)[\alpha_1] = \theta_2(\alpha_1) - \theta_2(\alpha_1\beta_1^{-d}) \\ = -\frac{d(d-1)}{2}[\beta_1]^{\otimes 2}.$$

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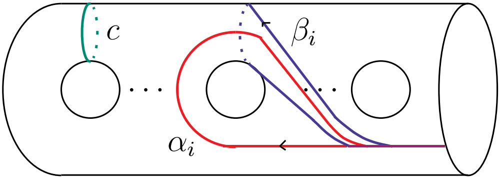

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For $x = \alpha_2, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$, we have $t_c^d(x) = x$.
Hence, we have

$$\tau_1(t_c^d) = \frac{d(d-1)}{2}[\beta_1]^{\otimes 3}.$$

Theorem (S.)

When $d = 2$, the sequence

$$0 \to H_1(\mathcal{I}_{g,1}; \mathbb{Z}/d\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) \to H_1(\Gamma_d(\Sigma_{g,1}); \mathbb{Z}) \to H_1(\mathrm{Sp}(2g; \mathbb{Z})[d]; \mathbb{Z}) \to 0$$

is exact.

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corollary

$$H_1(\Gamma_2(\Sigma_{g,1}); \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\binom{2g}{3}} \oplus (\mathbb{Z}/4\mathbb{Z})^{\binom{2g}{2}} \oplus (\mathbb{Z}/8\mathbb{Z})^{\binom{2g}{1}},$$
$$H_1(\Gamma_2(\Sigma_{g,1}); \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Im}\, \tau_1.$$

Theorem (Perron, Putman, S)

When d is odd,

$$H_1(\Gamma_d(\Sigma_{g,1});\mathbb{Z}) \cong H_1(\mathcal{I}_{g,1};\mathbb{Z}/d\mathbb{Z}) \otimes H_1(\mathrm{Sp}(2g;\mathbb{Z})[d];\mathbb{Z}).$$

Mapping class groups for nonorientable surfaces

Let $H := H_1(N_{g,1}; R)$.
We denote by

$$\mathcal{M}(N_g^*) = \frac{\mathrm{Diff}_+(N_g, *)}{\text{isotopy rel } *}$$

the mapping class group of a pointed surface N_g^*.

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Actually, $\mathrm{Im}(\mathcal{M}(N_{g,1}) \to \mathcal{M}(N_g^*)) \subset \mathcal{M}(N_g^*)$ is of index 2.

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Actually, $\mathrm{Im}(\mathcal{M}(N_{g,1}) \to \mathcal{M}(N_g^*)) \subset \mathcal{M}(N_g^*)$ is of index 2.
Instead, we define the Johnson homomorphism directly on $\mathcal{M}(N_g^*)$.

Let $\zeta \in \pi_1(N_{g,1})$ denote the boundary curve.
The map $\theta_2 : \pi_1(N_{g,1}) \to H^{\otimes 2}$ induces a map

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Thus, we also have a crossed homomorphism

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Since we have a forgetful exact sequence
(Earle-Eell showed $\mathrm{Diff}_0 N_g$ is contractible when $g \geq 3$)

$$1 \to \pi_1 N_g \to \mathcal{M}(N_g^*) \to \mathcal{M}(N_g) \to 1,$$

τ_1 induces a crossed homomorphism

$$\bar{\tau}_1 : \mathcal{M}(N_g) \to \mathrm{Hom}(H, H^{\otimes 2}/\theta_2(\zeta))/\tau_1(\pi_1 N_g).$$

Theorem

The image of the homomorphism

$$\bar{\tau}_1 : \Gamma_2(N_g) \to \mathrm{Hom}(H, H^{\otimes 2}/\theta_2(\zeta))/\tau_1(\pi_1 N_g)$$

is of $(\mathbb{Z}/2\mathbb{Z})$*-rank* $\binom{g}{3} + \binom{g}{2}$.

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Corollary

The homomorphism

$$(\bar{\tau}_1)_* : H_1(\Gamma_2(N_g); \mathbb{Z}) \to \mathrm{Hom}(H, H^{\otimes 2}/\theta_2(\zeta))/\tau_1(\pi_1 N_g)$$

is injective.
In particular, we have

$$H_1(\Gamma_2(N_g); \mathbb{Z}) \cong H_1(\Gamma_2(N_g); \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Im}\, \tau_1.$$

Lemma

Let a be an A-circle and m an M-circle which intersect transversely in one point. Then we have

$$\bar{\tau}_1(Y_{m,a}) = [a] \otimes [m] \otimes [m] + [m] \otimes [a] \otimes [m] + [m] \otimes [m] \otimes [a].$$

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Lemma

Let $\gamma \in \pi_1(N_g)$.
Then we have

$$\bar{\tau}_1(\gamma) = \sum_{i=1}^{g} ([\alpha_i] \otimes [\gamma] \otimes [\alpha_i] + [\alpha_i] \otimes [\alpha_i] \otimes [\gamma]).$$

Problems

For mapping class groups of orientable surfaces,

1. A minimal generating set for $\Gamma_2(\Sigma_g)$?
2. $H_1(\mathrm{IA}_n(m)[d]; \mathbb{Q})$ is trivial or not?

There are also interesting subgroups of $\mathcal{M}(N_g)$.

1. $H_1(\mathcal{I}(N_g))$? for the Torelli group $\mathcal{I}(N_g)$.
2. Other finite index subgroups of $\mathcal{M}(N_g)$?
   (twist subgroup, pin mapping class group, etc.)