On the rings of Fricke characters of free groups

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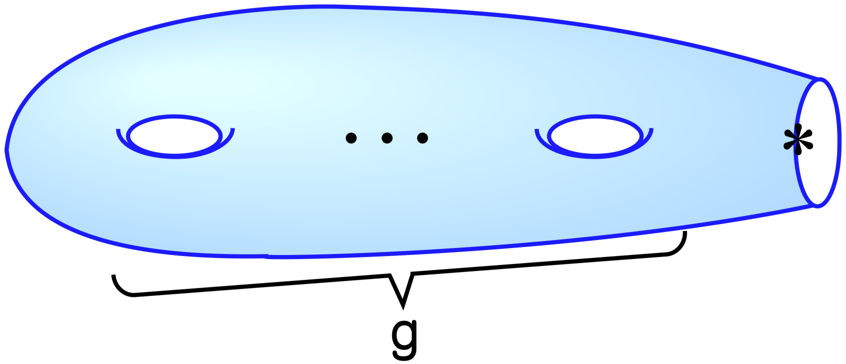
Introduction [1]

$F_n = \langle x_1, \cdots, x_n \rangle$, free group of rank n

$\mathrm{Aut}\, F_n = \{ F_n \to F_n \text{, isomorphisms} \}$

$\Sigma_{g,1} =$



The mapping class group

$\mathcal{M}_{g,1} := \mathrm{Diff}^+ (\Sigma_{g,1}, \partial)$ / isotopy fixing ∂ pointwise

$\pi_1 (\Sigma_{g,1}, *) \cong F_{2g} \curvearrowleft \mathcal{M}_{g,1}$

$\mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}\, F_{2g}$, subgroup

Introduction [2]

$F_n / [F_n, F_n]$, abelianization $\curvearrowleft$ Aut F_n

$$\begin{array}{cccc} \rho : \text{Aut}\, F_n & \longrightarrow & \text{Aut}\, F_n / [F_n, F_n] & (= \mathrm{GL}(n, \mathbb{Z})) \\ \Psi & & \Psi & \\ \sigma & \mapsto & ([x] \mapsto [x^\sigma]) & \end{array}$$

is surjective

$\mathrm{IA}_n := \mathrm{Ker}\, \rho$, IA - automorphism group

$\cup \; n = 2g$

$\mathrm{Ker}\, \rho \,|_{\mathcal{M}_{g,1}}$, Torelli group

Introduction [3]

[Nielsen, 1918]

$$\mathrm{IA}_2 = \mathrm{Inn}\, F_2$$

[Magnus, 1935]

IA_n is finitely generated

<u>Problem</u>

Find a presentation of IA_n, $n \geq 3$

Introduction [4]

$\Gamma_n(1) := F_n$

$\Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2$

$\Gamma_n(k)$ is generated by

$$\{ [y_1, \cdots, y_k] = [\cdots[[y_1, y_2], y_3]\cdots, y_k] \mid y_1, \cdots, y_k \in F_n \}$$

The lower central series of F_n

$$F_n = \Gamma_n(1) \supset \Gamma_n(2) \supset \Gamma_n(3) \supset \cdots,$$

$F_n / \Gamma_n(k+1) \curvearrowleft$ Aut F_n

$\mathcal{A}_n(k) :=$ Ker (Aut $F_n \to$ Aut $F_n / \Gamma_n(k+1)$, homomorphism)

<u>Andreadakis-Johnson filtration</u>

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

Introduction [5]

[Andreadakis, 1965]

(1) $\left[\mathcal{A}_n(k), \mathcal{A}_n(l)\right] \subset \mathcal{A}_n(k+l)$

(2) $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$

is a free abelian group of finite rank

Our research

The ring of Fricke characters of F_n $\curvearrowleft$ Aut F_n

$\cup$

ideal J, Aut F_n - invariant

A descending filtration $J \supset J^2 \supset J^3 \supset \cdots$

Result 1

A basis of $\mathrm{gr}^k(J) := J^k / J^{k+1}$, $k = 1,\ 2$

$$\mathcal{E}_n(k) = \mathrm{Ker}\left(\mathrm{Aut}\,F_n \to \mathrm{Aut}\,J/J^{k+1}\right)$$

Result 2

Relations between $\mathcal{A}_n(k)$ and $\mathcal{E}_n(k)$

Fricke characters [1]

$\mathcal{C}_n := \{ \mathrm{Hom}(F_n,\ \mathrm{SL}(2,\ \mathbb{C})) \to \mathbb{C},\ \text{map} \}$

$\mathcal{C}_n$ is a commutative ring

$$\left[\begin{array}{l} f, f' \in \mathcal{C}_n,\ \ \rho \in \mathrm{Hom}(F_n,\ \mathrm{SL}(2,\ \mathbb{C})) \\ (f + f')(\rho) = f(\rho) + f'(\rho) \\ (f \cdot f')(\rho) = f(\rho) \cdot f'(\rho) \end{array} \right]$$

$\mathcal{C}_n$ is a $\mathbb{C}$ - vector space

$\mathrm{Hom}(F_n,\ \mathrm{SL}(2,\ \mathbb{C})),\ \ \mathcal{C}_n \curvearrowleft \mathrm{Aut}\, F_n$, actions from right

$$\left[\begin{array}{l} \sigma \in \mathrm{Aut}\, F_n,\ \ \ x \in F_n \\ \rho^{\sigma}(x) := \rho\left(x^{\sigma^{-1}}\right),\ \ \ \ f^{\sigma}(\rho) := f\left(\rho^{\sigma^{-1}}\right) \end{array} \right]$$

Fricke characters　[2]

Fricke character　$\mathrm{tr}\, x \in \mathcal{C}_n$,　$\forall x \in F_n$

$$(\mathrm{tr}\, x)(\rho) := \mathrm{tr}\, \rho(x)$$

||

trace of 2×2 matricies

Ex.) $e \in F_n$, unit

$$(\mathrm{tr}\, e)(\forall \rho) = \mathrm{tr}\, \rho(e) = \mathrm{tr}\, E_2 = 2 \quad \text{(constant map)}$$

$\sigma \in \mathrm{Aut}\, F_n$,　$(\mathrm{tr}\, x)^{\sigma} = \mathrm{tr}\, x^{\sigma}$

Fricke characters [3]

Formulae

(1) $\mathrm{tr}\,x^{-1} = \mathrm{tr}\,x$

(2) $\mathrm{tr}\,xy = \mathrm{tr}\,yx$

(3) $\mathrm{tr}\,xy + \mathrm{tr}\,xy^{-1} = (\mathrm{tr}\,x)(\mathrm{tr}\,y)$

(4) $$\mathrm{tr}\,xyz + \mathrm{tr}\,yxz = (\mathrm{tr}\,x)(\mathrm{tr}\,yz) + (\mathrm{tr}\,y)(\mathrm{tr}\,xz) + (\mathrm{tr}\,z)(\mathrm{tr}\,xy) - (\mathrm{tr}\,x)(\mathrm{tr}\,y)(\mathrm{tr}\,z)$$

(5) $$\begin{aligned} 2\mathrm{tr}\,xyzw = {} & (\mathrm{tr}\,x)(\mathrm{tr}\,yzw) + (\mathrm{tr}\,y)(\mathrm{tr}\,zwx) \\ & + (\mathrm{tr}\,z)(\mathrm{tr}\,wxy) + (\mathrm{tr}\,w)(\mathrm{tr}\,xyz) \\ & + (\mathrm{tr}\,xy)(\mathrm{tr}\,zw) - (\mathrm{tr}\,xz)(\mathrm{tr}\,yw) + (\mathrm{tr}\,xw)(\mathrm{tr}\,yz) \\ & - (\mathrm{tr}\,x)(\mathrm{tr}\,y)(\mathrm{tr}\,zw) - (\mathrm{tr}\,y)(\mathrm{tr}\,z)(\mathrm{tr}\,xw) \\ & - (\mathrm{tr}\,x)(\mathrm{tr}\,w)(\mathrm{tr}\,yz) - (\mathrm{tr}\,z)(\mathrm{tr}\,w)(\mathrm{tr}\,xy) \\ & + (\mathrm{tr}\,x)(\mathrm{tr}\,y)(\mathrm{tr}\,z)(\mathrm{tr}\,w) \end{aligned}$$

Fricke characters [4]

The ring of Fricke characters over $\mathbb{Q}$

$\mathcal{X}_n$:= $\mathbb{Q}$ - vector subspace of $\mathcal{C}_n$ generated by all tr x $(x \in F_n)$

$\mathcal{X}_n$ is a subring of $\mathcal{C}_n$

The unit is $\frac{1}{2}$ tr $e = 1 \in \mathcal{C}_n$, constant map

$$\mathbb{Q}_n[t] := \mathbb{Q}\left[t_* \;\middle|\; \begin{array}{r} * = i \ (1 \le i \le n), \text{or } ij \ (1 \le i < j \le n), \\ \text{or } ijk \ (1 \le i < j < k \le n) \end{array} \right],$$

rational polynomial ring

$\mathbb{Q}_n[t]$ has $n + nC2 + nC3$ indeterminates

Fricke characters [5]

Define a ring homomorphism

$$\pi : \mathbb{Q}_n[t] \longrightarrow \mathcal{C}_n$$

$$\begin{array}{ccc} \cup & & \cup \\ 1 & \mapsto & \frac{1}{2}(\mathrm{tr}\, e) \\ t_i & \mapsto & \mathrm{tr}\, x_i \\ t_{ij} & \mapsto & \mathrm{tr}\, x_i x_j \\ t_{ijk} & \mapsto & \mathrm{tr}\, x_i x_j x_k \end{array}$$

[Horowitz, 1972]

$$\mathrm{Im}\ \pi = \mathcal{X}_n$$

$$\mathcal{X}_n \cong \mathbb{Q}_n[t] / \mathrm{Ker}\pi$$

Fricke characters [6]

<u>Facts</u>

(1) $n = 1, 2$, $\mathrm{Ker}\pi = (0)$, trivial ideal [Horowitz , 1972]

(2) $n = 3$, $\mathrm{Ker}\pi$ is principal [Horowitz , 1972]

(3) $n = 4$, $\mathrm{Ker}\pi$ is not principal [Wittemore , 1973]

$t_*' := t_* - 2 \in \mathbb{Q}_n[t]$, $* = i$, or $i\,j$, or $i\,j\,k$

ideal $J_0 := (t_*') \subset \mathbb{Q}_n[t]$

$\mathrm{Ker}\pi \subset J_0$

$J := J_0 / \mathrm{Ker}\,\pi$

Fricke characters [7]

Lemma [in the case $n = 3$, Magnus, 1980]

J is Aut F_n -inv.

A descending filtration of $\mathbb{Q}_n[t]/\ \mathrm{Ker}\,\pi = \mathcal{X}_n$

$J \supset J^2 \supset J^3 \supset \cdots,$ Aut F_n -inv

$\mathrm{gr}^k(J) := J^k / J^{k+1}$

is a finite dimensional $\mathbb{Q}$ -vector space

Fricke characters [8]

Theorem 1

$T := \{ t_*' \mid \quad * = i, \text{ or } i\,j, \text{or } i\,j\,k \}$

$\pi(T)$ forms a basis of $\mathrm{gr}^1(J) = J/J^2$

Theorem 2

We determined a basis of $\mathrm{gr}^2(J)$

$\exists S \subsetneqq \{ t_{*_1}' \, t_{*_2}' \mid *_1, *_2 = i, \text{ or } i\,j, \text{ or } i\,j\,k \}$

$\pi(S)$ forms a basis of $\mathrm{gr}^2(J) = J^2/J^3$

A proof of Theorem 1

It is sufficient to show that $\mathrm{Ker}\pi \subset J_0^{\,2}$

$\forall f \in \mathrm{Ker}\pi$, assume that

$$f = \sum_{i} a_i t_i' + \sum_{i<j} a_{ij} t_{ij}' + \sum_{i<j<k} a_{ijk} t_{ijk}'$$

$$+ (\text{terms of degree} \geq 2),$$

$$a_i ,\ a_{ij} ,\ a_{ijk} \in \mathbb{Q}$$

We want to show $a_i = a_{ij} = a_{ijk} = 0$

Fricke characters [10]

For $\forall s \in \mathbb{C}$, A := $\begin{pmatrix} s+2 & 1 \\ -1 & 0 \end{pmatrix} \in$ SL$(2, \mathbb{C})$

Then, tr $A^m - 2 = m^2 s$ + (higher degree terms of s)

$$\begin{array}{cccc} \rho_0 : & F_n & \rightarrow & \mathrm{SL}(2, \mathbb{C}) \\ & \Downarrow & & \Downarrow \\ & x_i & \mapsto & \begin{cases} A & \text{if } i = n \\ E_2 & \text{if } i \neq n \end{cases} \end{array}$$

From the equation

$$0 = \pi(f) = \left(a_n + \sum_{i<n} a_{i\,n} + \sum_{i<j<n} a_{i\,j\,n} \right) s$$

$$+ \text{(higher degree terms of } s \text{)},$$

we have $a_n + \sum_{i<n} a_{i\,n} + \sum_{i<j<n} a_{i\,j\,n} = 0$, for example.

Fricke characters [11]

Theorem 3

In the case of free abelian goup $F_n / [F_n, F_n]$ of rank n, we determined the $\mathbb{Q}$ - vector space structures of $\mathrm{gr}^k(J)$, $k \geq 1$.

In this case, $\mathbb{Q}_n[t] := \mathbb{Q}[t_i, t_{ij}]$

For $k \geq 1$ and $0 \leq l \leq k$, define

$$T_l := \left\{ t'_{p_1 q_1} \cdots t'_{p_l q_l} t'_{i_{l+1}} \cdots t'_{i_k} \;\middle|\; \begin{array}{l} 1 \leq p_1 < q_1 < \cdots < p_l < q_l \leq n, \\ 1 \leq i_{l+1} \leq \cdots \leq i_k \leq n \end{array} \right\}$$

Then $\displaystyle\bigcup_{l=0}^{k} \pi(T_l)$ forms a basis of $\mathrm{gr}^k(J)$

A filtration of Aut F_n [1]

$J/J^{k+1} \curvearrowleft$ Aut F_n

$$\mathcal{E}_n(k) = \mathrm{Ker}\left(\mathrm{Aut}\,F_n \to \mathrm{Aut}\left(J/J^{k+1}\right)\right)$$

$$\begin{array}{ccc} \mathrm{Aut}\,F_n & & \mathrm{Aut}\left(J/J^{k+1}\right) \\ \Cup & & \Cup \\ \sigma & \mapsto & ([f] \mapsto [f^{\sigma}]) \end{array}$$

A descending filtration $\mathcal{E}_n(1) \supset \mathcal{E}_n(2) \supset \cdots$

Proposition

$$[\,\mathcal{E}_n(k), \mathcal{E}_n(l)\,] \subset \mathcal{E}_n(k+l)$$

Theorem 4

(1) $n \geq 3$, $\mathcal{E}_n(1) = \mathrm{Inn}\,F_n \cdot \mathcal{A}_n(2)$

(2) $\mathcal{A}_n(2k) \subset \mathcal{E}_n(k)$, $k \geq 1$

A filtration of Aut F_n [2]

$$\mathrm{gr}^k(\mathcal{E}_n) := \mathcal{E}_n(k)/\mathcal{E}_n(k+1)$$

<u>Theorem 5</u> $(n \geq 3)$

(1) $\mathrm{gr}^k(\mathcal{E}_n)$ is torsion free

(2) $\dim_{\mathbb{Q}}\left(\mathrm{gr}^k(\mathcal{E}_n) \otimes_{\mathbb{Z}} \mathbb{Q}\right) < \infty$

In the proof, Johnson homomorphism like homomorphism is used.

$$\begin{array}{ccc} \eta_k : \mathrm{gr}^k(\mathcal{E}_n) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{gr}^1(J),\ \mathrm{gr}^{k+1}(J)\right) \\ \Cup & & \Cup \\ \sigma & \mapsto & \left(f \mapsto f^{\sigma} - f\right) \end{array}$$

This is an injective homomorphism between abelian groups