Whittaker models
of
degenerate principal series

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\( G = KAN \): a real reductive Lie group
\( g = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \): complexifications of Lie algebras
For \( \pi \in \hat{G}_{\text{ad}} \) and \( \varpi \in \hat{\mathscr{N}} \)
Whittaker model: \( \pi \hookrightarrow \text{Ind}_{\mathscr{N}}^{G} \varpi \)

\( \Sigma(g) \): the root system for the pair \((g, a)\) with the Weyl group \( W \)
\( \Sigma(g)^{+} \): the positive root system corresponding to \( \mathfrak{n} \)
\( \Psi(g) \): the fundamental system
\( g^\alpha := \{ X \in g ; \text{ad}(H)X = \alpha(H)X \ (\forall H \in \mathfrak{a}) \} \) for \( \alpha \in \Sigma(g) \).

For \( \Theta \subset \Psi(g) \) put
\( W_\Theta := \langle s_\alpha ; \alpha \in \Theta \rangle \) and \( W(\Theta) := \{ w \in W ; w\Theta \subset \Sigma(g)^{+} \} \). Then
\( W(\Theta) \times W_\Theta \cong W : (w_1, w_2) \mapsto w_1w_2 \)

\( P = MAN \): a minimal parabolic subgroup with \( M = Z_K(\mathfrak{a}) \)
\( P_\Theta := PW_\Theta P = M_\Theta A_\Theta N_\Theta = G_\Theta N_\Theta \).

Here \( G_\Theta = M_\Theta A_\Theta \) and \( \Psi(g_\Theta) = \Theta \).
Theorem. Let $\lambda \in \tilde{G}_\Theta$ and $\varpi \in \tilde{N}$.
Suppose $\dim \lambda < \infty$, $\dim \varpi = 1$ and $\lambda|_{A_\Theta}$ is generic. Then

$$\dim \text{Hom}_{(g,K)}(\text{Ind}_{G_{\Theta}N_{\Theta}}^G(\lambda \otimes 1), \text{Ind}_{N_{\Theta}}^G\varpi) = \#W(\text{supp}\varpi, \Theta) \cdot \#W_{\text{supp}\varpi} \cdot \dim_M \lambda$$

$$\dim \text{Hom}_{G_{\infty}}(\text{Ind}_{G_{\Theta}N_{\Theta}}^G(\lambda \otimes 1), \text{Ind}_{N_{\Theta}}^G\varpi) = \#W(\text{supp}\varpi, \Theta) \cdot \dim_M \lambda$$

if $\varpi$ is unitary

$$\text{supp}\varpi := \{\alpha \in \Psi(g) ; \varpi(g^\alpha) \neq \{0\}\}$$

$$W(\Upsilon, \Theta) := \{w \in W(\Upsilon) \cap W(\Theta)^{-1} ; w \Sigma(g_\Upsilon) \cap \Sigma(g_\Theta) = \emptyset\}$$

$\dim_M \lambda$: the dimension of the representation of $M$ with the same highest weight of $\lambda$.

$\varpi$ is non-degenerate $\iff$ $\text{supp}\varpi = \Psi(g)$.

"The radial parts of $K$-finite functions of this Whittaker model of $G$"

are

"those of the non-degenerate Whittaker model of $G_{\text{supp}\varpi}$".

Remark. i) $W(\Theta, \Upsilon) = W(\Upsilon, \Theta)^{-1}$.

ii) $W(\Upsilon, \Theta)$ corresponds to a subset of $P_\Theta \backslash G/P_{\Upsilon}$. 
\[
\dim \text{Hom}(\text{Ind}_{\mathcal{P}}^G(\lambda), \text{Ind}_{\mathcal{N}}^G(\varpi)) = \#W(\text{supp}\varpi, \Theta) \cdot \dim_{\mathcal{M}} \lambda \cdot \begin{cases} 
\#W_{\text{supp}\varpi} & \text{if } \Theta = \emptyset \\
1 & \text{otherwise}
\end{cases}
\]

\[
W(\gamma, \Theta) := \{w \in W(\gamma) \cap W(\Theta)^{-1}; w\Sigma(g_{\gamma}) \cap \Sigma(g_{\Theta}) = \emptyset\}
\]
with \(\gamma = \text{supp}\varpi\)

**Examples.**

1. \(\varpi\) is trivial. (\(\Rightarrow\) Imbeddings into principal series)

   \(\Rightarrow\) supp\(\varpi = \emptyset \Rightarrow W(\text{supp}\varpi) = W \Rightarrow W(\text{supp}\varpi, \Theta) = W(\Theta)^{-1}\)

   If \(\Theta = \emptyset\), then \(P_{\Theta} = P\), \(W(\Theta) = W\), \(W(\text{supp}\varpi, \Theta) = W\) and the imbeddings are obtained by standard intertwining operators between principal series.

   If \(G\) is compact, the result corresponds to Peter-Weyl theorem.

   For general \(\Theta\) they are similarly obtained through the natural imbedding of degenerate series into principal series.

2. \(\varpi\) is non-degenerate. (\(\Rightarrow\) well-studied)

   \(\Rightarrow W_{\text{supp}\varpi} = W\), \(W(\text{supp}\varpi) = \{e\}\), \(\Sigma(g_{\text{supp}\varpi}) = \Sigma(g)\).

   Hence \(W(\text{supp}\varpi, \Theta) \neq \emptyset \Rightarrow \Theta = \emptyset\) and \(W(\text{supp}\varpi, \emptyset) = \{e\}\).
3. $G = GL(n, \mathbb{R})$. $\Theta$ corresponds to a partition of $n$ and a Young diagram:

\[
G_\Theta = GL(2, \mathbb{R}) \times GL(4, \mathbb{R}) \times GL(1, \mathbb{R}) \Rightarrow 7 = 4 + 2 + 1 : \\
\#	ext{W}(\text{supp}\omega, \Theta) = 1 \quad \Leftrightarrow \quad \text{The partition sup}\omega \text{ equals the dual partition of } \Theta
\]

\[
7 = (4 + 2 + 1)' = 3 + 2 + 1 + 1 \Rightarrow 2 + 1 + 3 + 1: \\
G_{\text{supp}\omega} = GL(2, \mathbb{R}) \times GL(1, \mathbb{R}) \times GL(3, \mathbb{R}) \times GL(1, \mathbb{R})
\]

\[
n = 7 \\
= 2 + 4 + 1 \\
= 2 + 1 + 3 + 1
\]

\[
\lambda_2 : \ n_2' = 4 \quad \lambda_1 : \ n_1' = 2 \quad \lambda_3 : \ n_3' = 1
\]

\[
P_{2,4,1} = \\
p_3 \ p_1' \ p_2' \ p_4'
\]

\[
E_{2,4,1}(\lambda_1, \lambda_2, \lambda_3) := \text{Ind}_{P_{2,4,1}}^G (\lambda_1, \lambda_2, \lambda_3) \\
\mapsto E(\lambda_1, \lambda_2 + 1) \otimes E(\lambda_2 + 1) \otimes E(\lambda_1 - 2, \lambda_2, \lambda_3 + 1) \otimes E(\lambda_2 - 1)
\]
Suppose \( G = GL(n, \mathbb{R}) \) and \( \#W(\text{supp}\varpi, \Theta) = 1 \). Then “\( K \)-finite functions of the Whittaker model are reduced to the usual Whittaker functions”

\( \Leftrightarrow G_{\text{supp}\varpi} \) is a direct product of some copies of \( GL(2, \mathbb{R}) \) and/or \( GL(1, \mathbb{R}) \)

\( \Leftrightarrow P_\Theta \) is a maximal parabolic subgroup

\( x = (x_{ij}) \in GL(n, \mathbb{R}) \)

\( (E_{ij}\varphi)(x) = \frac{d}{dt}\varphi(xe^tE_{ij})|_{t=0}, \quad E_{ij} = \sum_{\nu=1}^{n} x_{\nu i} \frac{\partial}{\partial x_{\nu j}}, \)

\( n = \sum_{1 \leq j < i \leq n} CE_{ij} \)

\( \varpi\left(\exp\left(\sum_{i > j} t_{ij}E_{ij}\right)\right) = e^{\sqrt{-1}(c_1t_{21} + \cdots + c_{n-1}t_{n,n-1})} \)

\( \Theta = \{1, 2, \ldots, k-1, k+1, \ldots, n-1\} \) (\( 2 \leq 2k \leq n \)), \( P_\Theta = P_{k,n-k} \)

“Existence of the Whittaker model of \( \text{Ind}_{P_{k,n-k}}^{G}(\lambda, \mu) \)”

\( \Leftrightarrow c_{i}c_{i+1} = c_{1}c_{2} \cdots c_{k+1} = 0 \) (\( 1 \leq i < n, \ 1 \leq i_1 < \cdots < i_{k+1} < n \))
For example, put

\[
\begin{aligned}
c_i &= 0 \quad (i = 2, 4, \ldots, 2k, 2k + 1, 2k + 2, \ldots, n - 1), \\
c_{2j-1} &\neq 0 \quad (j = 1, \ldots, k)
\end{aligned}
\]

⇒ The $K$-fixed vector of the Whittaker model is the solution of

\[
\begin{aligned}
E_i v &= \mu v \quad (i = 2k + 1, 2k + 2, \ldots, n), \\
(E_{2j-1} + E_{2j}) v &= (\lambda + \mu - 2j + k + 1) v, \\
\left(\frac{E_{2j-1} - E_{2j}}{2}\right)^2 - \left(\frac{E_{2j-1} - E_{2j}}{2}\right) - c_{2j-1} e^{2(t_{2j-1} - t_{2j})} v &= \frac{\lambda - \mu - k + 1}{2} \left(\frac{\lambda - \mu - k + 1}{2} - 1\right) v,
\end{aligned}
\]

Here $j = 1, \ldots, k$, $E_\nu = \frac{\partial}{\partial t_\nu}$ ($\nu = 1, \ldots, n$).

on $A \ni \text{diag}(e^{t_1}, \ldots, e^{t_n})$.

The dimension of the solution space equals $2^k$.

There exists a unique solution with the moderate growth up to a constant multiple. It is written by the modified Bessel functions of the 2-nd kind.

(Similarly $K$-finite vectors ⇒ expressed by Whittaker functions.)
Key to prove Theorem.

1. Irreducibility of a Whittaker module.
The left $g = gl(2, \mathbb{C})$-module
\[
\begin{cases}
(E_{11} + E_{22} - \lambda - \mu)v = 0, \\
(E_{11}E_{22} - E_{12}E_{21} + E_{11} - \lambda(\mu + 1))v = 0 \\
(E_{21} - c_1)v = 0
\end{cases}
\]
is irreducible if $c_1 \neq 0$ (Kostant in general).

2. Twisted Harish-Chandra homomorphism.
$g = \bar{n} + a + n$: a complex reductive Lie algebra (or $G$ is a normal real form)
\[
\gamma_{\varpi} : U(g) \to U(a), \quad D \mapsto \gamma_{\varpi}(D) \text{ with } D - \gamma_{\varpi}(D) \in \bar{n}U(g) + U(g) \sum_{X \in n}(X - \varpi(X)),
\]
Remark. $I$: two-sided ideal of $U(g)$ ($\Rightarrow \gamma_{\varpi}(I)$ is an ideal of $U(a)$)
i) supp$\varpi$ $\subset$ supp$\varpi'$ $\Rightarrow$ $\gamma_{\varpi}(I)$ $\subset$ $\gamma_{\varpi'}(I)$
i) If gr$(I)$ does not vanish at $\sum_{\alpha \in \text{supp}\varpi} X_\alpha$, then $\gamma_{\varpi}(I) = U(a)$.

Characterize $\gamma_{\varpi}(\text{Ann(Ind}_{P}^{G}(\lambda)))$!
3. **A Boundary value problem on** $K_{\mathbb{C}} \times AN$

A boundary attached to infinite points of $\exp(\sqrt{-1}t + a)$ corresponding to a certain Weyl chamber. Here $t$ is a maximal torus of $m = Z_{\mathfrak{t}}(a)$.

4. **Integral expression of Whittaker model with moderate growth**

The kernel function of the intertwining operator is a distribution but its support has in general no inner point. $W(\text{supp}\omega, \Theta)$ gives the possibility of the support.

That's all!
Thank you.