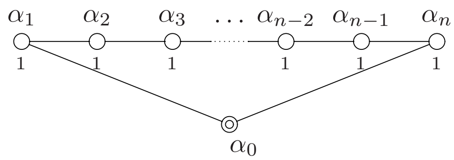
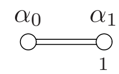
Irreducible root systems

A_n

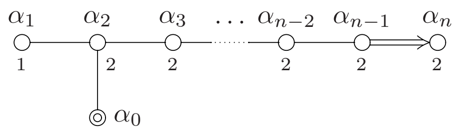


A_1

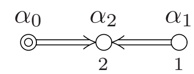


$$\Sigma = \big\{\pm(\epsilon_i - \epsilon_j)\,;\, 1 \le i < j \le n+1\big\}, \quad \#\Sigma = n(n+1),$$
$$\alpha_j = \epsilon_j - \epsilon_{j+1} \quad (j = 1, \dots, n),\ \alpha_0 = \epsilon_{n+1} - \epsilon_1 \quad (n > 1), \quad \#W = (n+1)!.$$

B_n

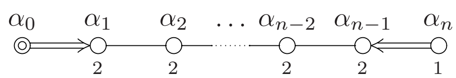


$B_2 = C_2$



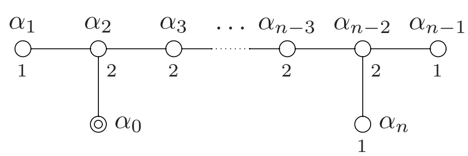
$$\Sigma = \big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm\epsilon_k\,;\, 1 \le i < j \le n,\ 1 \le k \le n\big\}, \quad \#\Sigma = 2n^2,$$
$$\alpha_j = \epsilon_j - \epsilon_{j+1} \quad (j = 1, \dots, n-1),\ \alpha_n = \epsilon_n,\ \alpha_0 = -\epsilon_1 - \epsilon_2,\ \alpha_0' = -\epsilon_1,$$
$$\#W = 2^n \cdot n!.$$

C_n

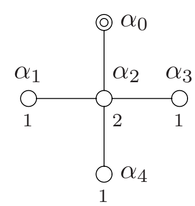


$$\Sigma = \big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm 2\epsilon_k\,;\, 1 \le i < j \le n,\ 1 \le k \le n\big\}, \quad \#\Sigma = 2n^2,$$
$$\alpha_j = \epsilon_j - \epsilon_{j+1} \quad (j = 1, \dots, n-1),\ \alpha_n = 2\epsilon_n,\ \alpha_0 = -2\epsilon_1,\ \alpha_0' = -\epsilon_1 - \epsilon_2,$$
$$\#W = 2^n \cdot n!.$$

D_n

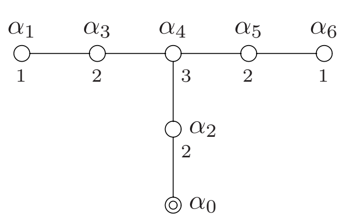


D_4



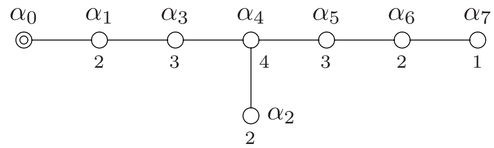
$$\Sigma = \big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j)\,;\, 1 \le i < j \le n\big\}, \quad \#\Sigma = 2n(n-1)$$
$$\alpha_j = \epsilon_j - \epsilon_{j+1} \quad (j = 1, \dots, n-1), \qquad \alpha_n = \epsilon_{n-1} + \epsilon_n,$$
$$\alpha_0 = -\epsilon_1 - \epsilon_2, \quad \#W = 2^{n-1} n!.$$

E_6



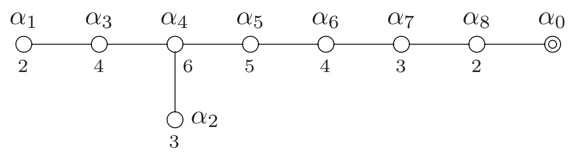
$$\Sigma = \Big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm\tfrac{1}{2}\big(\epsilon_8 - \epsilon_7 - \epsilon_6 + \textstyle\sum_{k=1}^{5}(-1)^{\nu(k)}\epsilon_k\big)\,;$$
$$1 \le i < j \le 5,\ \textstyle\sum_{k=1}^{5}\nu(k) \text{ is even}\Big\}, \quad \#\Sigma = 72$$
$$\alpha_1 = \tfrac{1}{2}(\epsilon_1 + \epsilon_8) - \tfrac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7),$$
$$\alpha_2 = \epsilon_1 + \epsilon_2, \quad \alpha_j = \epsilon_{j-1} - \epsilon_{j-2} \quad (3 \le j \le 6),$$
$$\alpha_0 = -\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8), \quad \#W = 2^7 \cdot 3^4 \cdot 5.$$

E_7



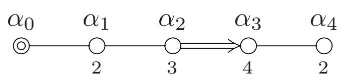
$$\Sigma = \big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm(\epsilon_7 - \epsilon_8),\ \pm\tfrac{1}{2}(\epsilon_7 - \epsilon_8 + \textstyle\sum_{k=1}^{6}(-1)^{\nu(k)}\epsilon_k)\,;$$
$$1 \le i < j \le 6,\ \textstyle\sum_{k=1}^{6}\nu(k) \text{ is odd}\big\}, \quad \#\Sigma = 126$$
$$\alpha_1 = \tfrac{1}{2}(\epsilon_1 + \epsilon_8) - \tfrac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7),$$
$$\alpha_2 = \epsilon_1 + \epsilon_2, \quad \alpha_j = \epsilon_{j-1} - \epsilon_{j-2} \quad (3 \le j \le 7),$$
$$\alpha_0 = \epsilon_7 - \epsilon_8, \quad \#W = 2^{10}\cdot 3^4 \cdot 5 \cdot 7.$$

E_8



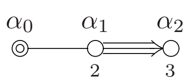
$$\Sigma = \big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm\tfrac{1}{2}\textstyle\sum_{k=1}^{8}(-1)^{\nu(k)}\epsilon_k\,;$$
$$1 \le i < j \le 8,\ \textstyle\sum_{k=1}^{8}\nu(k) \text{ is even}\big\}, \quad \#\Sigma = 240$$
$$\alpha_1 = \tfrac{1}{2}(\epsilon_1 + \epsilon_8) - \tfrac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7),$$
$$\alpha_2 = \epsilon_1 + \epsilon_2, \quad \alpha_j = \epsilon_{j-1} - \epsilon_{j-2} \quad (3 \le j \le 8),$$
$$\alpha_0 = -\epsilon_7 - \epsilon_8, \quad \#W = 2^{14}\cdot 3^5 \cdot 5^2 \cdot 7.$$

F_4



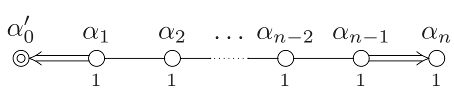
$$\Sigma = \big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm\epsilon_k,\ \pm\tfrac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\,;$$
$$1 \le i < j \le 4,\ 1 \le k \le 4\big\}, \quad \#\Sigma = 48,$$
$$\alpha_1 = \epsilon_2 - \epsilon_3,\ \alpha_2 = \epsilon_3 - \epsilon_4,\ \alpha_3 = \epsilon_4,\ \alpha_4 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4),$$
$$\alpha_0 = -\epsilon_1 - \epsilon_2,\ \alpha_0' = -\epsilon_1, \quad \#W = 2^7 \cdot 3^2.$$

G_2

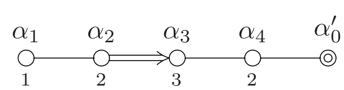


$$\Sigma = \big\{\pm(\epsilon_i - \epsilon_j),\ \mp(2\epsilon_1 - \epsilon_2 - \epsilon_3),\ \mp(2\epsilon_2 - \epsilon_1 - \epsilon_3),\ \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2)\,;$$
$$1 \le i < j \le 3\big\}, \quad \#\Sigma = 12,$$
$$\alpha_1 = -2\epsilon_1 + \epsilon_2 + \epsilon_3,\ \alpha_2 = \epsilon_1 - \epsilon_2,\ \alpha_0 = \epsilon_1 + \epsilon_2 - 2\epsilon_3,\ \alpha_0' = \epsilon_2 - \epsilon_3, \quad \#W = 12.$$

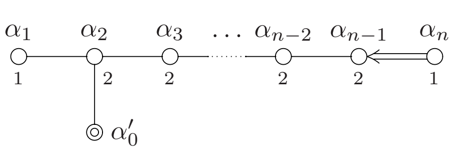
B_n' :



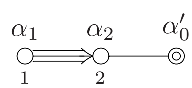
F_4' :



C_n' :



G_2' :



$D_1 = \emptyset, \quad D_2 \simeq A_1 + A_1, \quad D_3 \simeq A_3, \quad A_1 \simeq B_1 \simeq C_1$

Notation

$$\Xi,\ \Sigma : \text{reduced root systems}$$
$$\Psi = \{\alpha_1, \dots, \alpha_n\} \ : \ \text{a fundamental system of } \Sigma$$
$$W_\Sigma : \text{the Weyl group of } \Sigma$$
$$\mathrm{Hom}(\Xi, \Sigma) := \{\iota : \Xi \to \Sigma\, ;\, 2\tfrac{(\iota(\alpha)|\iota(\beta))}{(\iota(\alpha)|\iota(\alpha))} = 2\tfrac{(\alpha|\beta)}{(\alpha|\alpha)} \quad (\forall \alpha,\, \beta \in \Xi)\}$$
$$\mathrm{Aut}(\Sigma) := \mathrm{Hom}(\Sigma, \Sigma), \quad \mathrm{Out}(\Sigma) := \mathrm{Aut}(\Sigma)/W_\Sigma$$
$$\mathrm{Aut}'(\Xi) := \prod_{j=1}^{m} \mathrm{Aut}(\Xi_j) \subset \mathrm{Aut}(\Xi)$$
for the irreducible decomposition $\Xi = \Xi_1 + \cdots + \Xi_m$

$$\Xi,\, \Xi' : \ \text{subsystems of } \Sigma$$
$$G \ : \ \text{a subgroup of } \mathrm{Aut}(\Sigma)$$
$$N_G(\Xi) := \{g \in G\, ;\, g(\Xi) = \Xi\}, \quad Z_G(\Xi) := \{g \in G\, ;\, g|_\Xi = id\}$$

$$\Xi \underset{\Sigma}{\sim} \Xi' \overset{\text{def}}{\iff} \exists w \in W_\Sigma \text{ such that } \Xi' = w(\Xi)$$
$$\Xi \overset{w}{\underset{\Sigma}{\sim}} \Xi' \overset{\text{def}}{\iff} \exists g \in \mathrm{Aut}(\Sigma) \text{ such that } \Xi' = g(\Xi)$$
$$\mathrm{Hom}(\Xi, \Sigma)_o = \{\iota \in \mathrm{Hom}(\Xi, \Sigma)\, ;\, \iota(\Xi) \overset{w}{\underset{\Sigma}{\sim}} \Xi\}$$
$$O^w_\Xi := \{F \subset \Sigma\, ;\, F \overset{w}{\underset{\Sigma}{\sim}} \Xi\}$$
$$\simeq \mathrm{Hom}(\Xi, \Sigma)_o / \mathrm{Aut}(\Xi) \simeq \mathrm{Aut}(\Sigma)/N_{\mathrm{Aut}(\Sigma)}(\Xi)$$
$$O_\Xi := \{F \subset \Sigma\, ;\, F \underset{\Sigma}{\sim} \Xi\} \simeq W_\Sigma / N_{W_\Sigma}(\Xi)$$
$$\# \ : \ \#\big(W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)_o\big)$$
$$\#_\Xi \ : \ \#\big(W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)_o / \mathrm{Aut}(\Xi)\big)$$
$$\#_{\Xi'} \ : \ \#\big(W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)_o / \mathrm{Aut}'(\Xi)\big)$$
$$\#_\Sigma \ : \ \#\big(\mathrm{Aut}(\Sigma) \backslash \mathrm{Hom}(\Xi, \Sigma)_o\big)$$
$$\Xi^{\perp\perp} \ : \ \begin{cases} \circ & \big((\Xi, \Xi^\perp) : \text{a special dual pair} \Rightarrow \mathrm{Out}(\Xi) \simeq \mathrm{Out}(\Xi^\perp)\big) \\ \times & \big((\Xi, \Xi^\perp) : \text{a nonspecial dual pair}\big) \\ (\Xi^\perp)^\perp & \big((\Xi^\perp)^\perp \neq \Xi\big) \end{cases}$$
$$P \ : \ \begin{cases} \#\{\Theta \subset \Psi\, ;\, \langle \Theta \rangle \overset{w}{\underset{\Sigma}{\sim}} \Xi\} & \big(\text{if } \Xi \text{ is fundamental}\big) \\ \circ & \big(\text{if } \Xi \text{ is not fundamental but } L\text{-closed}\big) \\ \leftarrow & \big(L\text{-closure of } \Xi \text{ is given by } (\Xi^\perp)^\perp\big) \\ \rightarrow & \big(L\text{-closure of } \Xi \text{ is given in the right column}\big) \end{cases}$$
$$\Xi \text{ is fundamental} : \exists w \in W_\Sigma \text{ and } \exists \Theta \subset \Psi \text{ such that } \langle \Theta \rangle = w(\Xi)$$
$$S \ : \ S\text{-closure} \quad \big(S\text{-closed} : \alpha,\, \beta \in \Xi,\ \alpha + \beta \in \Sigma \Rightarrow \alpha + \beta \in \Xi\big)$$
$$L \ : \ L\text{-closure} \quad \Big(L\text{-closed} : \sum_{\alpha \in \Xi} \mathbb{R}\alpha \cap \Sigma = \Xi\Big)$$
$$\langle j_1, \dots, j_m \rangle \ : \ \langle \alpha_{j_1}, \dots, \alpha_{j_m} \rangle$$
$$\langle \backslash j \rangle \ : \ \langle \Psi \setminus \{\alpha_j\} \rangle$$
$$\#O^w_\Xi / \#O_\Xi = \#\big(W_\Xi \backslash \mathrm{Hom}(\Xi, \Sigma)_o / \mathrm{Aut}(\Xi)\big) = (\#_\Xi)$$
$$\#O_\Xi = (\#) \cdot \#W_\Xi \ / \ \big((\#_\Sigma) \cdot \#\,\mathrm{Out}(\Xi) \cdot \#W_\Xi \cdot \#W_{\Xi^\perp}\big)$$

Classical Type (Ξ: irreducible)

Σ	Ξ	$\#$	$\#_\Xi$	$\#_\Sigma$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
A_n	A_1	1	1	1	A_{n-2}	$\times$	n
$A_{n\ (1<m\leq n-2)}$	A_m	2	1	1	A_{n-m-1}	$\times$	$n-m+1$
$A_{n\ (n\geq 3)}$	A_{n-1}	2	1	1	$\emptyset$	Σ	2
$A_{n\ (n\geq 2)}$	A_n	2	1	1	$\emptyset$	Σ	1

$\Sigma\ (n\geq 5)$	Ξ	$\#$	$\#_\Xi$	$\#_\Sigma$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
D_n	A_1	1	1	1	$D_{n-2}+A_1$	$\times$	n
D_n	A_2	1	1	1	D_{n-3}	A_3	$n-1$
D_n	A_3	1	1	1	D_{n-4}	D_4	$n-2$
	(D_3)	1	1	1	$D_{n-3\ (n\neq 7)}$	$\circ$	1
					$D_{4\ (n=7)}$	$\times$	
$D_{n\ (4\leq k\leq n-3)}$	A_k	1	1	1	D_{n-k-1}	D_{k+1}	$n-k+1$
D_n	A_{n-2}	1	1	1	$\emptyset$	Σ	3
$D_{n\ (n:\text{odd})}$	A_{n-1}	2	1	1	$\emptyset$	Σ	2
$D_{n\ (n:\text{even})}$		2	2	1			
D_n	D_4	3	1	3	$D_{n-4\ (n\geq 6)}$	2	1
					$\emptyset_{\ (n=5)}$	Σ	
$D_{n\ (4<k\leq n-2)}$	D_k	1	1	1	D_{n-k}	$\circ_{\ (k\neq n-4)}$	1
						$\times_{\ (k=n-4)}$	
$D_{n\ (n\geq 6)}$	D_{n-1}	1	1	1	$\emptyset$	Σ	1
D_n	D_n	2	1	1	$\emptyset$	Σ	1

$\Sigma\ (n\geq 2)$	Ξ	$\#$	$\#_\Xi$	$\#_\Sigma$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
B_n	A_1^L	1	1	1	$B_{n-2}+A_1^L$	$\circ$	$n-1$
	A_1^S	1	1	1	B_{n-1}	$\circ$	1
$B_{n\ (n\geq 3)}$	A_2	1	1	1	B_{n-3}	B_3	$n-2$
$B_{n\ (n\geq 4)}$	A_3	1	1	1	B_{n-4}	B_4	$n-3$
$B_{n\ (n\geq 3)}$	(D_3)	1	1	1	B_{n-3}	B_3	$\leftarrow$
$B_{n\ (4\leq m<n)}$	A_m	1	1	1	B_{n-m-1}	B_{m+1}	$n-m$
$B_{n\ (n\geq 4)}$	D_4	3	1	3	B_{n-4}	B_4	$\leftarrow$
$B_{n\ (4<m\leq n)}$	D_m	1	1	1	B_{n-m}	B_m	$\leftarrow$
$B_{n\ (2\leq m\leq n)}$	B_m	1	1	1	B_{n-m}	$\circ$	1

$\Sigma\ (n\geq 2)$	Ξ	$\#$	$\#_\Xi$	$\#_{\Xi'}$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
C_n	A_1^S	1	1	1	$C_{n-2}+A_1^S$	$\circ$	$n-1$
	A_1^L	1	1	1	C_{n-1}	$\circ$	1
$C_{n\ (n\geq 3)}$	A_2	1	1	1	C_{n-3}	C_3	$n-2$
$C_{n\ (n\geq 4)}$	A_3	1	1	1	C_{n-4}	C_4	$n-3$
$C_{n\ (n\geq 3)}$	(D_3)	1	1	1	C_{n-3}	C_3	$\leftarrow\ S:C_3$
$C_{n\ (4\leq m<n)}$	A_m	1	1	1	C_{n-m-1}	C_{m+1}	$n-m$
$C_{n\ (n\geq 4)}$	D_4	3	1	3	C_{n-4}	C_4	$\leftarrow\ S:C_4$
$C_{n\ (4<m\leq n)}$	D_m	1	1	1	C_{n-m}	C_m	$\leftarrow\ S:C_m$
$C_{n\ (2\leq m\leq n)}$	C_m	1	1	1	C_{n-m}	$\circ$	1

Exceptional Type and D_4

Σ	Ξ	$\#$	$\#_{\Xi}$	$\#_{\Xi'}$	$\Xi^{\perp}$	$\Xi^{\perp\perp}$	P	$\#_{\Sigma}$
D_4	A_1	1	1	1	$3A_1$	$\times$	4	1
D_4	A_2	1	1	1	$\emptyset$	Σ	3	1
D_4	A_3	3	3	3	$\emptyset$	Σ	3	1
D_4	D_4	6	1	1	$\emptyset$	Σ	1	1
D_4	$2A_1$	3	3	3	$2A_1$	$\circ$	3	1
D_4	$3A_1$	6	1	6	A_1	$\times$	1	1
D_4	$4A_1$	6	1	6	$\emptyset$	Σ	$\leftarrow$	1

Σ	Ξ	$\#$	$\#_{\Xi'}$	$\#_{\Sigma}$	$\Xi^{\perp}$	$\Xi^{\perp\perp}$	P	
E_6	A_1	1	1	1	A_5	$\times$	6	
E_6	A_2	1	1	1	$2A_2$	$\times$	5	
E_6	A_3	1	1	1	$2A_1$	$\circ$	5	
E_6	A_4	2	1	1	A_1	A_5	4	
E_6	A_5	2	1	1	A_1	$\times$	1	$\langle\backslash 2\rangle$
E_6	D_4	1	1	1	$\emptyset$	Σ	1	
E_6	D_5	2	1	1	$\emptyset$	Σ	2	$\langle\backslash 1\rangle,\ \langle\backslash 6\rangle$
E_6	E_6	2	1	1	$\emptyset$	Σ	1	
E_6	$2A_1$	1	1	1	A_3	$\circ$	10	
E_6	$3A_1$	1	1	1	A_1	A_5	5	
E_6	$4A_1$	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L : D_4$
E_6	$A_2 + A_1$	2	1	1	A_2	$2A_2$	10	
E_6	$A_2 + 2A_1$	2	1	1	$\emptyset$	Σ	5	$\subset 3A_2$
E_6	$2A_2$	4	1	2	A_2	$\times$	1	
E_6	$2A_2 + A_1$	4	1	2	$\emptyset$	Σ	1	$\langle\backslash 4\rangle \subset 3A_2$
E_6	$3A_2$	8	1	4	$\emptyset$	Σ	$\leftarrow$	§8.2.5
E_6	$A_3 + A_1$	2	1	1	A_1	A_5	4	
E_6	$A_3 + 2A_1$	2	1	1	$\emptyset$	Σ	$\rightarrow$	§8.2.1, $L : D_5$
E_6	$A_4 + A_1$	2	1	1	$\emptyset$	Σ	2	$\langle\backslash 3\rangle,\ \langle\backslash 5\rangle$
E_6	$A_5 + A_1$	2	1	1	$\emptyset$	Σ	$\leftarrow$	

Σ	Ξ	$\#$	$\#_{\Xi}$	$\#_{\Xi'}$	$\Xi^{\perp}$	$\Xi^{\perp\perp}$	P	
E_7	A_1	1	1	1	D_6	$\times$	7	
E_7	A_2	1	1	1	A_5	$\circ$	6	
E_7	A_3	1	1	1	$A_3 + A_1$	$\circ$	6	
E_7	A_4	1	1	1	A_2	A_5	5	
E_7	A_5]''	1	1	1	A_2	$\circ$	1	$\langle 2,4,5,6,7\rangle$
	]'	1	1	1	A_1	D_6	2	$\langle 3,4,5,6,7\rangle$
E_7	A_6	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 2\rangle$
E_7	A_7	1	1	1	$\emptyset$	Σ	$\leftarrow$	
E_7	D_4	1	1	1	$3A_1$	$\circ$	1	
E_7	D_5	1	1	1	A_1	D_6	2	
E_7	D_6	2	1	1	A_1	$\times$	1	$\langle\backslash 1\rangle$
E_7	E_6	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 7\rangle$
E_7	E_7	1	1	1	$\emptyset$	Σ	1	
E_7	$2A_1$	1	1	1	$D_4 + A_1$	$\times$	15	
E_7	$3A_1$]''	1	1	1	D_4	$\circ$	1	$\langle 2,5,7\rangle$
	]'	1	1	1	$4A_1$	$\times$	10	$\langle 3,5,7\rangle$
E_7	$4A_1$]''	4	1	4	$3A_1$	$\times$	2	$\langle 2,3,5,7\rangle$
	]'	1	1	1	$3A_1$	D_4	$\leftarrow$	

E_7	$5A_1$	15	1	15	$2A_1$	D_4+A_1	$\leftarrow$	§8.2.3
E_7	$6A_1$	30	1	30	A_1	D_6	$\leftarrow$	§8.2.3
E_7	$7A_1$	30	1	30	$\emptyset$	Σ	$\leftarrow$	§8.2.3
E_7	A_2+A_1	1	1	1	A_3	A_3+A_1	18	
E_7	A_2+2A_1	1	1	1	A_1	D_6	12	
E_7	A_2+3A_1	1	1	1	$\emptyset$	Σ	1	
E_7	$2A_2$	2	1	1	A_2	A_5	4	
E_7	$2A_2+A_1$	2	1	1	$\emptyset$	Σ	3	$\subset 3A_2$
E_7	$3A_2$	4	1	1	$\emptyset$	Σ	$\rightarrow$	§8.2.5, $L: E_6$
E_7	A_3+A_1]″	1	1	1	A_3	$\circ$	2	$\langle 2,5,6,7\rangle$
	]′	1	1	1	$2A_1$	D_4+A_1	9	$\langle 3,5,6,7\rangle$
E_7	A_3+2A_1]″	2	1	2	A_1	D_6	3	$\exists(A_3+A_1)^\perp = A_3$
	]′	1	1	1				$\forall(A_3+A_1)^\perp = 2A_1$
							$\rightarrow$	$\subset D_3+D_2$ $L: D_5$
E_7	A_3+3A_1	3	1	3	$\emptyset$	Σ		$\subset 2A_3+A_1$
							$\rightarrow$	$L: D_5+A_1$
E_7	A_3+A_2	2	1	1	A_1	D_6	3	$\subset 2A_3+A_1$
E_7	$A_3+A_2+A_1$	2	1	1	$\emptyset$	Σ	1	$\langle\backslash 4\rangle$ $\subset 2A_3+A_1$
E_7	$2A_3$	2	1	1	A_1	D_6	$\leftarrow$	$\subset 2A_3+A_1$
E_7	$2A_3+A_1$	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_7	A_4+A_1	1	1	1	$\emptyset$	Σ	5	
E_7	A_4+A_2	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 5\rangle$
E_7	A_5+A_1]″	1	1	1	$\emptyset$	Σ	1	$A_5^\perp = A_2$, $\langle\backslash 3\rangle$
	]′	1	1	1	$\emptyset$	Σ	$\rightarrow$	$A_5^\perp = A_1$, $L: E_6$
E_7	A_5+A_2	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_7	D_4+A_1	3	1	1	$2A_1$	$\times$	1	
E_7	D_4+2A_1	6	1	1	A_1	D_6	$\leftarrow$	
E_7	D_4+3A_1	6	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.4
E_7	D_5+A_1	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 6\rangle$
E_7	D_6+A_1	2	1	1	$\emptyset$	Σ	$\leftarrow$	

Σ	Ξ	#	$\#_\Xi$	$\#_{\Xi'}$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P	
E_8	A_1	1	1	1	E_7	$\circ$	8	
E_8	A_2	1	1	1	E_6	$\circ$	7	
E_8	A_3	1	1	1	D_5	$\circ$	7	
E_8	A_4	1	1	1	A_4	$\circ$	6	
E_8	A_5	1	1	1	A_2+A_1	$\circ$	4	
E_8	A_6	1	1	1	A_1	E_7	3	
E_8	A_7]″	1	1	1	A_1	E_7	$\leftarrow$	
	]′	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 2\rangle$
E_8	A_8	1	1	1	$\emptyset$	Σ	$\leftarrow$	
E_8	D_4	1	1	1	D_4	$\circ$	1	
E_8	D_5	1	1	1	A_3	$\circ$	2	
E_8	D_6	1	1	1	$2A_1$	$\circ$	1	
E_8	D_7	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 1\rangle$
E_8	D_8	2	1	1	$\emptyset$	Σ	$\leftarrow$	
E_8	E_6	1	1	1	A_2	$\circ$	1	
E_8	E_7	1	1	1	A_1	$\circ$	1	$\langle\backslash 8\rangle$
E_8	E_8	1	1	1	$\emptyset$	Σ	1	
E_8	$2A_1$	1	1	1	D_6	$\circ$	21	
E_8	$3A_1$	1	1	1	D_4+A_1	$\circ$	21	

E_8	$4A_1$ $]''$	1	1	1	D_4	D_4	$\leftarrow$	
	$]'$	1	1	1	$4A_1$	$\circ$	7	$\langle 2,3,6,8\rangle$
E_8	$5A_1$	5	1	5	$3A_1$	D_4+A_1	$\leftarrow$	§8.2.3
E_8	$6A_1$	15	1	15	$2A_1$	D_6	$\leftarrow$	§8.2.3
E_8	$7A_1$	30	1	30	A_1	E_7	$\leftarrow$	§8.2.3
E_8	$8A_1$	30	1	30	$\emptyset$	Σ	$\leftarrow$	§8.2.3
E_8	A_2+A_1	1	1	1	A_5	$\circ$	28	
E_8	A_2+2A_1	1	1	1	A_3	D_5	28	
E_8	A_2+3A_1	1	1	1	A_1	E_7	7	
E_8	A_2+4A_1	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L: A_2+D_4$
E_8	$2A_2$	1	1	1	$2A_2$	$\circ$	8	
E_8	$2A_2+A_1$	2	1	1	A_2	E_6	9	
E_8	$2A_2+2A_1$	2	1	1	$\emptyset$	Σ	2	$\subset 4A_2$
E_8	$3A_2$	4	1	1	A_2	E_6	$\leftarrow$	
E_8	$3A_2+A_1$	4	1	1	$\emptyset$	Σ	$\rightarrow$	$\subset 4A_2$ $L: E_6+A_1$
E_8	$4A_2$	8	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.5
E_8	A_3+A_1	1	1	1	A_3+A_1	$\circ$	20	
E_8	A_3+2A_1 $]''$	1	1	1	A_3	D_5	$\leftarrow$	
	$]'$	1	1	1	$2A_1$	D_6	10	$\langle 2,3,4,6,8\rangle$
E_8	A_3+3A_1	3	1	3	A_1	E_7	$\rightarrow$	$L: D_5+A_1$
E_8	A_3+4A_1	3	1	3	$\emptyset$	Σ	$\rightarrow$	$\subset A_3+D_5$ $L: D_7$
E_8	A_3+A_2	1	1	1	$2A_1$	D_6	10	
E_8	$A_3+A_2+A_1$	2	1	1	A_1	E_7	4	
E_8	$A_3+A_2+2A_1$	2	1	1	$\emptyset$	Σ		$\subset D_6+2A_1$ $L: D_5+2A_1$
E_8	$2A_3$ $]''$	1	1	1	$2A_1$	D_6	$\leftarrow$	
	$]'$	1	1	1	$\emptyset$	Σ	2	$\langle 2,3,4,6,7,8\rangle$
E_8	$2A_3+A_1$	2	1	1	A_1	E_7	$\leftarrow$	
E_8	$2A_3+2A_1$	2	1	1	$\emptyset$	Σ	$\leftarrow$	$\subset D_6+2A_1$
E_8	A_4+A_1	1	1	1	A_2	E_6	12	
E_8	A_4+2A_1	1	1	1	$\emptyset$	Σ	5	
E_8	A_4+A_2	2	1	1	A_1	E_7	4	
E_8	$A_4+A_2+A_1$	2	1	1	$\emptyset$	Σ	1	$\langle\backslash 4\rangle$ $\subset 2A_4$
E_8	A_4+A_3	2	1	1	$\emptyset$	Σ	1	$\langle\backslash 5\rangle$ $\subset 2A_4$
E_8	$2A_4$	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	A_5+A_1 $]''$	1	1	1	A_2	E_6	$\leftarrow$	
	$]'$	1	1	1	A_1	E_7	3	$\langle 1,4,5,6,7,8\rangle$
E_8	A_5+2A_1	2	1	2	$\emptyset$	Σ	$\rightarrow$	$\subset A_5+A_2+A_1$ $L: E_6+A_1$
E_8	A_5+A_2	2	1	1	A_1	E_7	$\leftarrow$	
E_8	$A_5+A_2+A_1$	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	A_6+A_1	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 3\rangle$
E_8	A_7+A_1	1	1	1	$\emptyset$	Σ	$\leftarrow$	
E_8	D_4+A_1	1	1	1	$3A_1$	$\circ$	2	
E_8	D_4+2A_1	3	1	1	$2A_1$	D_6	$\leftarrow$	
E_8	D_4+3A_1	6	1	1	A_1	E_7	$\leftarrow$	
E_8	D_4+4A_1	6	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.4
E_8	D_4+A_2	1	1	1	$\emptyset$	Σ	1	
E_8	D_4+A_3	3	1	1	$\emptyset$	Σ	$\rightarrow$	$\subset 2D_4$ $L: D_7$
E_8	$2D_4$	6	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.4
E_8	D_5+A_1	1	1	1	A_1	E_7	3	

E_8	D_5+2A_1	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L: D_7$
E_8	D_5+A_2	2	1	1	$\emptyset$	Σ	1	$\langle\backslash 6\rangle$ $\subset D_5+A_3$
E_8	D_5+A_3	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	D_6+A_1	2	1	1	A_1	E_7	$\leftarrow$	
E_8	D_6+2A_1	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	E_6+A_1	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 7\rangle$
E_8	E_6+A_2	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	E_7+A_1	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	A_1^L	1	1	1	C_3	$\circ$	2	
	A_1^S	1	1	1	B_3	$\circ$	2	
F_4	A_2^L	1	1	1	A_2^S	$\circ$	1	
	A_2^S	1	1	1	A_2^L	$\circ$	1	
F_4	A_3^L	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L: B_3$
	A_3^S	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L,\, S: C_3$
F_4	D_4^L	1	1	1	$\emptyset$	Σ	$\leftarrow$	
	D_4^S	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: F_4$
F_4	B_2	1	1	1	B_2	$\circ$	1	
F_4	B_3	1	1	1	A_1^S	$\circ$	1	$\langle\backslash 4\rangle$
F_4	C_3	1	1	1	A_1^L	$\circ$	1	$\langle\backslash 1\rangle$
F_4	B_4	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	C_4	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: F_4$
F_4	F_4	1	1	1	$\emptyset$	Σ	1	
F_4	$2A_1^L$	1	1	1	B_2	B_2	$\leftarrow$	
	$2A_1^S$	1	1	1	B_2	B_2	$\leftarrow$	$S: B_2$
	$A_1^S+A_1^L$	1	1	1	$A_1^L+A_1^S$	$\times$	3	
F_4	$3A_1^L$	1	1	1	A_1^L	C_3	$\leftarrow$	
	$3A_1^S$	1	1	1	A_1^S	B_3	$\leftarrow$	$S: B_3$
	$A_1^S+2A_1^L$	1	1	1	A_1^S	B_3	$\leftarrow$	
	$2A_1^S+A_1^L$	1	1	1	A_1^L	C_3	$\leftarrow$	$S: B_2+A_1^L$
F_4	$4A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
	$4A_1^S$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: F_4$
	$2A_1^S+2A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: B_2+2A_1^L$
F_4	$A_2^L+A_1^S$	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 3\rangle$
	$A_2^S+A_1^L$	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 2\rangle$
F_4	$A_2^S+A_2^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	$B_2+A_1^L$	1	1	1	A_1^L	C_3	$\leftarrow$	
	$B_2+A_1^S$	1	1	1	A_1^S	B_3	$\leftarrow$	$S: B_3$
F_4	$B_2+2A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
	$B_2+2A_1^S$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: B_4$
F_4	$2B_2$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: B_4$
F_4	$A_3^S+A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: C_3+A_1^L$
F_4	$A_3^L+A_1^S$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	$C_3+A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	$B_3+A_1^S$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: B_4$
G_2	A_1^L	1	1	1	A_1^S	$\circ$	1	$\langle\backslash 2\rangle$
	A_1^S	1	1	1	A_1^L	$\circ$	1	$\langle\backslash 1\rangle$
G_2	A_2^L	1	1	1	$\emptyset$	Σ	$\leftarrow$	
	A_2^S	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: G_2$
G_2	G_2	1	1	1	$\emptyset$	Σ	1	
G_2	$A_1^S+A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	