A CLASSIFICATION OF SUBSYSTEMS OF A ROOT SYSTEM

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Abstract. We classify isomorphic classes of the homomorphisms of a root system Ξ to a root system Σ which do not change Cartan integers. We examine several types of isomorphic classes defined by the Weyl group of Σ, that of Ξ and the automorphisms of Σ or Ξ etc. We also distinguish the subsystem generated by a subset of a fundamental system. We introduce the concept of the dual pair for root systems which helps to study the action of the outer automorphism of Ξ on the homomorphisms.

Contents

1. Introduction

Root systems were introduced by W. Killing and E. Cartan for the study of semisimple Lie algebras and now they are basic in several fields of mathematics. In this note a subsystem of a root system means a subset of a root system which is stable under the reflections with respect to the roots in the subset. The purpose of this note is to study subsystems of a root system. It is not difficult to classify the subsystems if the root system is of the classical type but we do it in a unified way. The method used here will be useful in particular when the root system is of the exceptional type.

Let Ξ and Ξ' be subsystems of a root system Σ. We define that Ξ' is equivalent to Ξ by Σ and we write $\Xi \underset{\Sigma}{\sim} \Xi'$ if $w(\Xi) = \Xi'$ with an element w of the Weyl group W_Σ of Σ. By the classification in this note we will get complete answers to the following fundamental questions (cf. Remark 10.2 for the answers).

Q1. What kinds of subsystems of Σ exist as abstract root systems?

Q2. Suppose Ξ' is isomorphic to Ξ as abstract root systems, which is denoted by $\Xi' \simeq \Xi$. How do we know $\Xi' \underset{\Sigma}{\sim} \Xi$?

Q3. How many subsystems of Σ exist which are equivalent to Ξ?

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Q4. Does the outer automorphism of Ξ come from W_Σ?

Q5. Suppose σ is an outer automorphism of Ξ which stabilizes every irreducible component of Ξ. Is σ realized by an element of W_Σ?

Q6. Suppose that Ξ is transformed to Ξ' by an outer automorphism of Σ. Is $\Xi \underset{\Sigma}{\sim} \Xi'$ valid?

Q7. Is Ξ equivalent to a subsystem $\langle\Theta\rangle$ generated by a subset Θ of a fundamental system Ψ of Σ? How many elements exist in $\{\Theta \subset \Psi\,;\, \langle\Theta\rangle \underset{\Sigma}{\sim} \Xi\}$?

For example, Q4 may be interesting if Ξ has irreducible components which are mutually isomorphic to each other. An orthogonal system is its typical example (cf. Remark 8.2).

The first question of Q7 is studied by [1] and the answer is given there when Ξ is irreducible (cf. Remark 8.3 iii)).

To answer these questions we will study subsystems as follows.

Let Ξ and Σ be reduced root systems and let $\mathrm{Hom}(\Xi, \Sigma)$ denote the set of maps of Ξ to Σ which keep the Cartan integers $2\frac{(\alpha|\beta)}{(\beta|\beta)}$ invariant for the roots α and β. Since the map is injective and its image is a root system, the image is a subsystem of Σ isomorphic to Ξ.

Let W_Ξ and W_Σ denote the Weyl groups of Ξ and Σ respectively and put $\mathrm{Aut}(\Xi) = \mathrm{Hom}(\Xi, \Xi)$ and $\mathrm{Aut}(\Sigma) = \mathrm{Hom}(\Sigma, \Sigma)$. We will first study the most refined classification, that is, $W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)$ after the review of the standard materials for root systems in §2. In §3 we will give Theorem 3.5 which reduces the structure of $W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)$ to a simple graphic combinatorics in the extended Dynkin diagrams. It is a generalization of the fact that an element of $W_\Sigma \backslash \mathrm{Aut}(\Sigma)$ corresponds to a graph automorphism of the Dynkin diagram associated to Σ (cf. Example 3.6) and will be proved in §5 after the preparation in §4.

In §6 we define the dual pair of subsystems, which helps us to study the action of $\mathrm{Aut}(\Xi)$ on $\mathrm{Hom}(\Xi, \Sigma)$. In §10 we have the table of all the non-empty $\mathrm{Hom}(\Xi, \Sigma)$ with irreducible Σ. The table gives the numbers of the elements of the cosets

$$W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma),\ \mathrm{Aut}(\Sigma) \backslash \mathrm{Hom}(\Xi, \Sigma),$$
$$W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)/\mathrm{Aut}(\Xi),\ W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)/\mathrm{Aut}'(\Xi)$$

and the number of the subsystems generated by subsets of a fundamental system of Σ which correspond to a coset. Here $\mathrm{Aut}'(\Xi)$ is the subgroup of $\mathrm{Aut}(\Xi)$ defined by the direct product of the automorphisms of the irreducible components of Ξ. The table also determines certain closures of Ξ (cf. Definition 6.3, 6.6).

In many cases $\#\bigl(W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)/\mathrm{Aut}(\Xi)\bigr) = 1$, which is equivalent to say that the subsystems of Σ which are isomorphic to Ξ form a single W_Σ-orbit. We will also distinguish the orbits when the number is larger than one.

In §8 we give some remarks obtained by our study. For example, Q4 will be examined for the orthogonal systems of the root systems of type E_7 and E_8 .

In §9 we give the extended Dynkin diagrams and roots of the irreducible root systems following the notation in [3], which is for the reader's convenience and will be constantly used in this note. A proof of the classification of the root systems is also given for the completeness (cf. Proposition 9.3 and Remark 9.4 iv)).

Dynkin [4] classified regular subalgebras of a simple Lie algebra in his study of semisimple subalgebras. The classification is given by Table 9 and Table 11 in [4]. In Table 11, $A_6 + A_2$ and the second one of $A_7 + A_1$ should be replaced by $E_6 + A_2$ and $E_7 + A_1$, respectively. These tables describe the classification of $\mathrm{Aut}(\Sigma) \backslash \mathrm{Hom}(\Xi, \Sigma)/\mathrm{Aut}(\Xi)$ for S-closed subsystems (cf. Definition 6.6) in our table in §10 (cf. Remark 10.7 ii)) and were obtained from Dynkin diagrams given by successive procedures removing roots from extended Dynkin diagrams. The procedure is the way to classify maximal S-closed subsystems used by [2] (cf. Remark 8.4). The maximal S-closed subsystems are also classified by [8]. Our classification based on Theorem 3.5 gives a more refined classification of $W_\Sigma\backslash\mathrm{Hom}(\Xi,\Sigma)$. In fact we give a simple algorithm to give the complete representatives of the coset $W_\Sigma\backslash\mathrm{Hom}(\Xi,\Sigma)$.

The author would like to thank E. Opdam for pointing out (8.8) and related errors in the table in §10.

2. Notation

In this section we review the root systems and fix the notation related to them. All the materials in this section are elementary and found in [3].

Fix a standard inner product $(\ |\ )$ of $\mathbb{R}^n$ and an orthonormal basis $\{\epsilon_1,\dots,\epsilon_n\}$ of $\mathbb{R}^n$. For $\alpha\in\mathbb{R}^n\setminus\{0\}$ the *reflection* s_α *with respect to* α is defined by

$$\begin{array}{cccc} s_\alpha: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & \cup & & \cup \\ & x & \mapsto & s_\alpha(x) := x - 2\frac{(\alpha|x)}{(\alpha|\alpha)}\alpha \end{array} \tag{2.1}$$

and we put $|\alpha| = \sqrt{(\alpha|\alpha)}$.

Definition 2.1. A *reduced* root system of rank n is a finite subset Σ of $\mathbb{R}^n\setminus\{0\}$ which satisfies

$$\mathbb{R}^n = \textstyle\sum_{\alpha\in\Sigma}\mathbb{R}\alpha, \tag{2.2}$$

$$s_\alpha(\Sigma)=\Sigma \qquad (\forall\alpha\in\Sigma), \tag{2.3}$$

$$2\frac{(\alpha|\beta)}{(\alpha|\alpha)}\in\mathbb{Z} \qquad (\forall\alpha,\,\beta\in\Sigma), \tag{2.4}$$

$$\mathbb{R}\alpha\cap\Sigma=\{\pm\alpha\} \qquad (\forall\alpha\in\Sigma). \tag{2.5}$$

In general the rank of a root system Σ is denoted by $\operatorname{rank}\Sigma$.

Remark 2.2. i) In this note any non-reduced root system, which doesn't satisfy (2.5), doesn't appear except in §9 and hereafter for simplicity a root system always means a *reduced* root system.

ii) We use the notation $\mathbb{N}$ for the set $\{0,1,2,\dots\}$ of non-negative integers.

Definition 2.3. Let Σ be a root system of rank n. A *fundamental system* Ψ of Σ is a finite subset $\{\alpha_1,\dots,\alpha_n\}$ of Σ which satisfies

$$\mathbb{R}^n=\mathbb{R}\alpha_1+\mathbb{R}\alpha_2+\cdots+\mathbb{R}\alpha_n, \tag{2.6}$$

$$\alpha=\sum_{j=1}^n m_j(\alpha)\alpha_j\in\Sigma \Rightarrow \big(m_1(\alpha),\dots,\dots,m_n(\alpha)\big)\in\mathbb{N}^n \text{ or } -\mathbb{N}^n. \tag{2.7}$$

The fundamental system Ψ exists for any root system Σ and the root $\alpha\in\Sigma$ is *positive* (with respect to Ψ) if $m_j(\alpha)\ge 0$ for $j=1,\dots,n$, which is denoted by $\alpha>0$.

Definition 2.4. Let Θ be a finite subset of Σ and put

$$W_\Theta := \langle s_\alpha\,;\,\alpha\in\Theta\rangle = \text{the group generated by } \{s_\alpha\,;\,\alpha\in\Theta\}, \tag{2.8}$$

$$W := W_\Sigma = W_\Psi, \tag{2.9}$$

$$\langle\Theta\rangle := W_\Theta\Theta, \tag{2.10}$$

$$\Theta^\perp := \{\alpha\in\Sigma\,;\,(\alpha|\beta)=0 \quad (\forall\beta\in\Theta)\}. \tag{2.11}$$

The group W is called the *Weyl group* of Σ. A subset Ξ of Σ is called a *subsystem* of Σ if $s_\alpha(\Xi)=\Xi$ for any $\alpha\in\Xi$. Then Ξ is a root system with $\operatorname{rank}\Xi=\dim\sum_{\alpha\in\Xi}\mathbb{R}\alpha$.

We put $\alpha^\perp = \{\alpha\}^\perp$ for $\alpha \in \Sigma$. Note that $\langle\Theta\rangle$ and $\Theta^\perp$ are subsystems of Σ and

$$\operatorname{rank}\langle\Theta\rangle + \operatorname{rank}\Theta^\perp \le \operatorname{rank}\Sigma. \tag{2.12}$$

Definition 2.5. A map ι of a root system Ξ to a root system Σ is a *homomorphism* if ι keeps the Cartan integers:

$$2\frac{(\iota(\alpha)|\iota(\beta))}{(\iota(\alpha)|\iota(\alpha))} = 2\frac{(\alpha|\beta)}{(\alpha|\alpha)} \quad (\forall\alpha,\, \beta \in \Xi). \tag{2.13}$$

In this case ι is injective and $\iota(\Xi)$ is a subsystem of Σ.

The set of all homomorphisms of Ξ to Σ is denoted by $\operatorname{Hom}(\Xi,\Sigma)$ and define

$$\operatorname{Aut}(\Sigma) := \operatorname{Hom}(\Sigma,\Sigma). \tag{2.14}$$

Note that W_Σ and W_Ξ naturally act on $\operatorname{Hom}(\Xi,\Sigma)$ and

$$\iota \circ s_\alpha = s_{\iota(\alpha)} \circ \iota \quad (\iota \in \operatorname{Hom}(\Xi,\Sigma),\ \alpha \in \Xi). \tag{2.15}$$

Two homomorphisms ι and ι' of Ξ to Σ are W_Σ*-equivalent* if

$$\iota' = w \circ \iota \tag{2.16}$$

for a suitable $w \in W_\Sigma$ and we define

$$\overline{\operatorname{Hom}}(\Xi,\Sigma) := W_\Sigma\backslash\operatorname{Hom}(\Xi,\Sigma) \simeq W_\Sigma\backslash\operatorname{Hom}(\Xi,\Sigma)/W_\Xi, \tag{2.17}$$

$$\begin{aligned}\operatorname{Out}(\Sigma) &:= W_\Sigma\backslash\operatorname{Aut}(\Sigma) = \overline{\operatorname{Hom}}(\Sigma,\Sigma) \simeq \operatorname{Aut}(\Sigma)/W_\Sigma \\ &\xleftarrow{\sim} \{g \in \operatorname{Aut}(\Sigma)\,;\, g(\Psi) = \Psi\}.\end{aligned} \tag{2.18}$$

The root system Ξ is *isomorphic to* Σ, which is denoted by $\Xi \simeq \Sigma$, if there exists a surjective homomorphism of Ξ onto Σ.

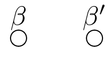
Suppose Σ_1 and Σ_2 are subsystems of Σ such that $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma_1 \perp \Sigma_2$. Then we say that Σ is a *direct sum* of Σ_1 and Σ_2, which is denoted by $\Sigma = \Sigma_1 + \Sigma_2$. A root system is *irreducible* if if has no non-trivial direct sum decomposition. Note that every root system is decomposed into a direct sum of irreducible root systems and

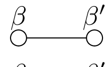
$$\operatorname{Aut}(\Sigma) \simeq \{g \in O(n)\,;\, g(\Sigma) = \Sigma\} \tag{2.19}$$

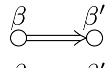
if Σ is an irreducible root system of rank n. Here $O(n)$ is the orthogonal group of $\mathbb{R}^n$ with respect to $(\ |\ )$.

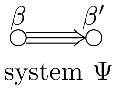
For root systems Σ_1 and Σ_2 there exists a root system $\Sigma = \Sigma_1' + \Sigma_2'$ such that $\Sigma_j \simeq \Sigma_j'$ for $j = 1$ and 2. This root system Σ is determined modulo isomorphisms and hence we simply write $\Sigma = \Sigma_1 + \Sigma_2$. When $\Sigma_1 = \Sigma_2$, we sometimes write $2\Sigma_1$ in place of $\Sigma_1 + \Sigma_2$.

For any two elements α and α' in Ψ, there exists an isomorphism ι of $\langle\alpha,\alpha'\rangle$ to one of the following four root systems with the fundamental system $\{\beta,\beta'\}$ such that $\iota(\alpha) = \beta$ and $\iota(\alpha') = \beta'$:

$A_1 + A_1 = 2A_1$: $(\beta,\beta') = (\epsilon_1,\epsilon_2)$ $\quad 2\frac{(\beta|\beta')}{(\beta|\beta)} = 0,\quad 2\frac{(\beta|\beta')}{(\beta'|\beta')} = 0$ 

A_2: $(\beta,\beta') = (\epsilon_1-\epsilon_2,\epsilon_2-\epsilon_3)$ $\quad 2\frac{(\beta|\beta')}{(\beta|\beta)} = -1,\quad 2\frac{(\beta|\beta')}{(\beta'|\beta')} = -1$ 

B_2: $(\beta,\beta') = (\epsilon_1-\epsilon_2,\epsilon_2)$ $\quad 2\frac{(\beta|\beta')}{(\beta|\beta)} = -1,\quad 2\frac{(\beta|\beta')}{(\beta'|\beta')} = -2$ 

G_2: $(\beta,\beta') = (-2\epsilon_1+\epsilon_2+\epsilon_3,\epsilon_1-\epsilon_2)$ $\quad 2\frac{(\beta|\beta')}{(\beta|\beta)} = -1,\quad 2\frac{(\beta|\beta')}{(\beta'|\beta')} = -3$ 

The *Dynkin diagram* $G(\Psi)$ of a root system Σ with the fundamental system Ψ is the graph which consists of vertices expressed by circles and edges expressed by some lines or arrows such that the vertices are associated to the elements of Ψ. The lines or arrows connecting two vertices represent the isomorphic classes of the corresponding two roots in Ψ according to the above Dynkin diagram of rank 2.

Here the number of lines which link β to β' in the diagram equals $\frac{-2(\beta|\beta')}{\min\{(\beta|\beta),(\beta'|\beta')\}}$. The arrow points toward a shorter root.

Definition 2.6. A root α of an irreducible root system Σ is called *maximal* and denoted by α_{max} if every number $m_j(\alpha)$ for $j = 1, \ldots, n$ in Definition 2.3 is maximal among the roots of Σ. It is known that the maximal root uniquely exists.

Let $\Psi = \{\alpha_1, \ldots, \alpha_n\}$ be a fundamental system of Σ. Define

$$\alpha_0 := -\alpha_{max}, \tag{2.20}$$
$$\tilde{\Psi} := \Psi \cup \{\alpha_0\}. \tag{2.21}$$

The *extended Dynkin diagram* of Σ in this note is the graph $G(\tilde{\Psi})$ associated to $\tilde{\Psi}$ which is defined in the same way as $G(\Psi)$ associated to Ψ. We call $\tilde{\Psi}$ the *extended fundamental system* of Σ. A *subdiagram* of $G(\tilde{\Psi})$ is the Dynkin diagram $G(\Theta)$ associated to a certain subset $\Theta \subset \tilde{\Psi}$.

In §9 the extended Dynkin diagrams of all the irreducible root systems are listed, which are based on the notation in [3]. The vertex expressed by a circled circle in the diagram corresponds to the special root α_0. If the vertex and the lines starting from it are removed from the diagram, we get the corresponding Dynkin diagram of the irreducible root system. The numbers below vertices α_j in the diagram in §9 are the numbers $m_j(\alpha_{max})$ given by (2.7). We define $m_0(\alpha_{\max}) = 1$ and then

$$\sum_{\alpha_j \in \tilde{\Psi}} m_j(\alpha_{\max})\alpha_j = 0. \tag{2.22}$$

Remark 2.7. i) There is a bijection between the isomorphic classes of root systems and the Dynkin diagrams.

The irreducible decomposition of a root system Σ corresponds to the decomposition of its Dynkin diagram $G(\Psi)$ into the connected components $G(\Psi_j)$. It also induces the decomposition of the fundamental system $\Psi = \Psi_1 \amalg \cdots \amalg \Psi_m$ such that $\Sigma = \langle\Psi_1\rangle + \cdots + \langle\Psi_m\rangle$ is the decomposition into irreducible root systems. Then we call each Ψ_j an *irreducible component* of Ψ.

The irreducible root systems are classified as follows (cf. §9):

$$A_n (n \geq 1),\ B_n (n \geq 2),\ C_n (n \geq 3),\ D_n (n \geq 4),\ E_6,\ E_7,\ E_8,\ F_4,\ G_2. \tag{2.23}$$

We will also use this notation $A_n, \ldots$ for a root system or a fundamental system. For example, $A_2 + 2B_3$ means a root system isomorphic to the direct sum of the root system of type A_2 and two copies of the root system of type B_3 or it means its fundamental system.

ii) $\mathrm{Out}(\Sigma)$ is naturally isomorphic to the group of graph automorphisms of the Dynkin diagram associated to Σ. If Σ is irreducible, it also corresponds to the graph automorphisms of the extended Dynkin diagram which fix the vertex corresponding to α_0. Here we give the list of irreducible root systems Σ with non-trivial $\mathrm{Out}(\Sigma)$:

$$\begin{cases} \mathrm{Out}(A_n) \simeq \mathbb{Z}/2\mathbb{Z} & (n \geq 2), \\ \mathrm{Out}(D_4) \simeq \mathfrak{S}_3 & (= \text{the symmetric group of degree 3}), \\ \mathrm{Out}(D_n) \simeq \mathbb{Z}/2\mathbb{Z} & (n \geq 5), \\ \mathrm{Out}(E_6) \simeq \mathbb{Z}/2\mathbb{Z}. \end{cases} \tag{2.24}$$

iii) The graph automorphism σ of the extended Dynkin diagram $G(\tilde\Psi)$ with the following property corresponds to a transformation by an element of W_Σ.

$$\begin{cases} \text{A rotation of } G(\tilde\Psi) & (\Sigma = A_n,\ E_6),\\ \text{Any automorphism} & (\Sigma = B_n,\ C_n,\ E_7),\\ \sigma\big((\alpha_0,\alpha_1,\alpha_{n-1},\alpha_n)\big) = (\alpha_1,\alpha_0,\alpha_n,\alpha_{n-1}) & (\Sigma = D_n),\\ \sigma\big((\alpha_0,\alpha_1,\alpha_{n-1},\alpha_n)\big) = (\alpha_n,\alpha_{n-1},\alpha_1,\alpha_0) & (\Sigma = D_n,\ n : \text{even} \ge 4),\\ \sigma\big((\alpha_0,\alpha_1,\alpha_{n-1},\alpha_n)\big) = (\alpha_n,\alpha_{n-1},\alpha_0,\alpha_1) & (\Sigma = D_n,\ n : \text{odd}). \end{cases} \tag{2.25}$$

When Σ is irreducible, we have the bijection:

$$\begin{array}{ccc} \{w \in W_\Sigma\,;\, w(\tilde\Psi) = \tilde\Psi\} & \xrightarrow{\sim} & \{\alpha_j \in \tilde\Psi\,;\, m_j(\alpha_{max}) = 1\} \\ \Psi & & \Psi \\ \sigma & \mapsto & \sigma(\alpha_0) \end{array} \tag{2.26}$$

To classify subsystems contained in a root system we prepare more definitions.

Definition 2.8. We put

$$\mathrm{Aut}'(\Xi) := \mathrm{Aut}(\Xi_1) \times \cdots \times \mathrm{Aut}(\Xi_m) \subset \mathrm{Aut}(\Xi), \tag{2.27}$$

$$\mathrm{Out}'(\Xi) := \mathrm{Aut}'(\Xi)/W_\Xi \tag{2.28}$$

for a root system Ξ with an irreducible decomposition $\Xi = \Xi_1 + \cdots + \Xi_m$.

Definition 2.9. Let Ξ, Ξ' and Θ be subsystems of Σ.

$$\Xi \underset{\Theta}{\sim} \Xi' \ \Leftrightarrow\ \exists w \in W_\Theta \text{ such that } \Xi' = w(\Xi), \tag{2.29}$$

$$\Xi \overset{w}{\underset{\Theta}{\sim}} \Xi' \ \Leftrightarrow\ \exists g \in \mathrm{Aut}(\Theta) \text{ such that } \Xi' = g(\Xi). \tag{2.30}$$

If $\Xi \underset{\Theta}{\sim} \Xi'$ (resp. $\Xi \overset{w}{\underset{\Theta}{\sim}} \Xi'$), we say that Ξ' is *equivalent* (resp. *weakly equivalent*) *to* Ξ *by* Θ. Since $\mathrm{Aut}(\Xi) \simeq \{\iota \in \mathrm{Hom}(\Xi,\Sigma)\,;\, \iota(\Xi) = \Xi\}$, we have

$$\begin{aligned} &\{\Xi' \subset \Sigma\,;\, s_\alpha(\Xi') = \Xi' \ (\forall\alpha \in \Xi') \text{ and } \Xi' \simeq \Xi\}/\underset{\Sigma}{\sim} \\ &\qquad \simeq W_\Sigma \backslash \mathrm{Hom}(\Xi,\Sigma)/\mathrm{Aut}(\Xi) \\ &\qquad \simeq \overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}(\Xi), \end{aligned} \tag{2.31}$$

$$\begin{aligned} &\{\Xi' \subset \Sigma\,;\, s_\alpha(\Xi') = \Xi' \ (\forall\alpha \in \Xi') \text{ and } \Xi' \simeq \Xi\}/\overset{w}{\underset{\Sigma}{\sim}} \\ &\qquad \simeq \mathrm{Aut}(\Sigma) \backslash \mathrm{Hom}(\Xi,\Sigma)/\mathrm{Aut}(\Xi) \\ &\qquad \simeq \mathrm{Out}(\Sigma) \backslash \overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}(\Xi), \end{aligned} \tag{2.32}$$

$$W_\Sigma \backslash \mathrm{Hom}(\Xi,\Sigma)/\mathrm{Aut}'(\Xi) \simeq \overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}'(\Xi). \tag{2.33}$$

Definition 2.10 (fundamental subsystems)**.** A subsystem Ξ of Σ is called *fundamental* if there exists $\Theta \subset \Psi$ such that $\Xi \underset{\Sigma}{\sim} \langle\Theta\rangle$.

Remark 2.11. Suppose Σ is of type A_n. Then it is clear that

$$\text{any subsystem of } \Sigma \text{ is fundamental,} \tag{2.34}$$

$$\big(\Xi \underset{\Sigma}{\sim} \Xi' \ \Leftrightarrow\ \Xi \simeq \Xi'\big) \ \text{ for subsystems } \Xi \text{ and } \Xi' \text{ of } \Sigma. \tag{2.35}$$

Our aim in this note is to clarify the structure of

$$\overline{\mathrm{Hom}}(\Xi,\Sigma),\ \mathrm{Out}(\Sigma)\backslash\overline{\mathrm{Hom}}(\Xi,\Sigma),\ \overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}(\Xi),\ \overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}'(\Xi),$$
$$\mathrm{Out}(\Sigma)\backslash\overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}(\Xi) \text{ and fundamental subsystems of } \Sigma.$$

For this purpose we prepare the following definition.

Definition 2.12. i) A root $\alpha \in \Psi$ (resp. $\tilde{\Psi}$) is an *end root* of Ψ (resp. $\tilde{\Psi}$) if

$$\#\{\beta \in \Psi \text{ (resp. } \tilde{\Psi}) \,;\, (\beta, \alpha) < 0\} \leq 1. \tag{2.36}$$

A root $\alpha \in \Psi$ (resp. $\tilde{\Psi}$) is called a *branching root* of Ψ (resp. $\tilde{\Psi}$) if

$$\#\{\beta \in \Psi \text{ (resp. } \tilde{\Psi}) \,;\, (\beta, \alpha) < 0\} \geq 3. \tag{2.37}$$

The corresponding vertex in the (extended) Dynkin diagram is also called an end vertex or a branching vertex, respectively.

ii) When Σ is irreducible, we put

$$\Sigma^L := \{\alpha \in \Sigma \,;\, |\alpha| = |\alpha_{max}|\} \tag{2.38}$$

and denote its fundamental system by $\tilde{\Psi}^L$. Then Σ^L is a subsystem of Σ and

$$\begin{gathered} A_n^L = A_n,\ B_n^L = D_n\ (n \geq 2),\ C_n^L = nA_1\ (n \geq 3),\ D_n^L = D_n\ (n \geq 4), \\ E_6^L = E_6,\ E_7^L = E_7,\ E_8^L = E_8,\ F_4^L = D_4,\ G_2^L = A_2. \end{gathered} \tag{2.39}$$

A root system whose Dynkin diagram contains no arrow is called *simply laced*.

3. A theorem

In this section we will give a simple procedure to clarify the set $\overline{\mathrm{Hom}}(\Xi, \Sigma) := W_\Sigma \backslash \mathrm{Hom}(\Xi, \Sigma)$ for root systems Ξ and Σ.

Remark 3.1. i) Note that

$$\overline{\mathrm{Hom}}\big(\Xi, \Sigma_1 + \Sigma_2\big) \simeq \coprod_{\Xi' \subset \Xi:\ \text{component}} \Big(\overline{\mathrm{Hom}}\big(\Xi', \Sigma_1\big), \overline{\mathrm{Hom}}\big((\Xi')^\perp, \Sigma_2\big)\Big), \tag{3.1}$$

$$\overline{\mathrm{Hom}}\big(\Xi_1 + \Xi_2, \Sigma\big) \simeq \coprod_{\bar{\iota} \in \overline{\mathrm{Hom}}(\Xi_1, \Sigma)} \Big(\bar{\iota}, \overline{\mathrm{Hom}}\big(\Xi_2, \iota(\Xi_1)^\perp\big)\Big). \tag{3.2}$$

Here $\bar{\iota}$ means a class of $\iota \in \mathrm{Hom}(\Xi_1, \Sigma)$ in $\overline{\mathrm{Hom}}(\Xi_1, \Sigma)$ and the *component* Ξ' of Ξ is the subsystem of Ξ such that $\Xi = \Xi' + (\Xi')^\perp$. The empty set and Ξ are also components of Ξ.

The identification (3.2) follows from

$$\{w \in W_\Sigma \,;\, w|_{\iota(\Xi)} = id\} = W_{\iota(\Xi)^\perp} \subset W_\Sigma \tag{3.3}$$

for any $\iota \in \mathrm{Hom}(\Xi, \Sigma)$ (cf. [3]).

ii) The identifications (3.1) and (3.2) assure that we may assume Ξ and Σ are irreducible. In fact, the study of the structure of $\overline{\mathrm{Hom}}(\Xi, \Sigma)$ is reduced to the study of $\bar{\iota} \in \overline{\mathrm{Hom}}(\Xi, \Sigma)$ and $\iota(\Xi)^\perp$ for irreducible Ξ and Σ.

iii) We may moreover assume $\iota(\Xi) \cap \Sigma^L \neq \emptyset$ by considering the dual root systems $\Xi^\vee := \left\{\frac{2\alpha}{(\alpha|\alpha)} \,;\, \alpha \in \Xi\right\}$ and $\Sigma^\vee := \left\{\frac{2\alpha}{(\alpha|\alpha)} \,;\, \alpha \in \Sigma\right\}$ in place of Ξ and Σ, respectively.

Definition 3.2. When $G(\Phi)$ is isomorphic to a subdiagram $G(\Theta)$ of $G(\tilde{\Psi})$ with a map $\bar{\iota} : \Phi \to \Theta \subset \tilde{\Psi}$, it is clear that $\bar{\iota}$ defines an element of $\mathrm{Hom}(\Xi, \Sigma)$. In this case we say that $\bar{\iota}$ is an *imbedding* of $G(\Phi)$ into $G(\tilde{\Psi})$.

Recalling Definition 2.4, 2.6 and 2.12, we now state a main lemma in this note, which will be proved in §5 by using lemmas in §4.

Lemma 3.3. *Let Ξ and Σ be irreducible root systems and let Φ and Ψ be their fundamental systems, respectively. Denoting*

$$\mathrm{Hom}'(\Xi, \Sigma) := \big\{\iota \in \mathrm{Hom}(\Xi, \Sigma) \,;\, \iota(\Xi) \cap \Sigma^L \neq \emptyset\big\}, \tag{3.4}$$

$$\overline{\mathrm{Hom}}'(\Xi, \Sigma) := W_\Sigma \backslash \mathrm{Hom}'(\Xi, \Sigma), \tag{3.5}$$

we have the following claims according to the type of Ξ:

1) Ξ *is of type* A_m.

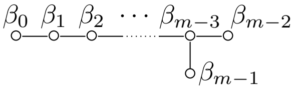
$$\overline{\mathrm{Hom}}'(\Xi,\Sigma) \overset{\sim}{\leftarrow} \big\{\textit{Imbeddings } \bar\iota \textit{ of } G(\Phi) \textit{ into } G(\tilde\Psi) \textit{ with the end vertex } \alpha_0\big\}.$$

Let $\bar\iota$ *be this graph imbedding corresponding to* $\iota \in \mathrm{Hom}(\Xi,\Sigma)$. *Then*

$$\iota(\Xi)^\perp \simeq \big\langle \alpha \in \tilde\Psi\,;\, \alpha \perp \bar\iota(\Phi)\big\rangle. \tag{3.6}$$

In the case $\#\overline{\mathrm{Hom}}'(\Xi,\Sigma) > 1$, *we have* $\#\overline{\mathrm{Hom}}'(\Xi,\Sigma) = 3$ *if* (Ξ,Σ) *is of type* (A_3, D_4) *and* 2 *if otherwise. Moreover for* $\bar\iota$, $\bar\iota' \in \overline{\mathrm{Hom}}'(\Xi,\Sigma)$

$$\begin{aligned} &\text{“}\bar\iota \textit{ and } \bar\iota' \textit{ are conjugate under an element of } \mathrm{Out}(\Sigma) \textit{ or } \mathrm{Out}(\Xi)\text{”}\\ &\qquad\qquad \Leftrightarrow\ \iota(\Xi)^\perp \simeq \iota'(\Xi)^\perp. \end{aligned} \tag{3.7}$$

2) Ξ *is of type* D_m $(m \ge 4)$.
Let $\Phi_m = \{\beta_0,\ldots,\beta_{m-1}\}$ *be a fundamental system of* Ξ *with the Dynkin diagram*  *and* m_Σ *denote the maximal integer* m *such that there is an imbedding* $\bar\iota_m$ *of* $G(\Phi_m)$ *into* $G(\tilde\Psi^L)$. *We put* $m_\Sigma = 0$ *if such an imbedding doesn't exist. Then*

$$m_\Sigma = \begin{cases} 0 & (\Sigma \textit{ is of type } A_n,\ C_n,\ G_2),\\ \mathrm{rank}\,\Sigma & (\Sigma \textit{ is of type } B_n,\ D_n,\ E_8,\ F_4),\\ 5 & (\Sigma \textit{ is of type } E_6),\\ 6 & (\Sigma \textit{ is of type } E_7) \end{cases}$$

and

$$\begin{aligned} \mathrm{Hom}'(D_m,\Sigma) \neq \emptyset\ &\Leftrightarrow\ (4 \le)m \le m_\Sigma\\ &\Leftrightarrow\ \#\big(\overline{\mathrm{Hom}}'(\Xi,\Sigma)/\mathrm{Out}(\Xi)\big) = 1. \end{aligned}$$

<u>Σ *is of type* E_6, E_7 *or* E_8.</u>

$$\#\overline{\mathrm{Hom}}(D_m,\Sigma) = \begin{cases} 2 & (m = m_\Sigma),\\ 1 & (4 \le m < m_\Sigma), \end{cases}$$

$$\iota(\Xi)^\perp \simeq D_{m_\Sigma - m} + \begin{cases} A_1 & (n = 7),\\ \emptyset & (n = 6,\,8). \end{cases}$$

<u>Σ *is of type* D_n, B_n *or* F_4 $(m \le n)$.</u>

$$\#\overline{\mathrm{Hom}}'(\Xi,\Sigma) = \begin{cases} 6 & (\Sigma : D_4\ (m = n = 4)),\\ 3 & (\Sigma : B_n \textit{ and } D_n\ (m = 4 < n)),\\ 2 & (\Sigma : D_n\ (4 < m = n)),\\ 1 & (\Sigma : F_4\ (4 = m),\ B_n\ (4 < m \le n),\ D_n\ (4 < m < n)), \end{cases}$$

$$\iota(\Xi)^\perp \simeq \begin{cases} D_{n-m} & (\iota \in \mathrm{Hom}(D_m, D_n)),\\ B_{n-m} & (\iota \in \mathrm{Hom}(D_m, B_n)),\\ \emptyset & (\iota \in \mathrm{Hom}(D_4, F_4)). \end{cases}$$

3) Ξ *is of type* B_m $(m \ge 2)$.

$$\mathrm{Hom}(\Xi,\Sigma) \neq \emptyset\ \Leftrightarrow\ \#\overline{\mathrm{Hom}}(\Xi,\Sigma) = 1 \textit{ and } \begin{cases} \Sigma \textit{ is of type } B_n \textit{ with } m \le n,\\ \Sigma \textit{ is of type } C_n \textit{ with } m = 2,\\ \Sigma \textit{ is of type } F_4 \textit{ with } m \le 4, \end{cases}$$

$$\iota(\Xi)^\perp \cap T_n \simeq T_{n-m} \quad (T = B,\,C,\,F,\ F_2 = B_2,\ F_1 = A_1 \textit{ and } \iota(B_3)^\perp \cap F_4 \not\subset \Sigma^L).$$

4) Ξ *is of type* C_m $(m \geq 3)$.

$$\mathrm{Hom}(\Xi,\Sigma)\neq\emptyset \;\Leftrightarrow\; \#\overline{\mathrm{Hom}}(\Xi,\Sigma)=1 \text{ and } \begin{cases}\Sigma \text{ is of type } C_n \text{ with } m\leq n,\\ \Sigma \text{ is of type } F_4 \text{ with } m\leq 4,\end{cases}$$

$$\iota(\Xi)^{\perp}\cap T_n\simeq T_{n-m} \quad (T=C,\ F,\ F_2=C_2,\ F_1=A_1 \text{ and } \iota(C_3)^{\perp}\cap F_4\subset\Sigma^L).$$

5) Ξ *is of type* E_m ($m=6$, 7 and 8).

$$\overline{\mathrm{Hom}}(\Xi,\Sigma)\overset{\sim}{\leftarrow}\bigl\{\text{Imbeddings } \bar\iota \text{ of } G(\Phi) \text{ into } G(\tilde\Psi)\bigr\}/\!\sim,$$
$$\iota(\Xi)^{\perp}\simeq\bigl\langle\alpha\in\tilde\Psi\,;\,\alpha\perp\bar\iota(\Phi)\bigr\rangle.$$

Here $/\!\sim$ *is interpreted that all the imbeddings of* $G(\Phi)$ *are considered to be isomorphic except for* $(\Xi,\Sigma)\simeq(E_6,E_6)$. *Namely* $\#\overline{\mathrm{Hom}}(\Xi,\Sigma)\leq 1$ *if* $(\Xi,\Sigma)\not\simeq(E_6,E_6)$.

6) Ξ *is of type* G_2 *or* F_4.

$$\mathrm{Hom}(\Xi,\Sigma)\neq\emptyset \;\Leftrightarrow\; \#\overline{\mathrm{Hom}}(\Xi,\Sigma)=1 \text{ and } \Xi\simeq\Sigma.$$

Remark 3.4. i) In the proof of Lemma 3.3 2) we will have

$$\iota(\Xi)^{\perp}\simeq\begin{cases}\bigl\langle\bar\iota_{m_\Sigma}(\Phi_{m_\Sigma})^{\perp}\cap\Psi\bigr\rangle & (m_\Sigma-1\leq m\leq m_\Sigma),\\ \bigl\langle\bar\iota_{m_\Sigma}(\Phi_{m_\Sigma})^{\perp}\cap\Psi,\ \bar\iota_{m_\Sigma}(\beta_m),\dots\bar\iota_{m_\Sigma}(\beta_{m_\Sigma-1})\bigr\rangle & (4\leq m\leq m_\Sigma-2)\end{cases}$$

for the imbedding $\bar\iota_{m_\Sigma}$ with $\bar\iota(\beta_{m_\Sigma})=\alpha_0$ if Σ is of type D_n, E_6, E_7 or E_8.

Let Θ_m be a subset of $\tilde\Psi$ such that $\langle\Theta_m\rangle\simeq D_m$. If Σ is of type B_n or D_n, we may assume that $\iota_m\in\overline{\mathrm{Hom}}(D_m,\Sigma)$ satisfies $\iota_m(\Xi)=\langle\Theta_m\rangle$ and then

$$\iota_m(\Phi_m)^{\perp}=\bigl\langle\Theta_m^{\perp}\cap\tilde\Psi\bigr\rangle. \tag{3.8}$$

Suppose Σ is of type E_6, E_7 or E_8. Let $\tilde\alpha_{max}$ be the maximal root of $\langle\Theta_{m_\Sigma}\rangle$. Put $\tilde\alpha_0=-\tilde\alpha_{max}$ and $\tilde\Theta_{m_\Sigma}=\Theta_{m_\Sigma}\cup\{\tilde\alpha_0\}$. We may assume $\iota_m\in\overline{\mathrm{Hom}}(D_m,\Sigma)$ satisfies $\iota_m(\Xi)=\langle\Theta_m\rangle$ and $\Theta_m\subset\tilde\Theta_{m_\Sigma}$. Then

$$\iota_m(\Phi_m)^{\perp}=\bigl\langle\Theta_m^{\perp}\cap\tilde\Theta_{m_\Sigma},\ \Theta_{m_\Sigma}^{\perp}\cap\tilde\Psi\bigr\rangle. \tag{3.9}$$

Note that $G(\tilde\Theta_{m_\Sigma})$ is the extended Dynkin diagram of $\langle\Theta_{m_\Sigma}\rangle\simeq D_{m_\Sigma}$. See Example 3.6 viii) and ix).

ii) Using a graph automorphism of $G(\tilde\Psi)$ corresponding to a suitable element of W_Σ, we may replace α_0 by another element α_j of Ψ with $m_j(\alpha_{max})=1$ in Theorem 3.5 and in the remark above (cf. Remark 2.7 ii)).

iii) The image $\iota(\Xi)$ corresponding to the graph automorphism $\bar\iota$ in Lemma 3.3 is obtained by Proposition 4.4.

Lemma 3.3 can be summarized in the following form.

Theorem 3.5. *Let* Σ *and* Ξ *be irreducible root systems and let* Ψ *and* Φ *be their fundamental systems, respectively. Retain the notation given in Definition 2.4–2.6 and 2.12. If* Σ *is not simply laced, we denote the maximal root in* $\Sigma\backslash\Sigma^L$ *by* α'_{max} *and the Dynkin diagram of* $\tilde\Psi'$ *by* $G(\tilde\Psi')$. *Here we put* $\alpha'_0=-\alpha'_{max}$ *and* $\tilde\Psi'=\Psi\cup\{\alpha'_0\}$.

i) *Suppose* Σ *is of the classical type or* $\Xi\simeq A_m$ *with* $m\geq 1$.

When $\Xi\not\simeq D_4$ *or* $(\Sigma,\Xi)\simeq(D_4,D_4)$,

$$\overline{\mathrm{Hom}}(\Xi,\Sigma)\overset{\sim}{\leftarrow}\{\text{Imbeddings } \bar\iota \text{ of } G(\Phi) \text{ to } G(\tilde\Psi) \text{ or } G(\tilde\Psi') \text{ such that } \beta_0 \text{ corresponds to } \alpha_0 \text{ or } \alpha'_0 \text{ by } \bar\iota\} \tag{3.10}$$

for a suitable root $\beta_0\in\Phi$. *Here we delete* $G(\tilde\Psi')$ *and* α'_0 *in the above if* Σ *is simply laced. Moreover* β_0 *is any root in* Φ *such that the right hand side of* (3.10) *is not empty and if such* β_0 *doesn't exit,* $\overline{\mathrm{Hom}}(\Xi,\Sigma)=\emptyset$.

When $\Xi\simeq D_4$,

$$\#\bigl(\overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}(\Xi)\bigr)\leq 1 \tag{3.11}$$

and the representative of $\overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}(\Xi)$ *is given by the above imbedding* $\bar\iota$ *and*

$$\#\overline{\mathrm{Hom}}(D_4,B_n)=\#\overline{\mathrm{Hom}}(D_4,C_n)=\#\overline{\mathrm{Hom}}(D_4,D_{n+1})=3\quad(n\ge 4). \tag{3.12}$$

For $\iota\in\mathrm{Hom}(\Xi,\Sigma)$ *corresponding to this imbedding* $\bar\iota$ *of* $G(\Phi)$ *we have*

$$\iota(\Xi)^\perp=\big\langle\alpha\in\Psi\,;\,\alpha\perp\bar\iota(\Phi)\big\rangle. \tag{3.13}$$

Moreover for $\bar\iota$, $\bar\iota'\in\overline{\mathrm{Hom}}(\Xi,\Sigma)$

$$\begin{aligned}&\text{“}\bar\iota\text{ and }\bar\iota'\text{ are conjugate under an element of }\mathrm{Out}(\Sigma)\text{ or }\mathrm{Out}(\Xi)\text{”}\\ &\quad\Leftrightarrow\ \iota(\Xi)\cap\Sigma^L\simeq\iota'(\Xi)\cap\Sigma^L\text{ and }\iota(\Xi)^\perp\simeq\iota'(\Xi)^\perp.\end{aligned} \tag{3.14}$$

ii) *Suppose* Σ *is of the exceptional type and* $\Xi=R_m$ *with* $R=B$, C, D, E, F *and* G. *Put* $m_0^R=2$, 3, 4, 6, 4 *and* 2 *according to* $R=B$, C, D, E, F *and* G, *respectively, and moreover suppose* $m\ge m_0^R$. *Let* m_Σ^R *be the maximal number* m *such that the Dynkin diagram* $G(R_m)$ *of the root system* R_m *is a subdiagram of* $G(\tilde\Psi)$ *or* $G(\tilde\Psi')$. *Thus for a subset* Φ_Σ^R *of* $\tilde\Psi$ *or* $\tilde\Psi'$ *we identify* $G(R_{m_\Sigma^R})$ *with the subdiagram* $G(\Phi_\Sigma^R)$. *Put* $m_\Sigma^R=0$ *if such a number* m *with* $m\ge m_0^R$ *does not exists.*

When $(\Sigma,R_m)\not\simeq(F_4,D_4)$, *we have* (3.11) *and*

$$\#\overline{\mathrm{Hom}}(R_m,\Sigma)=\begin{cases}0 & (m>m_\Sigma^R),\\ \#\,\mathrm{Out}(R_{m_\Sigma^R}) & (m=m_\Sigma^R),\\ 1 & (m_0^R\le m<m_\Sigma^R),\end{cases} \tag{3.15}$$

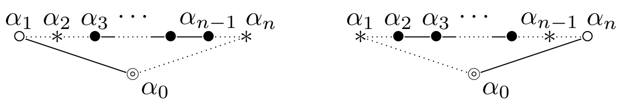
$$R_m^\perp\cap\Sigma=(R_m^\perp\cap R_{m_\Sigma^R})+\big\langle(\Phi_\Sigma^R)^\perp\cap\tilde\Psi\ (\text{or }\tilde\Psi')\big\rangle\qquad(m_0^R\le m\le m_\Sigma^R) \tag{3.16}$$

through the natural map $G(R_m)\subset G(R_{m_\Sigma^R})\simeq G(\Phi_\Sigma^R)\subset G(\tilde\Psi)$ (*or* $G(\tilde\Psi')$) *and* $R_m^\perp\cap R_{m_\Sigma^R}$ *is given by i) or Lemma 3.3 5). The coset* $\overline{\mathrm{Hom}}(D_4,F_4)$ *consists of the two elements corresponding to the identifications* $D_4\simeq F_4^L$ *and* $D_4\simeq F_4\setminus F_4^L$.

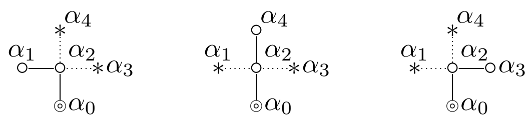
Proof. When Σ is of type R with $R=B_n$, C_n, F_4 or G_2, $G(\tilde\Psi')$ is the affine Dynkin diagram $\tilde R'$ given by Proposition 9.3. This theorem follows from Lemma 3.3, Remark 3.1 iv), Remark 9.4 iii) and Remark 4.2. □

Example 3.6. $\big(\overline{\mathrm{Hom}}(\Xi,\Sigma)$ and $\Xi^\perp\big)$

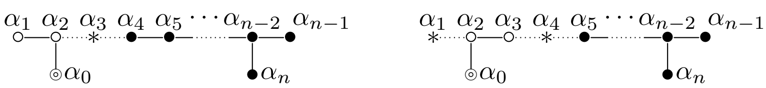
i) $\#\overline{\mathrm{Hom}}(A_2,A_n)=2$ and $A_2^\perp\cap A_n\simeq A_{n-3}$ $(n\ge 2)$.
Two elements of $\overline{\mathrm{Hom}}(A_2,A_n)$ are defined by $(\alpha_1,\alpha_2)\mapsto(\alpha_0,\alpha_1)$ and $(\alpha_1,\alpha_2)\mapsto(\alpha_0,\alpha_n)$, respectively. They are isomorphic to each other under $\mathrm{Out}(A_2)$. Note that the rotation of the extended Dynkin diagram corresponds to an element of W_{A_n}.



ii) $\#\overline{\mathrm{Hom}}(A_3,D_4)=3$, $\#\big(\mathrm{Out}(D_4)\backslash\overline{\mathrm{Hom}}(A_3,D_4)\big)=1$ and $A_3^\perp\cap D_4=\emptyset$.
The group $\mathrm{Out}(D_4)\simeq\mathfrak{S}_3$ corresponds to that of the graph automorphisms of the extended Dynkin diagram which fix α_0.

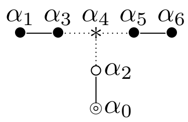


iii) $\#\overline{\mathrm{Hom}}(A_3,D_n)=\#\big(\mathrm{Out}(D_n)\backslash\overline{\mathrm{Hom}}(A_3,D_n)/\mathrm{Out}(A_3)\big)=2$ for $n>4$.

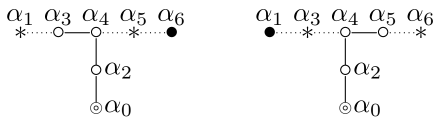


$A_3^\perp \cap D_n \simeq D_{n-3}$ or D_{n-4} according to the imbeddings $A_3 \subset D_n$.

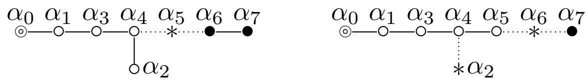
iv) $\#\overline{\mathrm{Hom}}(A_2, E_6) = 1$ and $A_2^\perp \cap E_6 \simeq 2A_2$. Then $3A_2 \subset E_6$ and $\#\overline{\mathrm{Hom}}(3A_2, E_6) = \#\overline{\mathrm{Hom}}(2A_2, 2A_2) = 8$ (cf. §8.2.5) .



v) $\#\overline{\mathrm{Hom}}(A_4, E_6) = 2$, $\#\big(\mathrm{Out}(E_6)\backslash\overline{\mathrm{Hom}}(A_4, E_6)\big) = 1$ and $A_4^\perp \cap E_6 \simeq A_1$.

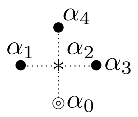


vi) $\#\overline{\mathrm{Hom}}(A_5, E_7) = \#\big(\mathrm{Out}(E_7)\backslash\overline{\mathrm{Hom}}(A_5, E_7)/\mathrm{Out}(A_5)\big) = 2$.



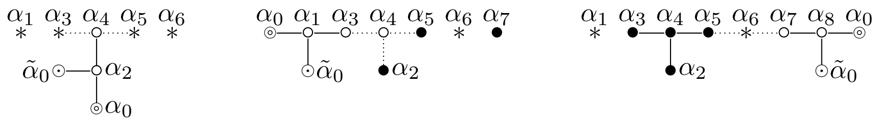
$A_5^\perp \cap E_7 \simeq A_2$ or A_1 according to the imbeddings $A_5 \subset E_7$.

vii) $\#\overline{\mathrm{Hom}}(4A_1, D_4) = 6$ and $\#\big(\mathrm{Out}(D_4)\backslash\overline{\mathrm{Hom}}(4A_1, D_4)\big) = 1$.



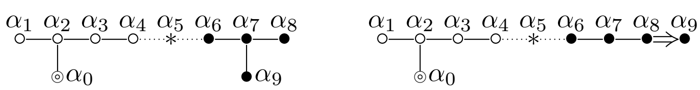
$$\alpha_0^\perp = \{\pm\alpha_1, \pm\alpha_3, \pm\alpha_4\}$$

viii) $\#\overline{\mathrm{Hom}}(D_4, E_n) = 1$ (cf. Remark 3.4 i))



$D_4^\perp \cap E_6 = \emptyset$ $\qquad$ $D_4^\perp \cap E_7 \simeq 3A_1$ $\qquad$ $D_4^\perp \cap E_8 \simeq D_4$

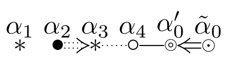
ix) $\#\overline{\mathrm{Hom}}(D_5, D_9) = \#\overline{\mathrm{Hom}}(D_5, B_9) = 1$ (cf. Remark 3.4 i)).



$D_5^\perp \cap D_9 \simeq D_4$ $\qquad$ $D_5^\perp \cap B_9 \simeq B_4$

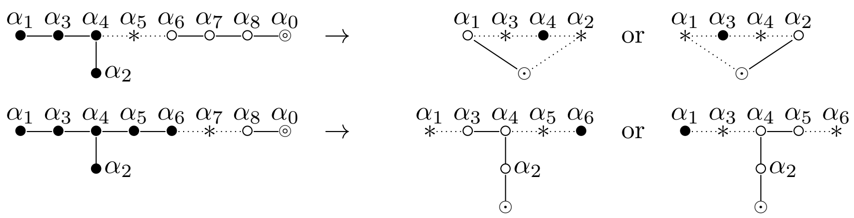
x) $\#\overline{\mathrm{Hom}}(A_2, F_4) = 2$, $(A_2^L)^\perp \cap F_4 \simeq A_2^S$, $(A_2^S)^\perp \cap F_4 \simeq A_2^L$ with $A_2^S = A_2 \setminus A_2^L$.

xi) $\#\overline{\mathrm{Hom}}(C_3, F_4) = 1$, $G(C_4) \subset G(\tilde{F}_4')$, $G(C_3) \subset G(\tilde{C}_4)$ and $C_3^\perp \cap F_4 = A_1^L$.



$$C_4 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_0'\},\ C_3 = \{\alpha_4, \alpha_0', \tilde{\alpha}_0\}.$$

xii) $\#\overline{\mathrm{Hom}}(A_4 + A_2, E_8) = 2$ and $\#\big(\overline{\mathrm{Hom}}(A_4 + A_2, E_8)/\mathrm{Out}(A_4 + A_2)\big) = 1$. Putting $(\Xi_1, \Xi_2, \Sigma) = (A_4, A_2, E_8)$ (resp. (A_2, A_4, E_8)) in the identification (3.2), we have the first (resp. second) line of diagrams below. These two reductions lead to the same result. In particular $(A_4 + A_2)^\perp \cap E_8 \simeq A_1$. Note that (A_4, A_4) and (A_2, E_6) are special dual pairs in E_8 (cf. Definition 6.3).



Corollary 3.7. i) *Suppose Σ is not of type A. Let $G(\{\alpha_0, \alpha_{j_1}, \dots, \alpha_{j_{m-1}}\})$ be a maximal subdiagram of $G(\tilde\Psi)$ isomorphic to $G(A_m)$ such that α_0 and $\alpha_{j_{m-1}}$ are the end vertices of the subdiagram and α_{j_ν} are not the branching vertex of $G(\tilde\Psi)$ for $\nu = 1, \dots, m-2$. Then*

$$\#\overline{\mathrm{Hom}}'(A_k, \Sigma) = 1 \qquad (k = 1, \dots, m),$$

$$\#\overline{\mathrm{Hom}}'(A_{m+1}, \Sigma) \begin{cases} = 0 & \big(\alpha_{j_{m-1}} \text{ is not a branching vertex of } G(\tilde\Psi)\big) \\ > 1 & \big(\alpha_{j_{m-1}} \text{ is a branching vertex of } G(\tilde\Psi)\big) \end{cases}$$

with

$$m = \begin{cases} 2 \quad (\Sigma = B_n,\ n \geq 3), \qquad 1 \quad (\Sigma = B_2,\ C_n), \\ 2 \quad (\Sigma = D_n,\ n \geq 4), \\ 3 \quad (\Sigma = E_6), \qquad 4 \quad (\Sigma = E_7), \qquad 6 \quad (\Sigma = E_8), \\ 3 \quad (\Sigma = F_4), \qquad 2 \quad (\Sigma = G_2). \end{cases}$$

Here $\alpha_{j_{m-1}}$ is the branching vertex if $\Sigma = B_n$ $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7 or E_8.

ii) *We consider the following procedure for a Dynkin diagram X:*

> *If X is connected, we replace it by the subdiagram X' of the extended Dynkin diagram $\tilde X$ of X where the vertices of X' correspond to the roots orthogonal to the maximal root of $\tilde X$. If an irreducible component of X' has no root with the length of the maximal root, we remove the component.*
> *If X is not connected, we choose one of the connected component of X and change the component by the above procedure.*

Then $\overline{\mathrm{Hom}}'(rA_1, \Sigma)$ corresponds to the totality of r steps of the above procedures starting from $G(\Psi)$. The existence of these steps implies $\mathrm{Hom}'(rA_1, \Sigma) \neq \emptyset$ and in this case $\#\overline{\mathrm{Hom}}'(rA_1, \Sigma) = 1$ if and only if any non-connected Dynkin diagram does not appear except for the final step. In particular, we have the following:

Let $r(\Sigma)$ be the maximal integer r satisfying $\mathrm{Hom}'(rA_1, \Sigma) \neq \emptyset$. Then

$$r(\Sigma) = 1 + \sum_j r(\Sigma'_j). \tag{3.17}$$

Here $\{\Sigma'_j\}$ is the set of irreducible components of $\alpha_0^\perp$ such that $\Sigma'_j \cap \Sigma^L \neq \emptyset$ and

$$\begin{aligned}
&r(A_n) = 1 + r(A_{n-2}) = [\tfrac{n+1}{2}] && (n \geq 2), && r(A_1) = 1,\ r(A_0) = 0, \\
&r(B_n) = 2 + r(B_{n-2}) = 2[\tfrac{n}{2}] && (n \geq 4), && r(B_3) = r(B_2) = 2, \\
&r(C_n) = 1 + r(C_{n-1}) = n && (n \geq 3), && r(C_2) = 2, \\
&r(D_n) = 2 + r(D_{n-2}) = 2[\tfrac{n}{2}] && (n \geq 4), && r(D_3) = r(D_2) = 2, \\
&r(E_6) = 1 + r(A_5) = 4, && && r(E_7) = 1 + r(D_6) = 7, \\
&r(E_8) = 1 + r(E_7) = 8, \\
&r(F_4) = 1 + r(C_3) = 4, && && r(G_2) = 1.
\end{aligned}$$

Remark 3.8. i) If Σ is of type A, D or E, then $\overline{\mathrm{Hom}}(rA_1,\Sigma)$ is figured as follows according to the procedures in Corollary 3.7 ii) and the notation in §9.

$$A_n \longrightarrow A_{n-2} \longrightarrow A_{n-4} \longrightarrow A_{n-6} \longrightarrow \cdots \qquad D_n \longrightarrow D_{n-2}+A_1 \longrightarrow \cdots$$

$$E_6 \xrightarrow{\alpha_0} A_5 \xrightarrow{\frac{\epsilon_1+\cdots+\epsilon_4-\epsilon_5+\epsilon_6+\epsilon_7-\epsilon_8}{2}} A_3 \xrightarrow{\epsilon_4-\epsilon_1} A_1 \xrightarrow{\epsilon_3-\epsilon_2} \emptyset \qquad E_8 \xrightarrow{-\epsilon_7-\epsilon_8} E_7$$

$$\begin{tikzcd}
 & & & 4A_1 \arrow[r] \arrow[dd, Rrightarrow] & 3A_1 \arrow[dr, Rrightarrow] & & & \\
E_7 \arrow[r, "\epsilon_7-\epsilon_8"] & D_6 \arrow[r, "-\epsilon_5-\epsilon_6"] & D_4+A_1 \arrow[ur, "-\epsilon_3-\epsilon_4"] \arrow[dr, "\epsilon_6-\epsilon_5"'] & & & 2A_1 \arrow[r, Rightarrow] & A_1 \arrow[r] & \emptyset \\
 & & & D_4 \arrow[r] & 3A_1 \arrow[ur, Rrightarrow] & & &
\end{tikzcd}$$

There appear the subsystems $3A_1$ of E_7 twice in the above. They are distinguished by the structure of $(3A_1)^\perp \cap E_7$ but they are in the same W_{E_8}-orbit under the above inclusion $E_7 \subset E_8$ (cf. §7.2, §7.3 and §8.2.3).

For example, it follows from the procedures shown above that

$$\begin{aligned}
\#\overline{\mathrm{Hom}}(5A_1,E_7) &= \#\overline{\mathrm{Hom}}(4A_1,D_6) = \#\overline{\mathrm{Hom}}(3A_1,D_4+A_1)\\
&= \#\overline{\mathrm{Hom}}(2A_1,D_4) + \#\overline{\mathrm{Hom}}(2A_1,4A_1)\\
&= \#\overline{\mathrm{Hom}}(A_1,3A_1) + 4\#\overline{\mathrm{Hom}}(A_1,3A_1) = 3+4\cdot 3 = 15.
\end{aligned} \tag{3.18}$$

ii) For an irreducible root system Σ, we can easily calculate $\#\overline{\mathrm{Hom}}(\Xi,\Sigma)$ and $\Xi^\perp \cap \Sigma$ for any root system Ξ in virtue of Theorem 3.5 together with Remark 3.1 (cf. Example 3.4 x)). The complete list for non-trivial $\overline{\mathrm{Hom}}(\Xi,\Sigma)$ is given in §10. More refined structures related to the actions of $\mathrm{Out}(\Sigma)$ and $\mathrm{Out}(\Xi)$ etc. are also given in §10, which will be studied in later sections.

4. Lemmas

In this section we prepare some lemmas to prove Lemma 3.3 and we always assume that Ψ is a fundamental system of an *irreducible* root system Σ and $\tilde\Psi$ is the corresponding extended fundamental system.

First note that for $\alpha \in \tilde\Psi \cap \Sigma^L$ we have

$$2\frac{(\alpha|\beta)}{(\alpha|\alpha)} \in \begin{cases} \{0,-1\} & (\forall\beta \in \langle\Psi\setminus\{\alpha\}\rangle \text{ and } \beta>0),\\ \{0,1\} & (\forall\beta \in \langle\Psi\setminus\{\alpha\}\rangle \text{ and } \beta<0).\end{cases} \tag{4.1}$$

Here we put $\Psi\setminus\{\alpha\} = \Psi$ if $\alpha \notin \Psi$.

Lemma 4.1. *If a subset Θ of $\tilde\Psi$ contains α_0 and the diagram $G(\Theta)$ is connected,*

$$\Theta^\perp = \bigl\langle \alpha \in \tilde\Psi\,;\, \alpha \perp \Theta \bigr\rangle. \tag{4.2}$$

Proof. Note that $(\alpha_i|\alpha_j) \le 0$ for $0 \le i < j \le n$.
We will prove the lemma by the induction on $\#\Theta$.

We may put $\Theta = \{\alpha_0,\alpha_1,\ldots,\alpha_m\}$ and we may assume $\Theta' := \{\alpha_0,\alpha_1,\ldots,\alpha_{m-1}\}$ is empty or forms a connected subdiagram.

Let $\alpha = \sum_{j=1}^n m_j(\alpha)\alpha_j \in \Theta^\perp$ with $m_j(\alpha) \ge 0$. Then the induction hypothesis for Θ' implies $m_j(\alpha) = 0$ for $j \le m$ and

$$0 = (\alpha_m|\alpha) = \textstyle\sum_{j=m+1}^n m_j(\alpha)(\alpha_m|\alpha_j).$$

Hence $(\alpha_m|\alpha_j) \ne 0$ means $m_j(\alpha) = 0$. $\square$

Remark 4.2. In Lemma 4.1 we may replace α_0 by any element α_0' satisfying $(\alpha_0'|\alpha) \le 0$ for all $\alpha \in \Psi$. Then the Dynkin diagram $G(\tilde\Psi')$ of $\tilde\Psi' = \Psi \cup \{\alpha_0'\}$ is an affine Dynkin diagram in Proposition 9.3.

Lemma 4.3. *Fix $\Theta \subset \Psi$ and $\mathbf{m} \in \mathbb{Z}^{\#\Theta} \setminus \{0\}$. Define the map*

$$\begin{array}{rccc} p_\Theta : & \Sigma & \to & \mathbb{Z}^{\#\Theta} \\ & \Cup & & \Cup \\ & \beta = \sum_{\alpha_i \in \Psi} m_i(\beta)\alpha_i & \mapsto & \bigl(m_i(\beta)\bigr)_{\alpha_i \in \Theta} \end{array}.$$

Then $p_\Theta^{-1}(\mathbf{m}) \cap \Sigma^L$ is empty or a single $W_{\Psi\setminus\Theta}$-orbit. Moreover $p_\Theta^{-1}(\mathbf{m}) \setminus \Sigma^L$ is also empty or a single $W_{\Psi\setminus\Theta}$-orbit.

Proof. Fix $0 \neq \mathbf{m} = (m_i)_{\alpha\in\Theta}$ in the image of p_Θ.

Let $\mathfrak{g}$ be the complex simple Lie algebra with the root system Σ and let $X_\alpha \in \mathfrak{g}$ be a root vector for $\alpha \in \Sigma$. We denote by $\mathfrak{g}_{\Psi\setminus\Theta}$ the semisimple Lie algebra generated by $\{X_\alpha\,;\,\alpha \in \Psi \setminus \Theta\}$. Then the space

$$V_{\mathbf{m}} := \textstyle\sum_{\alpha \in p_\Theta^{-1}(\mathbf{m})} \mathbb{C}X_\alpha \subset \mathfrak{g}$$

is a $\mathfrak{g}_{\Psi\setminus\Theta}$-stable subset under the adjoint representation of $\mathfrak{g}$, which is an irreducible representation of $\mathfrak{g}_{\Psi\setminus\Theta}$ as is shown in [6, Proposition 2.39 ii)].

Let π_Θ be the orthogonal projection of $\sum_{\alpha\in\Psi}\mathbb{R}\alpha$ onto $\sum_{\alpha\in\Psi\setminus\Theta}\mathbb{R}\alpha$ with respect to $(\ |\ )$. Put $v_{\mathbf{m}} = \sum_{\alpha_i\in\Theta} m_i\alpha_i - \pi_\Theta(\sum_{\alpha_i\in\Theta} m_i\alpha_i)$. Then

$$\pi_\Theta(\alpha) = \alpha - v_{\mathbf{m}},\ (\pi_\Theta(\alpha)|\pi_\Theta(\alpha)) = (\alpha|\alpha) - (v_{\mathbf{m}}|v_{\mathbf{m}}) \quad (\forall \alpha \in p_\Theta^{-1}(\mathbf{m})).$$

The set of the weights of the irreducible representation $(\mathfrak{g}_{\Psi\setminus\Theta}, V_{\mathbf{m}})$ is $\pi_\Theta(p_\Theta^{-1}(\mathbf{m}))$ and the set of the weights with the longest length is $\pi_\Theta(p_\Theta^{-1}(\mathbf{m}) \cap \Sigma^L)$. Hence $p_\Theta^{-1}(\mathbf{m}) \cap \Sigma^L$ is a single $W_{\Psi\setminus\Theta}$-orbit.

When $p_\Theta^{-1}(\mathbf{m}) \not\subset \Sigma^L$, we have the last statement in the lemma by combining the above argument with [6, Proposition 2.37 ii)]. □

Proposition 4.4. *For a proper subset Θ of the extended fundamental system $\tilde\Psi$ of Σ we have*

$$\langle\Theta\rangle = \begin{cases} p_{\Psi\setminus\Theta}^{-1}\bigl(0\bigr) & (\alpha_0 \notin \Theta), \\ p_{\Psi\setminus\Theta}^{-1}\bigl(\{0, \pm p_{\Psi\setminus\Theta}(\alpha_0)\}\bigr) & (\alpha_0 \in \Theta) \end{cases} \tag{4.3}$$

under the notation in Lemma 4.3.

Proof. Note that $\langle\Theta\rangle \supset p_{\Psi\setminus\Theta}^{-1}\bigl(0\bigr)$. We assume $\alpha_0 \in \Psi$ because the claim is clear when $\alpha_0 \notin \Theta$. Then (4.1) implies $\langle\Theta\rangle \subset p_{\Psi\setminus\Theta}^{-1}\bigl(\{0, \pm p_{\Psi\setminus\Theta}(\alpha_0)\}\bigr)$. Let Θ_0 be the irreducible component of Θ containing α_0. Since $\langle\Theta\rangle$ is $W_{\Theta\setminus\{\alpha_0\}}$-invariant, Lemma 4.3 implies that

$$\langle\Theta\rangle \setminus p_{\Psi\setminus\Theta}^{-1}\bigl(0\bigr) = p_{\Psi\setminus\Theta}^{-1}\bigl(\{\pm p_{\Psi\setminus\Theta}(\alpha_0)\}\bigr) \text{ or } p_{\Psi\setminus\Theta}^{-1}\bigl(\{\pm p_{\Psi\setminus\Theta}(\alpha_0)\}\bigr) \cap \Sigma^L.$$

Hence the proposition is clear if Σ is simply laced or if Θ_0 is of type B_n or C_n. It is also easy to check $p_{\Psi\setminus\Theta}^{-1}\bigl(p_{\Psi\setminus\Theta}(\alpha_0)\bigr) \subset \Sigma^L$ in any other case when $(\Psi, \Theta_0) = (B_n, A_{m-1})$, (B_n, D_m), (C_n, A_1) or (F_4, A_k) with $m \le n$ and $k \le 3$. □

Lemma 4.5 (roots orthogonal to the end root)**.** *Suppose α_1 is an end root of Ψ with $\alpha_1 \in \Sigma^L$. Then the set*

$$Q = \bigl\{\alpha = \alpha_1 + m_2(\alpha)\alpha_2 + m_3(\alpha)\alpha_3 + \cdots + m_n(\alpha)\alpha_n \in \Sigma^L\,;\,(\alpha|\alpha_1) = 0\bigr\} \tag{4.4}$$

is empty if Ψ is of type A and it is a single $W_{\Psi\cap\alpha_1^\perp}$-orbit if otherwise.

Proof. We may assume $\#\Psi > 1$. Then there is a unique $\beta \in \Psi$ with $(\alpha_1|\beta) < 0$. We may assume $\beta = \alpha_2$ and we have

$$Q = \bigl\{\alpha = \alpha_1 + 2\alpha_2 + m_3(\alpha)\alpha_3 + \cdots + m_n(\alpha)\alpha_n \in \Sigma^L\bigr\}.$$

Then $Q = \emptyset$ if and only if Ψ is of type A. If Ψ is not of type A, Lemma 4.3 assures that Q is a single $W_{\Psi\setminus\{\alpha_1,\alpha_2\}}$-orbit. Note that $\Psi \cap \alpha_1^\perp = \Psi \setminus \{\alpha_1, \alpha_2\}$. □

Lemma 4.6 (special imbeddings of A_2 and A_3)**.** *Let $\Psi' \subset \Psi$. If $\Psi' \neq \Psi$, we assume that we can choose $\alpha' \in \Psi \cap \Sigma^L$ with $\alpha' \notin \Psi'$. If $\Psi' = \Psi$, we put $\alpha' = \alpha_0$. Define*

$$\begin{aligned}
Q_1 &:= \big\{\beta \in \langle\Psi'\rangle \cap \Sigma^L\,;\,(\beta|\alpha') < 0\big\},\\
Q_2 &:= \big\{(\beta_1,\beta_2) \in \langle(\Psi'\rangle \cap \Sigma^L) \times (\langle\Psi'\rangle \cap \Sigma^L)\,;\\
&\qquad (\beta_1,\alpha') = (\beta_2|\alpha') < 0 \text{ and } (\beta_1|\beta_2) = 0\big\},\\
\Theta &:= \big\{\alpha \in \Psi'\,;\,(\alpha|\alpha') < 0\big\},\\
\Theta^L &:= \Theta \cap \Sigma^L.
\end{aligned}$$

Then Θ^L is the set of complete representatives of $Q_1/W_{\Psi'\setminus\Theta}$. Moreover if $\Psi' \neq \Psi$,

$$\begin{aligned}
Q_2/\#W_{\Psi'\setminus\Theta} &= \#\Theta^L\big(\#\Theta^L - 1\big)\\
&\quad + \big\{\alpha \in \Theta^L\,;\,G(\Psi'_\alpha) \text{ is not of type } A \text{ or not an end root of } G(\Psi'_\alpha)\big\}.
\end{aligned}$$

Here Ψ'_α is the irreducible component of Ψ' containing $\alpha \in \Theta$.

Proof. Let $\beta \in Q_1$. It follows from (4.1) that there exists $\alpha_m \in \Psi'$ satisfying

$$\beta = \alpha_m + \sum_{\alpha_j \in \Psi'\setminus\Theta} m_j(\beta)\alpha_j, \tag{4.5}$$

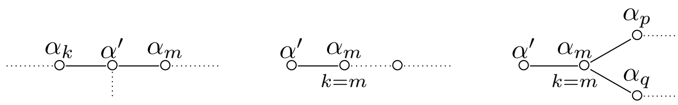
$$(\alpha_m|\beta) < 0. \tag{4.6}$$

If $\alpha_m \notin \Sigma^L$, Ψ'_{α_m} is of type A or C and therefore β of the form (4.5) does not belong to Σ^L. Hence $\alpha_m \in \Theta^L$ and $\alpha_m \in W_{\Psi'\setminus\Theta}\beta$ by Lemma 4.3.

Let α_m, $\alpha_{m'} \in \Theta^L$ with $m \neq m'$. We have $\alpha_{m'} \notin W_{\Psi'\setminus\Theta}\alpha_m$ and therefore Θ^L is the set of complete representatives of $Q_1/W_{\Psi'\setminus\Theta}$.

Let $(\beta_1,\beta) \in Q_2$. We may assume $\beta_1 = \alpha_k \in \Theta^L$ by the argument above and β is of the form (4.5) with $\alpha_m \in \Theta^L$.

If $k \neq m$, we may similarly assume $\beta = \alpha_m$ and $(\alpha_k,\alpha_m) \in Q_2$.



Suppose $k = m$. If α_m is the end root of Ψ_{α_m}, it follows from Lemma 4.5 that Ψ'_{α_m} is not of type A and (α_k,β) corresponds to a unique element of $Q_2/W_{\Psi'\setminus\Theta}$.

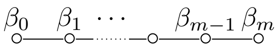
If α_m is not the end root of Ψ'_{α_m}, Ψ'_{α_m} is of type A and it is easy to see that (α_k,β) also corresponds to a unique class in $Q_2/W_{\psi'\setminus\Theta}$. In fact, we may put $\{\alpha \in \Psi_{\alpha_m}\,;\,(\alpha|\alpha_m) < 0\} = \{\alpha_p,\alpha_q\}$ and

$$\beta = \alpha_m + \alpha_p + \alpha_q + \sum_{\alpha_j \in \Psi'\setminus\{\alpha_m,\alpha_p,\alpha_q\}} m_j(\beta)\beta \in \Sigma.$$

Note that the roots β with this expression are in a single $W_{\Psi'\setminus\{\alpha_m,\alpha_p,\alpha_q\}}$-orbit. Thus we have the lemma. □

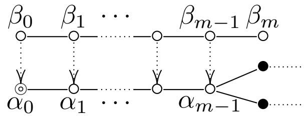
5. Proof of the main Lemma

Retain the notation in Lemma 3.3 to prove it.

1) Let Ξ be of type A_{m+1} with the fundamental system $\Phi = \{\beta_0,\dots,\beta_m\}$ and the Dynkin diagram  .

First note that $\bar\iota$ naturally corresponds to an element of $\mathrm{Hom}'(\Xi,\Sigma)$ and then (3.6) follows from Lemma 4.1. We will prove the lemma by the induction on m.

Let $\iota \in \mathrm{Hom}'(\Xi,\Sigma)$. Since $\{\alpha \in \Sigma\,;\,|\alpha| = |\alpha_{max}|\} = W_\Sigma\alpha_{max}$, the lemma is clear when $m = 0$. Suppose $m \geq 1$. By the induction hypothesis we may assume that there exists a unique sequence $(\alpha_0, \ldots, \alpha_{m-1})$ of element of $\tilde{\Psi}$ and an element $w \in W_\Sigma$ such that $w \circ \iota(\beta_j) = \alpha_j$ for $j = 0, \ldots, m-1$.

$w \circ \iota$: 

Put $\alpha'_m = w \circ \iota(\beta_m)$ and

$$\Psi' = \{\alpha \in \tilde{\Psi}\,;\, (\alpha|\alpha_j) = 0 \quad (j = 0, \ldots, m-2)\},$$
$$\Theta = \{\alpha \in \Psi'\,;\, (\alpha|\alpha_{m-1}) < 0\}.$$

Since $(\alpha'_m|\alpha_j) = 0$ for $j = 0, \ldots, m-2$, $\alpha'_m \in \langle \Psi' \rangle$. Applying Lemma 4.6 to $\alpha' := \alpha_{m-1}$, we have $\alpha_m \in \Theta \cap \Sigma^L$ and $w' \in W_{\Psi' \setminus \Theta}$ such that $w'(\alpha'_m) = \alpha_m$. Hence $w'w \circ \iota$ corresponds to a required imbedding of $G(\Phi)$ into $G(\Psi)$.

The uniqueness of $\alpha_m \in \Theta \cap \Sigma^L$ is proved as follows. Suppose there exists $w \in W_\Sigma$ such that

$$w\alpha_j = \alpha_j \quad \text{for } j = 0, \ldots, m-1 \quad \text{and} \quad w\alpha_m \in \Theta \cap \Sigma^L.$$

Then $w \in W_{\Psi' \setminus \Theta}$ and Lemma 4.6 assures $w\alpha_m = \alpha_m$.

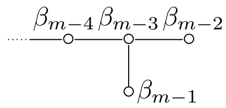
Thus we have proved the first claim and then Lemma 4.1 assures (3.6). The last claim is easily obtained by applying the claims we have proved to the extended Dynkin diagrams in §9.

2) Let Ξ is of type D_m with $m \geq 4$. We may assume that Σ is of type B_n, D_n, E_n or F_n. Let $\iota \in \mathrm{Hom}'(\Xi, \Sigma)$. Lemma 3.3 1) assures that there exists a unique sequence $\alpha_0, \alpha_{j_1}, \ldots, \alpha_{j_{m-3}}$ in $\tilde{\Psi}$ and an element $w \in W_\Sigma$ such that

$$w \circ \iota(\beta_\nu) = \alpha_{j_\nu} \quad (\nu = 0, \ldots, m-3) \text{ with } j_0 = 0. \tag{5.1}$$

Putting

$$\Psi' = \bigl\{\alpha \in \Psi,\ (\alpha|\alpha_{j_\nu}) = 0 \quad (\nu = 0, \ldots, m-4)\bigr\},$$
$$\Theta = \bigl\{\alpha \in \Psi'\,;\, (\alpha, \alpha_{j_{m-3}})) < 0\bigr\},$$
$$\alpha' = \alpha_{j_{m-3}},$$
$$(\beta, \beta') = \bigl(w \circ \iota(\beta_{m-2}), w \circ \iota(\beta_{m-1})\bigr),$$



we have β, $\beta' \in \langle \Psi' \rangle$ and we can apply Lemma 4.6 as in the case when Ξ is of type A. Thus

$$\begin{aligned}&\#W_\Sigma \backslash \bigl\{\iota \in \mathrm{Hom}'(\Xi, \Sigma)\,;\, \exists w \in W_\Sigma \text{ such that (5.1) is satisfied.}\bigr\}\\ &= \bigl(\#(\Theta \cap \Sigma^L)\bigr)\bigl(\#(\Theta \cap \Sigma^L) - 1\bigr) + \#\bigl\{\alpha \in \Theta \cap \Sigma^L \ :\ \text{the irreducible component of}\\ &\quad \Psi' \text{ containing } \alpha \text{ is not of type } A \text{ or } \alpha \text{ is not an end vertex of the component}\bigr\}.\end{aligned}$$

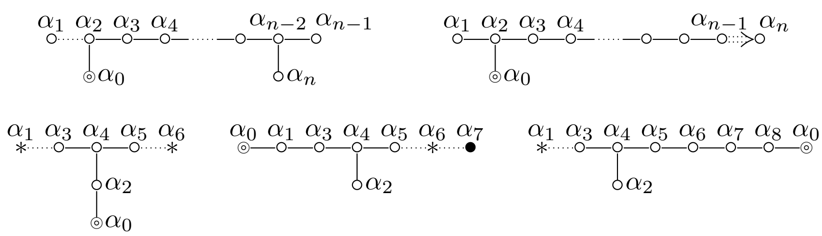
Hence $\mathrm{Hom}'(D_m, \Sigma) = \emptyset$ if Σ is of type A_n, C_n or G_2 or $m > \operatorname{rank} \Sigma$. Moreover we have $\#\overline{\mathrm{Hom}}(D_m, \Sigma)$ shown in the following table under the notation in §10.

Σ	Ξ	Ψ'	#
D_4	D_4	$\{\alpha_1, \alpha_3, \alpha_4\} \simeq 3A_1$	6
D_5	D_4	$\Psi \setminus \{\alpha_2\} \simeq A_1 + A_3 (\ni \alpha_3 :$ not an end root)	3
$D_{n\ (n\geq 6)}$	D_4	$\Psi \setminus \{\alpha_2\} \simeq A_1 + D_{n-2}$	3
B_4	D_4	$\Psi \setminus \{\alpha_2\} \simeq A_1 + B_2$	3
F_4	D_4	$\Psi \setminus \{\alpha_1\} \simeq C_3$	1
$D_{n\ (4<m=n)}$	D_m	$\{\alpha_{n-1}, \alpha_n\} \simeq 2A_1$	2
$D_{n\ (4<m=n-1)}$	D_m	$\{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\} \simeq A_3 (\ni \alpha_{n-2} :$ not an end root)	1
$D_{n\ (4<m\leq n-2)}$	D_m	$\{\alpha_{m-1}, \ldots, \alpha_n\} \simeq D_{n-m+1}$	1
$B_{n\ (4<m\leq n)}$	D_m	$\{\alpha_{m-1}, \ldots, \alpha_n\} \simeq B_{n-m+1}$	1
E_6	D_4	$\Psi \setminus \{\alpha_2\} \simeq A_5 (\ni \alpha_4 :$ not an end root)	1
	D_5	$\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\} \simeq 2A_2$	2
E_7	D_4	$\Psi \setminus \{\alpha_1\} \simeq D_6$	1
	D_5	$\{\alpha_2, \alpha_4, \ldots, \alpha_7\} \simeq A_5 (\ni \alpha_4 :$ not an end root)	1
	D_6	$\{\alpha_2, \alpha_5, \alpha_6, \alpha_7\} \simeq A_1 + A_3 (\ni \alpha_5 :$ an end root)	2
E_8	D_4	$\Psi \setminus \{\alpha_8\} \simeq E_7$	1
	D_5	$\{\alpha_1, \ldots, \alpha_6\} \simeq E_6$	1
	D_6	$\{\alpha_1, \ldots, \alpha_5\} \simeq D_5$	1
	D_7	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \simeq A_4 (\ni \alpha_4 :$ not an end root)	1
	D_8	$\{\alpha_2, \alpha_1, \alpha_3\} \simeq A_1 + A_2$	2

Here m_Σ is the rank of the maximal subdiagram of type D_m contained in the extended Dynkin diagram of Σ^L and then

$$m_\Sigma = \begin{cases} n & (\Sigma \text{ is of type } B_n \text{ or } D_n), \\ 5,\ 6,\ 8 & (\Sigma \text{ is of type } E_6,\ E_7 \text{ or } E_8, \text{ respectively}). \end{cases} \tag{5.2}$$

(5.3)



Fix $\iota \in \mathrm{Hom}(D_4, F_4)$. Since $\mathrm{Hom}(D_4, F_4)$ is a single W_{F_4}-orbit, for any $g \in \mathrm{Aut}(D_4)$ there exists $w_g \in W_{F_4}$ with $\iota \circ g = w_g \circ \iota$. Here w_g is uniquely determined by g because $\operatorname{rank} F_4 = \operatorname{rank} D_4$. Hence we have

$$\begin{aligned} &\mathrm{Aut}(D_4) \simeq W_{F_4} \supset \mathrm{Aut}(B_4) = W_{B_4} \supset W_{D_4}, \\ &\mathrm{Out}(D_4) \simeq \mathfrak{S}_3, \quad W_{B_4}/W_{D_4} \simeq \mathbb{Z}/2\mathbb{Z}. \end{aligned} \tag{5.4}$$

Let $\iota : D_4 \subset D_n$ $(\subset B_n)$ be the natural imbedding given by the realization in §9 and let $g \in \mathrm{Aut}(D_4)$ be a non-trivial rotation of $G(D_4)$. Then it is easy to see $\iota \circ g \neq w \circ \iota$ for any $w \in W_{B_n}$. Hence if $n \geq 4$, we have

$$\#\big(\overline{\mathrm{Hom}}(D_4, D_n)/\mathrm{Out}(D_4)\big) = \#\big(\overline{\mathrm{Hom}}(D_4, B_n)/\mathrm{Out}(D_4)\big) = 1 \tag{5.5}$$

because $\#\big(\overline{\mathrm{Hom}}(D_4, D_n)\big) = \#\big(\overline{\mathrm{Hom}}(D_4, B_n)\big) = 3$ and moreover we have

$$D_m^\perp \cap D_n \simeq D_{n-m}, \quad D_m^\perp \cap B_n \simeq B_{n-m} \tag{5.6}$$

for $m = 4$. Here $D_0 = D_1 = B_0 = \emptyset$, $D_2 \simeq 2A_1$, $B_1 \simeq A_1^S$ and A_1^S is the root space of type A_1 such that $A_1^S \cap (B_n)^L = \emptyset$.

Note that

$$\mathrm{Aut}(D_n) \simeq W_{B_n} \text{ and } \mathrm{Out}(D_n) := \mathrm{Aut}(D_n)/W_{D_n} \simeq \mathbb{Z}/2\mathbb{Z} \quad (n \geq 5) \tag{5.7}$$

under the natural imbedding $D_n \subset B_n$ of root spaces. Thus we have

$$\#\big(\overline{\mathrm{Hom}}(D_m,\Sigma)/\mathrm{Out}(D_m)\big) = 0 \text{ or } 1 \text{ if } m \ge 4 \text{ and } \Sigma \text{ is irreducible} \tag{5.8}$$

and therefore $\{\iota(D_m)^\perp\,;\,\iota \in \mathrm{Hom}(D_m,\Sigma)\}$ is a single W_Σ-orbit if it is non-empty.

Thus we have (5.6) for $4 \le m \le n$ since it does not depend on the imbedding of D_m.

Let $n \in \{6,7,8\}$ and put $m = m_{E_n}$. There exists $\iota \in \mathrm{Hom}(D_m,E_n)$ such that ι corresponds to the imbedding of Φ_{m_Σ} to $\tilde\Psi$ with $\iota(\beta_0) = \alpha_0$. Then we have $D_5^\perp \cap E_6 = \emptyset$, $D_6^\perp \cap E_7 \simeq A_1$ and $D_8^\perp \cap E_8 = \emptyset$ from Lemma 4.1.

Moreover there exists $\iota' \in \mathrm{Hom}(D_{m-1},E_n)$ such that

$$\iota'(D_{m-1}) = \big\{\iota(\beta_0),\dots,\iota(\beta_{m-3}),\iota(\beta_{m-3})+\iota(\beta_{m-2})+\iota(\beta_{m-1})\big\}$$

and it is clear that $D_{m-1}^\perp \cap E_n \simeq D_m^\perp \cap E_n$.

Let $4 \le k \le 6$. Then $D_k^\perp \cap E_8 \supset D_k^\perp \cap D_8 \simeq D_{8-k}$ and we can conclude $D_k^\perp \cap E_8 \simeq D_{8-k}$ because $\mathrm{rank}(D_k^\perp \cap E_8) \le 8-k$ and there is no root system containing D_{8-k} as a proper subsystem such that its roots have the same length and its rank is not larger than $8-k$.

Since $D_6^\perp \cap E_7 \simeq A_1$ and $D_4^\perp \cap D_6 \simeq 2A_1$, $D_4^\perp \cap E_7 \supset 3A_1$ and we have $D_4^\perp \cap E_7 \simeq 3A_1$ by the same argument as above.

Thus we have obtained the claims in the lemma and therefore Remark 3.4 i) is also clear.

3) Suppose Ξ is of type B_m with $m \ge 2$.

Note that for any $\beta \in \Xi \setminus \Xi^L$, there exists β_1, $\beta_2 \in \Xi^L$ such that $\beta = \frac12(\beta_1+\beta_2)$ and $(\beta_1|\beta_2) = 0$. Hence $\iota \in \mathrm{Hom}(\Xi,\Sigma)$ is determined by $\iota|_{\Xi^L}$. Note that Ξ^L is of type D_m with $D_2 \simeq 2A_1$ and $D_3 \simeq A_3$.

Then $\mathrm{Hom}(\Xi,\Sigma) \ne \emptyset$ means Σ is of type B_n $(n \ge m)$ or F_4 if $m > 2$.

If $m > 2$ or if Σ is of type F_4, $\#\overline{\mathrm{Hom}}(\Xi^L,\Sigma) = 1$ and therefore $\#\overline{\mathrm{Hom}}(\Xi,\Sigma) = 1$. If $m = 2$ and $\Sigma = B_n$ or C_n, it is easy to see that

$$\big\{\iota \in \mathrm{Hom}(2A_1,\Sigma)\,;\,\tfrac12\big(\iota(\beta_1)+\iota(\beta_2)\big) \in \Sigma\big\}$$

is a single W_Σ-orbit and we have also $\#\overline{\mathrm{Hom}}(\Xi,\Sigma) = 1$. Here $\Xi^L = \langle\beta_1\rangle + \langle\beta_2\rangle \simeq 2A_1$.

4) When Σ is of type C_n, we have the lemma from the case 3) by considering the dual root systems $\Xi^\vee$ and $\Sigma^\vee$.

5) We first examine $\overline{\mathrm{Hom}}(E_6,E_8)$ and $\overline{\mathrm{Hom}}(E_7,E_8)$ under the notation in §9. Since $\#\overline{\mathrm{Hom}}(A_5,E_8) = \#\overline{\mathrm{Hom}}(A_6,E_8) = 1$, we may assume

$$E_6^o \supset \Psi_{A_5} = \{\alpha_0 = -\epsilon_7-\epsilon_8,\ \alpha_8 = \epsilon_7-\epsilon_6,\ \alpha_7 = \epsilon_6-\epsilon_5,\ \alpha_6 = \epsilon_5-\epsilon_4,\ \alpha_5 = \epsilon_4-\epsilon_3\}$$

for the imbedding $E_6 \simeq E_6^o \subset E_8$. Let $\tilde\alpha \in \Phi \setminus \Psi_{A_5}$. We have

$$\tilde\alpha = \sum_{j=1}^{8} c_j\epsilon_j \in E_6^o \subset E_8 :\ \langle\tilde\alpha,\alpha_j\rangle = \begin{cases} 0 & (j = 0,\,8,\,6,\,5),\\ -1 & (j = 7). \end{cases}$$

Thus

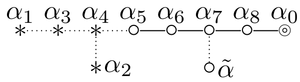
$$\tilde\alpha = c_1\epsilon_1 + c_2\epsilon_2 + c(\epsilon_3+\epsilon_4+\epsilon_5) + (c-1)(\epsilon_6+\epsilon_7-\epsilon_8).$$

Since $\tilde\alpha$ is a root of E_8, we have $c = \frac12$ and hence

$$\tilde\alpha = \alpha_\pm := \tfrac12(\pm(\epsilon_1+\epsilon_2)+\epsilon_3+\epsilon_4+\epsilon_5-\epsilon_6-\epsilon_7+\epsilon_8).$$

Since $\alpha_2 = \epsilon_1 + \epsilon_2$ is orthogonal to α_j $(j = 0, 5, 6, 7, 8)$ and $s_{\alpha_2}\alpha_+ = \alpha_-$, we have $\#\overline{\mathrm{Hom}}(E_6, E_8) = 1$ and

$$\begin{aligned}(E_6^o)^\perp \cap E_8 &\simeq \Psi_{A_5}^\perp \cap \alpha_+^\perp \cap E_8\\ &= (\langle \alpha_1, \alpha_3\rangle + \langle \alpha_2\rangle) \cap \alpha_+^\perp\\ &= \langle \alpha_1, \alpha_3\rangle \simeq A_2.\end{aligned}$$



Let $E_7 \simeq E_7^o \subset E_8$. Then we may moreover assume $\alpha_4 = \epsilon_3 - \epsilon_2 \in E_7$ and the condition $\tilde\alpha \perp \alpha_4$ implies $\tilde\alpha = \alpha_+$. Hence $\#\overline{\mathrm{Hom}}(E_7, E_8) = 1$ and

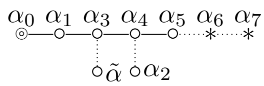
$$(E_7^o)^\perp \cap E_8 \simeq \langle \alpha_1, \alpha_3\rangle \cap \alpha_4^\perp = \langle \alpha_1\rangle \simeq A_1.$$

Now we examine $\overline{\mathrm{Hom}}(E_6, E_7)$. Since $A_5 \subset E_6 \simeq E_6^o \subset E_7$, the argument in 1) assures that we may assume

$$\begin{aligned}&E_6^o \supset \Psi'_{A_5} := \Psi_{A_4} \cup \{\alpha_2 = \epsilon_1 + \epsilon_2\} \text{ or } E_6^o \supset \Psi_{A_5} := \Psi_{A_4} \cup \{\alpha_5 = \epsilon_4 - \epsilon_3\}\\ &\Psi_{A_4} := \{\alpha_0 = \epsilon_8 - \epsilon_7,\ \alpha_1 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8),\\ &\qquad \alpha_3 = \epsilon_2 - \epsilon_1,\ \alpha_4 = \epsilon_3 - \epsilon_2\}.\end{aligned}$$

Then there exists $\tilde\alpha = \sum_{j=1}^8 c_j\epsilon_j \in E_6^o \subset E_7$ such that

$$\begin{aligned}(\tilde\alpha|\alpha_j) &= 0 \quad (j = 0,\ 1,\ 4),\\ (\tilde\alpha|\alpha_3) &= -1,\\ (\tilde\alpha|\alpha_2)(\tilde\alpha|\alpha_5) &= 0.\end{aligned}$$



Then the condition $(\tilde\alpha|\alpha_0) = 0$ implies $c_7 = c_8 = 0$ and

$$\begin{gathered}\tilde\alpha = (c+1)\epsilon_1 + c(\epsilon_2 + \epsilon_3) + c_4\epsilon_4 + c_5\epsilon_5 + c_6\epsilon_6,\\ 1 - c - c_4 - c_5 - c_6 = 0,\\ (2c+1)(c - c_4) = 0.\end{gathered}$$

Hence $c = 0$, $E_6^0 \supset \Psi_{A_5}$ and $\tilde\alpha = \epsilon_1 + \epsilon_5$ or $\epsilon_1 + \epsilon_6$. Since

$$\Psi_{A_5}^\perp \cap E_7 = \langle \alpha_7\rangle = \langle \epsilon_6 - \epsilon_5\rangle$$

and $s_{\epsilon_6 - \epsilon_5}(\epsilon_1 + \epsilon_5) = \epsilon_1 + \epsilon_6$, we have $\#\overline{\mathrm{Hom}}(\Xi, \Sigma) = 1$ and $(E_6^o)^\perp \cap E_7 = \emptyset$.

If $\#\,\mathrm{Hom}(\Xi, \Sigma) = 1$, any element of $\mathrm{Hom}(\Xi, \Sigma)$ is isomorphic to the imbedding ι corresponding to the graphic imbedding $\bar\iota$ given in the claim. Since $\iota(\Xi)^\perp \supset \langle \Psi \cap \bar\iota(\Xi)^\perp\rangle$ and $\iota(\Xi)^\perp \simeq \langle \Psi \cap \bar\iota(\Xi)^\perp\rangle$, we have $\iota(\Xi)^\perp = \langle \Psi \cap \bar\iota(\Xi)^\perp\rangle$.

6) If Ξ is of type G_2 or F_4, the lemma is clear and thus we have completed the proof of the lemma.

6. Dual pairs and closures

Definition 6.1. For a subsystem Ξ of a root system Σ and a subgroup G of $\mathrm{Aut}(\Sigma)$ we put

$$N_G(\Xi) := \{g \in G\,;\, g(\Xi) = \Xi\}, \quad Z_G(\Xi) := \{g \in G\,;\, g|_\Xi = id\}, \tag{6.1}$$

$$\mathrm{Aut}_\Sigma(\Xi) := N_{W_\Sigma}(\Xi)/Z_{W_\Sigma}(\Xi) \subset \mathrm{Aut}(\Xi), \tag{6.2}$$

$$\mathrm{Out}_\Sigma(\Xi) := \mathrm{Aut}_\Sigma(\Xi)/W_\Xi \simeq N_{W_\Sigma}(\Xi)/(W_\Xi \times W_{\Xi^\perp}) \subset \mathrm{Out}(\Xi). \tag{6.3}$$

Note that the isomorphism in (6.3) follows from the equality $Z_{W_\Sigma}(\Xi) = W_{\Xi^\perp}$.

Proposition 6.2. *Let Ξ_1 be a subsystem of Σ. Put $\Xi_2 = \Xi_1^\perp$. Then there is a homomorphism*

$$\varpi\ :\ \mathrm{Out}_\Sigma(\Xi_1) \to \mathrm{Out}_\Sigma(\Xi_2) \simeq \mathrm{Out}_\Sigma(\Xi_2^\perp) \tag{6.4}$$

and

$$\varpi \text{ is bijective if } \Xi_2^\perp = \Xi_1, \tag{6.5}$$

$$\mathrm{Out}_\Sigma(\Xi_1) \xrightarrow{\sim} \mathrm{Out}(\Xi_1) \text{ if } \#\big(\overline{\mathrm{Hom}}(\Xi_1,\Sigma)/\mathrm{Out}(\Xi_1)\big) = \#\overline{\mathrm{Hom}}(\Xi_1,\Sigma), \tag{6.6}$$

$$\mathrm{Out}_\Sigma(\Xi_2) \xrightarrow{\sim} \mathrm{Out}(\Xi_2) \text{ if } \#\big(\overline{\mathrm{Hom}}(\Xi_2,\Sigma)/\mathrm{Out}(\Xi_2)\big) = \#\overline{\mathrm{Hom}}(\Xi_2,\Sigma). \tag{6.7}$$

Proof. Since $N_{W_\Sigma}(\Xi_1) \subset N_{W_\Sigma}(\Xi_2)$ and $\Xi_2^\perp \supset \Xi_1$, (6.4) is well-defined and (6.5) is clear. Suppose $\#\big(\overline{\mathrm{Hom}}(\Xi_1,\Sigma)/\mathrm{Out}(\Xi_1)\big) = \#\overline{\mathrm{Hom}}(\Xi_1,\Sigma)$. Then for any $g \in \mathrm{Aut}(\Xi_1)$ there exists $w \in W_\Sigma$ with $w|_{\Xi_1} = g|_{\Xi_1}$ and (6.6) is clear. We similarly have (6.7). The isomorphism in (6.4) follows from (6.5) and the relation $(\Xi_2^\perp)^\perp = \Xi_2$. □

Definition 6.3 (dual pairs)**.** A pair (Ξ_1,Ξ_2) of subsystems of a root system Σ is called a *dual pair* in Σ if

$$\Xi_1^\perp = \Xi_2 \quad\text{and}\quad \Xi_2^\perp = \Xi_1. \tag{6.8}$$

If (Ξ_1,Ξ_2) is a dual pair, the map ϖ in Proposition 6.2 is an isomorphism. The dual pair is called *special* if the map ϖ is the isomorphism

$$\varpi : \mathrm{Out}(\Sigma_1) \xrightarrow{\sim} \mathrm{Out}(\Sigma_2). \tag{6.9}$$

For a subsystem Ξ of Σ, its $\perp$-*closure* $\overline{\Xi}$ is defined by $\overline{\Xi} := (\Xi^\perp)^\perp$. Then $(\Xi,\Xi^\perp)$ is a dual pair if and only if Ξ is $\perp$-closed (i.e. $(\Xi^\perp)^\perp = \Xi$) and hence $(\overline{\Xi},\Xi^\perp)$ is always a dual pair. We say that Ξ is $\perp$-*dense* in Σ if $\Xi^\perp = \emptyset$.

Corollary 6.4. *Let* (Ξ_1,Ξ_2) *be a dual pair in* Σ*. Then*

$$\mathrm{Out}(\Xi_1) \not\simeq \mathrm{Out}(\Xi_2) \ \Rightarrow \begin{cases} \#\big(\mathrm{Hom}(\Xi_1,\Sigma)/\mathrm{Out}(\Xi_1)\big) < \#\overline{\mathrm{Hom}}(\Xi_1,\Sigma) \\ \text{or} \\ \#\big(\mathrm{Hom}(\Xi_2,\Sigma)/\mathrm{Out}(\Xi_2)\big) < \#\overline{\mathrm{Hom}}(\Xi_2,\Sigma). \end{cases} \tag{6.10}$$

Suppose $\#\overline{\mathrm{Hom}}(\Xi_2,\Sigma) = 1$*. Let* $\iota \in \mathrm{Hom}(\Xi_1,\Sigma)$*. Then we have*

$$(\Xi_1,\Xi_2) \text{ is a special dual pair} \Leftrightarrow \#\,\mathrm{Out}(\Xi_1) = \#\,\mathrm{Out}(\Xi_2), \tag{6.11}$$

$$\exists w \in W_\Sigma \text{ such that } \iota(\Xi_1) = w(\Xi_1) \ \Leftrightarrow\ \iota(\Xi_1)^\perp \simeq \Xi_2. \tag{6.12}$$

Proof. Note that (6.10) is the direct consequence of Proposition 6.2.

Suppose $\#\overline{\mathrm{Hom}}(\Xi_2,\Sigma) = 1$. Then Proposition 6.2 implies

$$\mathrm{Out}(\Xi_1) \supset \mathrm{Out}_\Sigma(\Xi_1) \xrightarrow{\sim} \mathrm{Out}(\Xi_2)$$

and (6.11) is clear. Then if $\iota(\Xi_1)^\perp \simeq \Xi_2$, there exists $w \in W_\Sigma$ with $\iota(\Xi_1^\perp) = w(\Xi_2)$ and therefore $\iota(\Xi_1) = w(\Xi_1)$, which implies the claim. □

Example 6.5. i) The followings are examples of the triplets (Σ,Ξ_1,Ξ_2) such that (Ξ_1,Ξ_2) are special dual pairs in Σ.

$$\begin{gathered}(D_{m+n},D_m,D_n)\quad (m\ge 2,\ n\ge 2,\ m\ne 4,\ n\ne 4),\\ (E_6,A_3,2A_1),\ (E_7,A_5,A_2),\ (E_7,A_3+A_1,A_3),\ (E_7,3A_1,D_4),\\ (E_8,E_6,A_2),\ (E_8,A_5,A_2+A_1),\ (E_8,A_4,A_4),\ (E_8,D_6,2A_1),\ (E_8,D_5,A_3),\\ (E_8,D_4,D_4),\ (E_8,D_4+A_1,3A_1),\ (E_8,2A_2,2A_2),\\ (E_8,A_3+A_1,A_3+A_1),\ (E_8,4A_1,4A_1),\ (F_4,A_2,A_2).\end{gathered}$$

In these examples except for $(D_4,2A_1,2A_1)$ and $(E_8,4A_1,4A_1)$, $\#\overline{\mathrm{Hom}}(\Xi_2,\Sigma) = 1$ and the triplet is uniquely determined by the data (Σ,Ξ_1,Ξ_2) up to the automorphisms defined by W_Σ. If the imbedding $4A_1 \subset E_8$ satisfies $(4A_1)^\perp \simeq 4A_1$, we have a special dual pair $(4A_1,4A_1)$ in E_8, which is also uniquely defined. The imbedding $2A_1 \subset D_4$ is unique up to $\mathrm{Aut}(D_4)$.

ii) The isomorphism $\varpi \in \mathrm{Out}(2A_2)$ defined by the dual pair $(2A_2, 2A_2)$ in E_8 satisfies

$$\varpi\bigl(\mathrm{Out}(A_2) \times \mathrm{Out}(A_2)\bigr) \neq \mathrm{Out}(A_2) \times \mathrm{Out}(A_2) \tag{6.13}$$

because $\#\bigl(\overline{\mathrm{Hom}}(4A_2, E_8)/\mathrm{Out}'(4A_2)\bigr) = \#\overline{\mathrm{Hom}}(2A_2, E_8) = 1$. See §8.2.5.

iii) It happens that any dual pair of E_8 is special. But for example, if (Σ, Ξ_1, Ξ_2) is (E_6, A_5, A_1) or (E_7, D_6, A_1), (Ξ_1, Ξ_2) is a dual pair in Σ satisfying $\mathrm{Out}(\Xi_1) \not\simeq \mathrm{Out}(\Xi_2)$ and $\#\overline{\mathrm{Hom}}(\Xi_2, \Sigma) = 1$, which implies $\#\overline{\mathrm{Hom}}(\Xi_1, \Sigma) > 1$.

Definition 6.6 (S-closure and L-closure)**.** Let Ξ be a subsystem of Σ. Then Ξ is *S-closed* if and only if

$$\alpha,\, \beta \in \Xi \ \text{ and } \ \alpha + \beta \in \Sigma \ \Rightarrow \ \alpha + \beta \in \Xi \tag{6.14}$$

and *L-closed* if and only if

$$\beta \in \Sigma \cap \sum_{\alpha \in \Xi} \mathbb{R}\alpha \ \Rightarrow \ \beta \in \Xi. \tag{6.15}$$

The smallest S-closed (resp. L-closed) subsystem of Σ containing Ξ is called the S-closure (resp. L-closure) of Ξ.

Remark 6.7. i) We have the following relation for a subsystem Ξ of Σ:

$$\perp\text{-closed} \ \Rightarrow \ L\text{-closed} \ \Rightarrow S\text{-closed}. \tag{6.16}$$

ii) Let $\mathfrak{g}$ be a complex semisimple Lie algebra with the root system Σ and let X_α be root vectors corresponding to $\alpha \in \Sigma$. Then the root system of the semisimple Lie algebra $\mathfrak{g}_\Xi$ generated by $\{X_\alpha\,;\, \alpha \in \Xi\}$ is the S-closure of Ξ.

Let Ξ_1 and Ξ_2 be S-closed subsystems of Σ. Then

$$[\mathfrak{g}_{\Xi_1}, \mathfrak{g}_{\Xi_2}] = 0 \ \Leftrightarrow \ \Xi_1 \perp \Xi_2. \tag{6.17}$$

Hence if (Ξ_1, Ξ_2) is a dual pair with $\operatorname{rank}\Xi_1 + \operatorname{rank}\Xi_2 = \operatorname{rank}\Sigma$, the dual pair of root systems gives a dual pair in semisimple Lie algebras (cf. [7]).

iii) Suppose Σ is irreducible and there exist α, $\beta \in \Sigma$ with $\alpha + \beta \in \Sigma \setminus \Xi$. Then $\langle \alpha, \beta, \alpha + \beta \rangle$ is of type B_2 or of type G_2, which implies that Σ is not simply laced. For example, $D_n \subset C_n$ is not S-closed and the S-closure of D_n equals C_n $(n \geq 2)$.

iv) Let Ξ be an L-closed subsystem of Σ. Then for any subsystem Ξ' of Σ

$$W_\Xi \cap W_{\Xi'} = W_{\Xi \cap \Xi'}. \tag{6.18}$$

This is proved as follows. Choose a generic element v of the orthogonal complement of $\sum_{\alpha \in \Xi} \mathbb{R}\alpha$ in $\sum_{\alpha \in \Sigma} \mathbb{R}\alpha$ so that $\{\alpha \in \Sigma\,;\, (\alpha|v) = 0\} = \Xi$. Since $W_\Xi = \{w \in W_\Sigma\,;\, wv = v\}$, $W_\Xi \cap W_{\Xi'} = \{w \in W_{\Xi'}\,;\, wv = v\} = W_{\{\alpha \in \Xi'\,;\,(\alpha|v)=0\}} = W_{\Xi \cap \Xi'}$.

v) Put $\Xi = \{\pm\epsilon_1 \pm \epsilon_2, \pm\epsilon_3 \pm \epsilon_4\} \simeq 4A_1$ and $\Xi' = \{\pm\epsilon_1 \pm \epsilon_3, \pm\epsilon_2 \pm \epsilon_4\} \simeq 4A_1$. Then the subsystems Ξ and Ξ' of D_4 under the notation in §9 do not satisfy (6.18). When $\Sigma = B_n$, C_n, F_4 or G_2 and $\Xi = \Sigma^L$ and $\Xi' = \Sigma \setminus \Xi$, (6.18) is not valid.

7. Making tables

We are ready to answer the questions in the introduction by completing the tables in §10. In this section we do it when the root system Σ is of the exceptional type. Following the argument in §3, we easily get all Ξ satisfying $\mathrm{Hom}(\Xi, \Sigma) \neq \emptyset$ together with $\#\overline{\mathrm{Hom}}(\Xi, \Sigma)$ and $\Xi^\perp$ by Theorem 3.5. In fact, we start from the irreducible Ξ and then examine other Ξ by using (3.2) in a suitable lexicographic order (as in the tables) to avoid confusion (cf. Example 3.6 xii)).

As a result we finally get $(\Xi^\perp)^\perp$ and the dual pairs. Moreover (6.11) tells us whether the dual pair is special or not. We will calculate $\#\{\Theta \subset \Phi\,;\, \langle\Theta\rangle \simeq \Xi\}$ in §7.5.

Now we prepare the lemma to examine the action of W_Σ on the imbeddings of a root system Ξ into Σ.

Lemma 7.1. *Let* Ξ_1 *and* Ξ_2 *be subsystems of* Σ *with* $\Xi_2 \subset \Xi_1^\perp$. *Then*

$$\begin{aligned}&\#\big(\overline{\mathrm{Hom}}(\Xi_1,\Sigma)/\mathrm{Out}(\Xi_1)\big) = \#\big(\mathrm{Hom}(\Xi_2,\Xi_1^\perp)/\mathrm{Out}(\Xi_2)\big) = 1\\ &\qquad \Rightarrow \quad \#\big(\overline{\mathrm{Hom}}(\Xi_1+\Xi_2,\Sigma)/\mathrm{Out}(\Xi_1+\Xi_2)\big) = 1,\end{aligned}\tag{7.1}$$

$$\begin{aligned}&\#\big(\overline{\mathrm{Hom}}(\Xi_1,\Sigma)/\mathrm{Out}'(\Xi_1)\big) = \#\big(\overline{\mathrm{Hom}}(\Xi_2,\Xi^\perp)/\mathrm{Out}'(\Xi_2)\big) = 1\\ &\qquad \Rightarrow \quad \#\big(\overline{\mathrm{Hom}}(\Xi_1+\Xi_2,\Sigma)/\mathrm{Out}'(\Xi_1+\Xi_2)\big) = 1,\end{aligned}\tag{7.2}$$

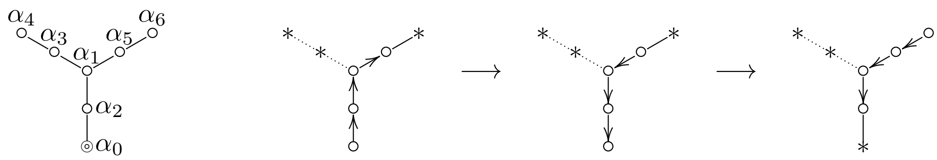
$$\begin{aligned}&\begin{cases}\#\big(\overline{\mathrm{Hom}}(\Xi_1,\Sigma)/\mathrm{Out}(\Xi_1)\big) = \#\big(\mathrm{Out}(\Xi_1^\perp)\backslash\overline{\mathrm{Hom}}(\Xi_2,\Xi_1^\perp)/\mathrm{Out}'(\Xi_2)\big) = 1,\\ \mathrm{Out}'(\Xi_1)\simeq \mathrm{Out}(\Xi_1) \ \textit{and}\ \ (\Xi_1,\Xi_1^\perp) \textit{ is a special dual pair}\end{cases}\\ &\qquad \Rightarrow \ \#\big(\overline{\mathrm{Hom}}(\Xi_1+\Xi_2,\Sigma)/\mathrm{Out}'(\Xi_1+\Xi_2)\big) = 1,\end{aligned}\tag{7.3}$$

$$\begin{aligned}&\#\big(\overline{\mathrm{Hom}}(\Xi_1,\Sigma)/\mathrm{Out}(\Xi_1)\big) = \#\big(\overline{\mathrm{Hom}}(\Xi_2,\Xi_1^\perp)/\mathrm{Out}(\Xi_2)\big) = 1\\ &\textit{and}\ \ \iota(\Xi_2)^\perp \simeq \Xi_1\ (\forall\iota\in\mathrm{Hom}(\Xi_2,\Sigma)) \ \Rightarrow\ \#\big(\overline{\mathrm{Hom}}(\Xi_2,\Sigma)/\mathrm{Out}(\Xi_2)\big) = 1.\end{aligned}\tag{7.4}$$

Proof. The claims (7.1) and (7.2) are clear because for $\iota \in \mathrm{Hom}(\Xi_1+\Xi_2,\Sigma)$ the assumptions assure that there exists $w \in W_\Sigma$ such that $\iota(\Xi_1) = w(\Xi_1)$ and hence we may assume $\iota(\Xi_1) = \Xi_1$ in $\overline{\mathrm{Hom}}(\Xi_1+\Xi_2,\Sigma)$. Under the assumption in (7.3) there exists $w \in W_\Sigma$ such that $w\circ\iota$ stabilizes every irreducible component of Ξ_2 and therefore it also stabilizes Ξ_1 and we have (7.3).

The claim (7.4) is also clear because for $\iota \in \mathrm{Hom}(\Xi_2,\Sigma)$, $\exists w \in W_\Sigma$ such that $w\circ\iota(\Xi_2)^\perp = \Xi_1$, which implies $w\circ\iota(\Xi_2)\subset\Xi_1^\perp$. □

7.1. **Type E_6.** The automorphism group of $G(\tilde\Psi)$ is of order 6, which is generated by a rotation and a reflection. Since the rotation has order 3, it corresponds to an element of W_{E_6} and the reflection corresponds to a non-trivial element of $\mathrm{Out}(E_6)$.



The set $\overline{\mathrm{Hom}}(A_4,E_6)$ has two elements which are shown in Example 3.6 v). It also shows that $\mathrm{Out}(E_6)$ non-trivially acts on this set. If A_4 is imbedded to E_6 given as in the above imbedding $G(A_4)\subset G(\tilde E_6)$ with the starting vertex $\{\alpha_0\}$, the non-trivial action by $\mathrm{Out}(A_4)$ changes the starting vertex as is shown above. Then by an element of W_{A_5} with $A_5 = \langle\alpha_0,\alpha_2,\alpha_1,\alpha_5,\alpha_6\rangle$ the imbedding is transformed as is shown by the second arrow. Then the result corresponds to a reflection, which implies that $\mathrm{Out}(A_4)$ also acts non-trivially on $\overline{\mathrm{Hom}}(A_4,E_6)$ and hence $\#\big(\overline{\mathrm{Hom}}(A_4,E_6)/\mathrm{Out}(A_4)\big) = 1$.

The same argument works for $\Xi = A_5$, $A_2 + A_1$ and $A_3 + A_1$. Similarly $(\alpha_2,\alpha_0,\alpha_5,\alpha_6)$ is transformed to $(\alpha_6,\alpha_5,\alpha_0,\alpha_2)$ by an element of W_{A_5} and furthermore to $(\alpha_0,\alpha_2,\alpha_4,\alpha_3)$ by a rotation. Hence a non-trivial element of $\mathrm{Out}(A_2)$ for $A_2 = \langle\alpha_0,\alpha_2\rangle$ induces the transposition of two irreducible components of $A_2^\perp \simeq A_2 + A_2$, which implies $\#\big(\overline{\mathrm{Hom}}(2A_2,E_6)/\mathrm{Out}'(2A_2)\big) = 1$.

From our construction of the representatives of $\overline{\mathrm{Hom}}(\Xi,E_6)$ it is obvious to have $\#\big(\overline{\mathrm{Hom}}(\Xi,E_6)/\mathrm{Out}'(\Xi)\big) = 1$ for $\Xi = D_5$ (cf. (5.3)) and E_6 and we can easily calculate $\#\big(\mathrm{Out}(E_6)\backslash\overline{\mathrm{Hom}}(\Xi,E_6)\big)$. Put $\Sigma = E_6$ and let (Ξ_1,Ξ_2) be any one of the pairs (A_2+A_1,A_1), $(2A_2,A_1)$, $(2A_2,A_2)$, $(2A_1,A_3)$, (A_4,A_1) and (A_5,A_1). Then applying (7.2) to Σ and (Ξ_1,Ξ_2), we have $\#\big(\overline{\mathrm{Hom}}(\Xi_1+\Xi_2,\Sigma)/\mathrm{Out}'(\Xi_1+\Xi_2)\big) = 1$.

7.2. **Type E_7.** Note that $G(\tilde{E}_7)$ has an automorphism of order 2 and it corresponds to an element of W_{E_7} because $W_{E_7} = \mathrm{Aut}(E_7)$.

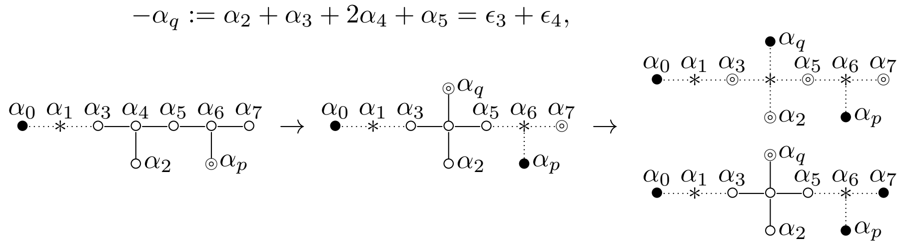
Let $\Sigma = E_7$ and let (Ξ_1, Ξ_2) be any one of (A_1, D_6), (A_2, A_2), $(A_2, A_2 + A_1)$, (A_2, A_3), $(A_2, A_3 + A_1)$, (A_3, A_3) and $(A_3, A_3 + A_1)$. We have $\#\bigl(\overline{\mathrm{Hom}}(\Xi_1 + \Xi_2, \Sigma)/\mathrm{Out}'(\Xi_1 + \Xi_2)\bigr) = 1$ by (7.2). Here we note that $A_1^\perp \simeq D_6$, $A_2^\perp \simeq A_5$ and $A_3^\perp \simeq A_3 + A_1$. We can apply (7.3) to $(\Xi_1, \Xi_2) = (D_4, kA_1)$ with $1 \le k \le 3$ and we have the same conclusion. Applying (7.1) to $(A_3, 3A_1)$, we have $\#\bigl(\overline{\mathrm{Hom}}(A_3 + 3A_1, E_7)/\mathrm{Out}(A_3 + 3A_1)\bigr) = 1$.

The subsystems Ξ of E_7 which are isomorphic to $3A_1$ and satisfy $\Xi^\perp \simeq 4A_1$ are mutually equivalent by Σ. Hence $\Xi^\perp \simeq 4A_1$ also have this property. Namely

$$\#\bigl(W_{E_7}\backslash\{\iota \in \mathrm{Hom}(4A_1, E_7)\,;\, (\iota(4A_1)^\perp)^\perp = \iota(4A_1)\}/\,\mathrm{Aut}(4A_1)\bigr) = 1.$$

Put $(A_1)_o = \langle \alpha_0 \rangle$. We have $G\bigl(\widetilde{(A_1)_o^\perp}\bigr)$ as is given in the following first diagram. Put $(2A_1)_o = \langle \alpha_0, \alpha_p \rangle$. Then the extended Dynkin diagrams of the components of $(2A_1)_o^\perp = \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7 \rangle \simeq D_4 + A_1$ are also given by the following second diagram. These diagrams correspond to the last figure in Remark 3.8 i). Here

$$-\alpha_p := (\alpha_2 + \alpha_3 + \cdots + \alpha_7) + (\alpha_4 + \alpha_5 + \alpha_6) = \epsilon_5 + \epsilon_6,$$
$$-\alpha_q := \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 = \epsilon_3 + \epsilon_4,$$



In the above diagrams the vertices expressed by asterisks are considered to be removed and the diagrams are (extended) Dynkin diagrams for other roots.

There are two equivalence classes in the imbeddings of $3A_1$ to E_7, whose representatives are

$$(3A_1)_1 = \langle \alpha_0, \alpha_p, \alpha_q \rangle, \qquad (3A_1)_2 = \langle \alpha_0, \alpha_p, \alpha_7 \rangle,$$

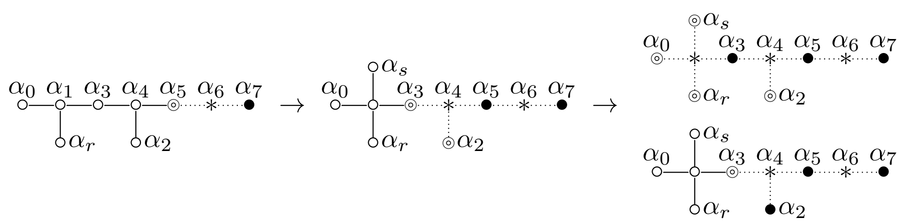
which satisfy

$$(3A_1)_1^\perp = \langle \alpha_2, \alpha_3, \alpha_5, \alpha_7 \rangle \simeq 4A_1, \qquad (3A_1)_2^\perp = \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle \simeq D_4.$$

Thus the image of the imbedding of $4A_1$ to E_7 is equivalent to one of the following subsystems $(4A_1)_j$ of E_7:

$$(4A_1)_1 = \langle \alpha_0, \alpha_p, \alpha_q, \alpha_7 \rangle, \qquad (4A_1)_1^\perp = \langle \alpha_2, \alpha_3, \alpha_5 \rangle,$$
$$(4A_1)_2 = \langle \alpha_0, \alpha_p, \alpha_q, \alpha_5 \rangle, \qquad (4A_1)_2^\perp = \langle \alpha_2, \alpha_3, \alpha_7 \rangle,$$
$$(4A_1)_3 = \langle \alpha_0, \alpha_p, \alpha_q, \alpha_2 \rangle, \qquad (4A_1)_3^\perp = \langle \alpha_3, \alpha_5, \alpha_7 \rangle,$$
$$(4A_1)_4 = \langle \alpha_0, \alpha_p, \alpha_q, \alpha_3 \rangle, \qquad (4A_1)_4^\perp = \langle \alpha_2, \alpha_5, \alpha_7 \rangle.$$

In view of Remark 3.4 ii) the above procedures for $3A_1 \subset E_6$ can be also explained by the following isomorphic ones.



Here $\alpha_r = -\alpha_p$ and $\alpha_s = -\alpha_q$. In fact α_p, $\alpha_r \in \langle\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_7\rangle^\perp = (D_4 + A_1)^\perp \simeq A_1$. As $m_7(\alpha_p) < 0$ and $m_7(\alpha_r) > 0$, we have $\alpha_r = -\alpha_p$. Similarly we have $\alpha_s = -\alpha_q$ from α_q, $\alpha_s \in \langle\alpha_0,\alpha_2,\alpha_3,\alpha_5,\alpha_7,\alpha_p\rangle^\perp \simeq A_1$. This is also easily verified by the Dynkin diagrams with the coefficients $m_j(-\alpha_0)$ in §9.

Note that

$$\langle\alpha_2,\alpha_3,\alpha_5\rangle \underset{\langle\alpha_5,\alpha_6,\alpha_7\rangle}{\sim} \langle\alpha_2,\alpha_3,\alpha_7\rangle \underset{\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle}{\sim} \langle\alpha_1,\alpha_4,\alpha_7\rangle \underset{\langle\alpha_1,\alpha_3,\cdots,\alpha_7\rangle}{\sim} \langle\alpha_3,\alpha_5,\alpha_7\rangle.$$

Thus we can conclude

$$\big((4A_1)_j^\perp\big)^\perp \begin{cases} \simeq \langle\alpha_3,\alpha_5,\alpha_7\rangle^\perp = \langle\alpha_0,\alpha_2,\alpha_r,\alpha_s\rangle \simeq 4A_1 & (j=1,2,3),\\ = \langle\alpha_0,\alpha_1,\alpha_3,\alpha_r,\alpha_s\rangle \simeq D_4 & (j=4).\end{cases} \tag{7.5}$$

Since $(4A_1)_j^\perp$ for $j=1,2,3$ are equivalent to each other by E_8, so are the subsystems $(4A_1)_j = ((4A_1)_j^\perp)^\perp$ for $j=1,2,3$. Moreover we have

$$\langle\alpha_0,\alpha_p,\alpha_q,\alpha_3\rangle \underset{\langle\alpha_2,\alpha_3,\ldots,\alpha_7\rangle}{\sim} \langle\alpha_0,\alpha_2,\alpha_q,\alpha_7\rangle \underset{\langle\alpha_2,\alpha_3,\alpha_4,\alpha_5\rangle}{\sim} \langle\alpha_0,\alpha_3,\alpha_5,\alpha_7\rangle. \tag{7.6}$$

Put $P_\Xi = \{\Theta \subset \Psi\,;\,\langle\Theta\rangle \simeq \Xi\}$. It is easy to see that if $\Theta \in P_{3A_1}$ satisfies $\Theta\cap\{\alpha_1,\alpha_3\} \neq \emptyset$, $\langle\Theta\rangle \underset{E_7}{\sim} (3A_1)_1$. Moreover if $\Theta \in P_{3A_1}$ satisfies $\Theta\cap\{\alpha_1,\alpha_3\} = \emptyset$, then $\Theta = \{\alpha_2,\alpha_5,\alpha_7\}$. We will have $\#B_{3A_1} = 11$ in §7.5.

Applying (7.4) to $(\Xi_1,\Xi_2) = (2A_1, 5A_1)$ and $(A_1, 6A_1)$ with $\Sigma = E_7$, we have $\#\big(\overline{\mathrm{Hom}}(\Xi_2,\Sigma)/\mathrm{Out}(\Xi_2)\big) = 1$, respectively. Similarly applying (7.1) to $(\Xi_1,\Xi_2) = (5A_1, A_1)$ and $(5A_1, 2A_1)$, we have $\#\big(\overline{\mathrm{Hom}}(\Xi_1+\Xi_2,\Sigma)/\mathrm{Out}(\Xi_1+\Xi_2)\big) = 1$, respectively.

7.3. **Type E_8.** Applying (7.4) to $(3A_1, 5A_1)$ and then (7.1) to $(5A_1, A_1)$, $(5A_1, 2A_1)$ and $(5A_1, 3A_1)$, respectively, we have $\#\big(\overline{\mathrm{Hom}}(kA_1,E_8)/\mathrm{Out}(kA_1)\big) = 1$ for $k = 5, 6, 7$ and 8. See §8.2.3 to get further results on $\mathrm{Hom}(kA_1, E_8)$ with $1 \le k \le 8$.

If $(\Xi_1, \Xi_1^\perp)$ is any one of the pairs (A_2, E_6), (A_4, A_4), (D_4, D_4), (D_5, A_3) and (D_6, A_1), we have

$$\mathrm{Hom}(\Xi_2, \Xi_1^\perp) \neq \emptyset \ \Rightarrow\ \#\big(\overline{\mathrm{Hom}}(\Xi_1+\Xi_2, E_8)/\mathrm{Out}'(\Xi_1+\Xi_2)\big) = 1$$

by applying (7.3). Hence if Ξ contains A_2, A_4, D_4, D_5 or D_6 as an irreducible component, the value of the column indicated by $\#_{\Xi'}$ equals one. Moreover (7.1) can be applied to $(\Xi_1,\Xi_2) = (A_3, 3A_1)$, $(A_3, 4A_1)$ and $(A_5, 2A_1)$. The number $\#\big(\overline{\mathrm{Hom}}(\Xi,E_8)/\mathrm{Out}'(\Xi)\big)$ for $\Xi = A_3 + 3A_1$, $A_3 + 4A_1$ and $A_5 + 2A_1$ is easily obtained from $(A_3 + A_1)^\perp \simeq A_3 + A_1$ and $A_5^\perp \simeq A_2 + A_1$.

Put

$$\begin{aligned}(A_7)_o &= \langle\alpha_1,\alpha_3,\ldots,\alpha_8\rangle \subset (A_8)_o = \langle(A_7)_o, \alpha_0\rangle,\\ (D_6)_o &= \{\pm\epsilon_i \pm \epsilon_j\,;\, 3 \le i < j \le 8\} = \{\alpha_2,\alpha_3\}^\perp,\\ P_\Xi &= \{\Theta \subset \{\alpha_1,\ldots,\alpha_8\}\,;\,\langle\Theta\rangle \simeq \Xi\}.\end{aligned}$$

Then we note the following for Θ_1, $\Theta_2 \in P_\Xi$.

$\Theta_1 \underset{\Sigma}{\sim} \Theta_2$ if $\Theta_j \subset (A_8)_o$ for $j = 1$, 2 and $\Theta_1 \simeq \Theta_2$.

If $\Theta_1 \ni \{\alpha_2,\alpha_3\}$, then $\Theta_1^\perp = \Theta_1^\perp \cap (D_6)_o$.

Using these facts, we can easily examine P_Ξ. For example, any $\Theta \in P_{4A_1}$ satisfies $\langle\Theta\rangle \underset{E_8}{\sim} (4A_1)_o := \langle\alpha_2,\alpha_3,\alpha_6,\alpha_8\rangle$. Here $(4A_1)_o^\perp = \langle\alpha_6,\alpha_8\rangle^\perp \cap D_6 \simeq 4A_1$.

7.4. **Type F_4 and G_2.** It is easy to examine the cases when $\Sigma = F_4$ and G_2 by using Theorem 3.5 together with Remark 3.1, (2.39) and (5.4).

7.5. **Fundamental subsystems.** We will give the number of the elements $P_\Xi := \{\Theta \subset \Psi\,;\, \langle\Theta\rangle \simeq \Xi\}$ for a subsystem Ξ of Σ when Σ is of the exceptional type. If $\#\Theta = \#\Psi - 1$, it is easy to specify Ξ that is isomorphic to $\langle\Theta\rangle$ and we get the corresponding $\#P_\Xi$. Other $\langle\Theta\rangle$ are fundamental subsystems of these maximal ones and hence it is also easy to know whether $P_\Xi = \emptyset$ or not. Note that $\mathrm{rank}\langle\Theta\rangle = \#\Theta$.

The number $\#P_\Xi$ can be inductively calculated as follows. Let denote the number by $[\Xi, \Sigma]$. For simplicity $\sum_j m_j A_j$ may be denoted $1^{m_1} \cdot 2^{m_2} \cdots$ with omitting the terms satisfying $m_j = 0$.

If Σ is of type E_n, we divide P_Ξ into the subsets according to the relation with the end root α_n. For example, suppose $\Sigma = E_6$ and $\Theta \subset \Psi$ satisfies $\langle\Theta\rangle \simeq 2A_1$. Then if $\alpha_6 \in \Theta$, the other element of Θ is in $\alpha_6^\perp \simeq A_4$. If $\alpha_6 \notin \Theta$, Θ is contained in $\Psi \setminus \{\alpha_6\} \simeq D_5$. Thus we have $[1^2, E_6] = [1, A_4] + [1^2, D_5]$. Now it is quite easy to have $[1, A_4] = 4$ and $[1^2, D_5] = 6$. Note that $[1^2, D_5] = [1, A_2 + A_1] + [1^2, A_4] = 3 + 3 = 6$. We will show such calculations except for quite easy cases.

$$
\begin{aligned}
[1^2, E_6] &= [1, A_4] + [1^2, D_5] = 4 + 6 = 10,\\
[1^2, E_7] &= [1, D_5] + [1^2, E_6] = 5 + 10 = 15,\\
[1^2, E_8] &= [1, E_6] + [1^2, E_7] = 6 + 15 = 21,\\
[1^3, E_6] &= [1^2, A_4] + [1^3, D_5] = 3 + 2 = 5,\\
[1^3, E_7] &= [1^2, D_5] + [1^3, E_6] = 6 + 5 = 11,\\
[1^3, E_8] &= [1^2, E_6] + [1^3, E_7] = 10 + 11 = 21,\\
[1^4, E_7] &= [1^3, D_5] + [1^4, E_6] = 2 + 0 = 2,\\
[1^4, E_8] &= [1^3, E_6] + [1^4, E_7] = 5 + 2 = 7,\\
[2 \cdot 1, E_6] &= [1, A_2 + A_1] + [2, A_4] + [2 \cdot 1, D_5] = 3 + 3 + 4 = 10,\\
[2 \cdot 1, E_7] &= [1, A_4] + [2, D_5] + [2 \cdot 1, E_6] = 4 + 4 + 10 = 18,\\
[2 \cdot 1, E_8] &= [1, D_5] + [2, E_6] + [2 \cdot 1, E_7] = 5 + 5 + 18 = 28,\\
[2 \cdot 1^2, E_7] &= [1^2, A_4] + [2 \cdot 1, D_5] + [2 \cdot 1^2, E_6] = 3 + 4 + 5 = 12,\\
[2 \cdot 1^2, E_8] &= [1^2, D_5] + [2 \cdot 1, E_6] + [2 \cdot 1^2, E_7] = 6 + 10 + 12 = 28,\\
[2 \cdot 1^3, E_8] &= [1^3, D_5] + [2 \cdot 1^2, E_6] + [2 \cdot 1^3, E_7] = 2 + 5 + 0 = 7,\\
[2^2, E_7] &= [2, A_4] + [2^2, E_6] = 3 + 1 = 4,\\
[2^2, E_8] &= [2, D_5] + [2^2, E_7] = 4 + 4 = 8,\\
[2^2 \cdot 1, E_8] &= [2 \cdot 1, D_5] + [2^2, E_6] + [2^2 \cdot 1, E_7] = 4 + 1 + 4 = 9,\\
[2^2 \cdot 1^2, E_8] &= [2 \cdot 1^2, D_5] + [2^2 \cdot 1, E_6] + [2^2 \cdot 1, E_7] = 1 + 1 + 0 = 2,\\
[3 \cdot 1, E_7] &= [1, A_2 + A_1] + [3, D_5] + [3 \cdot 1, E_6] = 3 + 4 + 4 = 11,\\
[3 \cdot 1, E_8] &= [1, A_4] + [3, E_6] + [3 \cdot 1, E_7] = 4 + 5 + 11 = 20,\\
[3 \cdot 1^2, E_8] &= [1^2, A_4] + [3 \cdot 1, E_6] + [3 \cdot 1^2, E_7] = 3 + 4 + 3 = 10,\\
[3 \cdot 2, E_8] &= [2, A_4] + [3, D_5] + [3 \cdot 2, E_7] = 3 + 4 + 3 = 10,\\
[3 \cdot 2 \cdot 1, E_8] &= [2 \cdot 1, A_4] + [3 \cdot 1, D_5] + [3 \cdot 2, E_6] + [3 \cdot 2 \cdot 1, E_7]\\
&= 2 + 1 + 0 + 1 = 4,\\
[3^2, E_8] &= [3, A_4] + [3^2, E_7] = 2 + 0 = 2,\\
[4 \cdot 1, E_7] &= [1, A_1] + [4, D_5] + [4 \cdot 1, E_6] = 1 + 2 + 2 = 5,\\
[4 \cdot 1, E_8] &= [1, A_2 + A_1] + [4, E_6] + [4 \cdot 1, E_7] = 3 + 4 + 5 = 12,\\
[4 \cdot 2, E_8] &= [2, A_2 + A_1] + [4, D_5] + [4 \cdot 2, E_7] = 1 + 2 + 1 = 4.
\end{aligned}
$$

8. Some remarks

8.1. **Some results from the tables.** In this section we always assume that Ξ is a subsystem of an irreducible root system Σ.

By our classification we have the following remarks.

Remark 8.1. i) The numbers of equivalence classes of certain subsystems Ξ (cf. Remark 8.4) and their pairs are as follows. Here we don't count the empty subsystem.

Σ	E_6	E_7	E_8	F_4	G_2
equivalence (isomorphic) classes	20 (20)	46 (40)	76 (71)	36 (22)	6 (4)
S-closed subsystems	20	46	76	23	5
L-closed ($\perp$-closed) subsystems	16 (7)	31 (13)	40 (18)	11 (9)	3 (3)
$\Xi^\perp = \emptyset$ ($\operatorname{rank}\Xi = \operatorname{rank}\Sigma$)	10 (3)	19 (7)	33 (13)	20 (16)	4 (4)
maximal (S-closed) subsystems	3 (3)	4 (4)	5 (5)	3 (3)	3 (2)
dual pairs (special dual pairs)	3 (1)	6 (3)	11 (11)	5 (4)	1 (1)

ii) Let σ be an outer automorphism of Σ. Then $\sigma(\Xi) \underset{\Sigma}{\sim} \Xi$ if (Σ, Ξ) does not satisfy the following condition.

$$\Sigma \simeq D_n \text{ with an even } n,\ \Xi \underset{\Sigma}{\sim} \textstyle\sum_j m_j A_j \text{ and } \sum_j (j+1)m_j = n. \tag{8.1}$$

iii) Suppose Ξ is irreducible. Then $\Xi^\perp \cap \Sigma$ is also irreducible if (Σ, Ξ) is not isomorphic to any one in the following list:

Σ	Ξ	$\Xi^\perp$	Σ	Ξ	$\Xi^\perp$
$B_{n\ (n\ge3)}$	A_1	$B_{n-2}+A_1$ or B_{n-1}	$C_{n\ (n\ge3)}$	A_1	$C_{n-2}+A_1$ or C_{n-1}
$D_{n\ (n\ge4)}$	A_1	$D_{n-2}+A_1$			
D_5	A_3	$2A_1$ or $\emptyset$	D_6	A_3	A_3 or $2A_1$
$D_{n\ (n\ge7)}$	A_{n-3}	$2A_1$	$D_{n\ (n\ge6)}$	D_{n-2}	$2A_1$
E_6	A_2	$2A_2$	E_6	A_3	$2A_1$
E_7	A_3	A_3+A_1	E_7	D_4	$3A_1$
E_8	A_5	A_2+A_1	E_8	D_6	$2A_2$

iv) The L-closure $\tilde{\Xi}$ of Ξ in Σ (cf. Definition 6.6 and Remark 6.7) can be easily obtained from the table in §10. Note that $\tilde{\Xi}$ is the maximal subsystem satisfying

$$\Xi \subset \tilde{\Xi} \subset (\Xi^\perp)^\perp \text{ and } \operatorname{rank}\Xi = \operatorname{rank}\tilde{\Xi}. \tag{8.2}$$

Remark 8.2 (orthogonal systems). A subsystem Ξ of Σ or the fundamental system of Ξ is called an *orthogonal system* of Σ if Ξ is isomorphic to mA_1 with a certain positive integer m. An orthogonal system Ξ is called *maximal* if $\Xi^\perp = \emptyset$ and called *strongly orthogonal* if Ξ is S-closed.

Suppose Σ is irreducible. Let $\Xi = \langle \alpha_1, \ldots, \alpha_m\rangle$ and $\Xi' = \langle \alpha'_1, \ldots, \alpha'_m\rangle$ be orthogonal systems of Σ with rank m.

i) The rank of a maximal orthogonal system is given in Corollary 3.7 i) when Σ is simply laced. If Σ is not simply laced, the rank equals $\operatorname{rank}\Sigma$.

ii) If one of the following conditions is satisfied, then $\Xi \underset{\Sigma}{\sim} \Xi'$.

(8.3) Ξ and Ξ' are strongly orthogonal maximal systems,

(8.4) Σ is of type A_n, E_6, E_7 or E_8 and (Σ, Ξ) is not isomorphic to $(E_7, 3A_1)$, $(E_7, 4A_1)$ or $(E_8, 4A_1)$.

iii) Let ι be a bijective map of Ξ to Ξ' with $(\iota(\alpha_j)|\iota(\alpha_j)) = (\alpha_j|\alpha_j)$ for $j = 1, \ldots, m$. Suppose $m \ge 2$. Then there exists $w \in W_\Sigma$ with $\iota = w|_\Xi$ if one of the following conditions is satisfied.

$$\Sigma \text{ is of type } A_n,\ B_2,\ E_6,\ F_4 \text{ or } G_2. \tag{8.5}$$

(8.6) Σ is of type E_7 with $m \le 2$.

(8.7) Σ is of type E_8 with $m \le 3$.

See §8.2.2 and §8.2.3 for more details.

Remark 8.3 (fundamental subsystem). i) We have

(8.8) Ξ is L-closed $\Leftrightarrow$ Ξ is fundamental,

(8.9) Ξ is $\perp$-closed $\Rightarrow$ Ξ is fundamental.

The minimal fundamental subsystem containing Ξ is the L-closure of Ξ.

ii) For a subset $\Theta \subset \Psi$

(8.10) $$\langle\Theta\rangle \simeq \Xi \text{ and } \langle\Theta\rangle \cap \Sigma^L \simeq \Xi \cap \Sigma^L \quad \Rightarrow \quad \langle\Theta\rangle \underset{\Sigma}{\overset{w}{\sim}} \Xi$$

if (Σ, Ξ) is not isomorphic to any one of the following list.

(8.11) Σ is of type B_n $(n \ge 2)$, C_n $(n \ge 3)$ or D_n $(n \ge 4)$
and Ξ has at least one A_3-component or two A_1-components.

(8.12) $(E_7, 4A_1)$, $(E_7, A_3 + 2A_1)$, $(E_7, A_5 + A_1)$,
$(E_8, 4A_1)$, $(E_8, A_3 + 2A_1)$, $(E_8, 2A_3)$, $(E_8, A_5 + A_1)$, (E_8, A_7).

(8.13) $(E_7, 3A_1)$, (E_7, A_5), $(E_7, A_3 + A_1)$.

If (Σ, Ξ) is isomorphic to one of the pairs in (8.12) and Ξ is a fundamental subsystem, then (8.10) is valid.

If (Σ, Ξ) is isomorphic to one of the pairs in (8.13), there exist Θ_1, $\Theta_2 \subset \Psi$ such that $\Xi \simeq \langle\Theta_1\rangle \simeq \langle\Theta_2\rangle$, $\langle\Theta_1\rangle \underset{\Sigma}{\not\sim} \langle\Theta_2\rangle$ and $\Xi \underset{\Sigma}{\sim} \langle\Theta_1\rangle$ or $\langle\Theta_2\rangle$.

Hence if (Σ, Ξ) is not isomorphic to any one of the pairs in (8.11) and (8.12),

(8.14) $$\langle\Theta\rangle \simeq \Xi \text{ and } \langle\Theta\rangle \cap \Sigma^L \simeq \Xi \cap \Sigma^L \quad \Rightarrow \quad \exists \Theta' \subset \Psi \text{ such that } \langle\Theta'\rangle \underset{\Sigma}{\sim} \Xi.$$

Note that (8.14) is still valid even if Σ is of type D_n except for the case

(8.15) Σ is of type D_n $(n \ge 4)$ and $m_1 + 2m_3 \ge 4$,

where m_j is the number of A_j-components of Ξ. In fact the subsystems

(8.16) $$\begin{aligned} \langle\epsilon_1 \pm \epsilon_2,\, \epsilon_3 \pm \epsilon_4\rangle &\underset{D_n}{\sim} 2D_2 \simeq 4A_1, \\ \langle\epsilon_1 \pm \epsilon_2,\, \epsilon_3 - \epsilon_4,\, \epsilon_4 \pm \epsilon_5\rangle &\underset{D_n}{\sim} D_2 + D_3 \simeq 2A_1 + A_3, \\ \langle\epsilon_1 - \epsilon_2,\, \epsilon_2 \pm \epsilon_3,\, \epsilon_4 - \epsilon_5,\, \epsilon_5 \pm \epsilon_6\rangle &\underset{D_n}{\sim} 2D_3 \simeq 2A_3 \end{aligned}$$

of D_n and the subsystems

(8.17) $$\begin{aligned} \langle\epsilon_1 \pm \epsilon_2\rangle &\underset{B_n}{\sim} D_2 \simeq 2A_1, \\ \langle\epsilon_1 - \epsilon_2,\, \epsilon_2 \pm \epsilon_3\rangle &\underset{B_n}{\sim} D_3 \simeq A_3 \end{aligned}$$

of B_n are not fundamental.

iii) Given a subset Φ of Σ, we examine the condition

(8.18) $\exists w \in W_\Sigma$ such that $w(\Phi) \subset \Psi$,

namely, the condition that Φ can be extended to a fundamental system of Σ.

A subset Φ of Σ is called a Π-*system* by [4] if Φ satisfies the two conditions

(8.19) $\alpha \in \Phi,\ \beta \in \Phi \Rightarrow \alpha - \beta \notin \Sigma$,

(8.20) the elements of Φ are linearly independent.

It is easy to see that (8.18) implies that Φ is a Π-system.

Suppose Φ is a Π-system of Σ. Put $\Xi = \langle\Phi\rangle$. Then Φ is a fundamental system of Ξ, which is shown by [4, Theorem 5.1]. Hence we note that the fundamental system of a subsystem of Σ is a Π-system if and only if the subsystem is S-closed. Therefore if $G(\Phi)$ is a subdiagram of the Dynkin diagram of Σ and the condition

$$\begin{cases}(\Sigma,\Xi) \text{ is isomorphic to one of (8.15) and (8.12),}\\ \text{or } \Sigma \text{ is of type } B_n \text{ with (8.11)}\end{cases} \tag{8.21}$$

is not satisfied, it follows from our table that (8.18) is valid. When Ξ is irreducible, (8.21) is equivalent to the condition that $(\Sigma,\Xi) \simeq (E_8, A_7)$ or (B_n, A_3), which coincides with the result given by [1].

Remark 8.4 (maximal subsystems). i) A proper subsystem Ξ of Σ is called *maximal* if there is no subsystem Ξ' satisfying $\Xi \subsetneqq \Xi' \subsetneqq \Sigma$. We have the following list of the maximal subsystems of irreducible root systems.

Σ	Ξ: maximal, $\operatorname{rank}\Xi = \operatorname{rank}\Sigma$	$\operatorname{rank}\Xi = \operatorname{rank}\Sigma - 1$
A_n		$A_{m-1} + A_{n-m} = \Xi_m$ $_{(2\le 2m\le n+1)}$
B_n	$B_m + B_{n-m}$ $_{(2\le 2m\le n)}$, D_n^L	
C_n	$C_m + C_{n-m}$ $_{(2\le 2m\le n)}$, D_n^S	
D_n	$D_m + D_{n-m} = \Xi_m$ $_{(4\le 2m\le n)}$	$D_{n-1} = \Xi_1,\ A_{n-1} = \Xi_n$
E_6	$A_1 + A_5 = \Xi_2,\ 3A_2 = \Xi_3$	$D_5 = \Xi_1$
E_7	$A_1 + D_6 = \Xi_1,\ A_7 = \Xi_2,\ A_2 + A_5 = \Xi_3$	$E_6 = \Xi_7$
E_8	$D_8 = \Xi_1,\ A_8 = \Xi_2,\ 2A_4 = \Xi_5,$ $A_2 + E_6 = \Xi_7,\ A_1 + E_7 = \Xi_8$	
F_4	$C_4,\ A_2^L + A_2^S,\ B_4$	
G_2	$A_1^L + A_1^S,\ A_2^L,\ A_2^S$	

Note that $D_2 \simeq 2A_1$ and $D_3 \simeq A_3$ in the above.

ii) A proper subsystem Ξ of Σ is called a *maximal S-closed subsystem* if Ξ is S-closed and if there is no S-closed subsystem Ξ' satisfying $\Xi \subsetneqq \Xi' \subsetneqq \Sigma$. The list of the maximal S-closed subsystems of the irreducible root system Σ is same as in i) if Σ is simply laced. In the other cases we have

Σ	Ξ: S-closed maximal, $\operatorname{rank}\Xi = \operatorname{rank}\Sigma$	$\operatorname{rank}\Xi = \operatorname{rank}\Sigma - 1$
B_n	$D_m^L + B_{n-m} = \Xi_m$ $_{(2\le m\le n)}$	$B_{n-1} = \Xi_1$
C_n	$C_m + C_{n-m} = \Xi_m$ $_{(2\le 2m\le n)}$	$A_{n-1}^S = \Xi_n$
F_4	$A_1^L + C_3 = \Xi_1,\ A_2^L + A_2^S = \Xi_2,\ B_4 = \Xi_4$	
G_2	$A_1^L + A_1^S = \Xi_1,\ A_2^L = \Xi_2$	

They are studied by [2] and [8] (cf. [4]) and the Dynkin diagram of Ξ is

$$\begin{cases} G(\Psi \setminus \{\alpha_j\}) & (m_j(\alpha_{max}) = 1),\\ G(\tilde\Psi \setminus \{\alpha_j\}) & (m_j(\alpha_{max}) \text{ is a prime number } \ge 2)\end{cases} \tag{8.22}$$

with a suitable $\alpha_j \in \Psi$ satisfying $m_j(\alpha_{\max}) = 1,\ 2,\ 3$ or 5.

iii) Let Ξ_j be the maximal subsystem of Σ defined by (8.22) with $\alpha_j \in \Psi$. Then Proposition 4.4 (cf. [8, Theorem 3.1]) implies

$$\Xi_j = \bigl\{\alpha = \textstyle\sum_{\alpha_\nu\in\Psi} m_\nu(\alpha)\alpha_\nu \in \Sigma\,;\ m_j(\alpha) \equiv 0 \mod \max\{2, m_j(\alpha_{\max})\}\bigr\}. \tag{8.23}$$

Note that $\iota(A_m) \cap \Xi_j \ne \emptyset$ for any $\iota \in \operatorname{Hom}(A_m, \Sigma)$ if $m \ge \max\{2, m_j(\alpha_{max})\}$.

This claim follows from the following fact with putting $n_i = m_j\bigl(\iota(\epsilon_i - \epsilon_{i+1})\bigr)$ for the i-th root $\epsilon_i - \epsilon_{i+1}$ in the fundamental system of A_m.

For a positive integer $m \ge 2$ and a sequence of integers $n_1, \dots, n_m$ we can choose $1 \le k \le k' \le m$ such that $n_k + n_{k+1} + \cdots + n_{k'} \equiv 0 \mod m$.

For example, when $\Sigma = E_8$, we have $\Xi_5 \simeq 2A_4$, $m_5(\alpha_{max}) = 5$ and

$$E_8 \setminus \Xi_5 \supset \Xi' := \Bigl\langle \overset{0001000}{\underset{0}{}},\ \overset{0121100}{\underset{1}{}},\ \overset{1111110}{\underset{0}{}},\ \overset{0011111}{\underset{1}{}} \Bigr\rangle \simeq A_4 \tag{8.24}$$

under the notation in §10. Here the roots $\alpha \in \Sigma$ are indicated by the numbers $m_\nu(\alpha)$ arranged according to the Dynkin diagram of Σ. We have $W_{\Xi_5} \cap W_{\Xi'} = \{e\}$ because Ξ' is L-closed in Σ (cf. Remark 6.7 iv)).

8.2. **Further study of the action of the Weyl group.** As for Q4 in §1, that is, "Is $\mathrm{Out}(\Xi)$ realized by W_Σ?" can be answered from the table in §10 by the condition $\# = \#_\Xi$ and the answer is "yes" in most cases in the table. We will consider the cases when the answer is "no", namely, we will study the group $\mathrm{Out}_\Sigma(\Xi)$ in $\mathrm{Out}(\Xi)$ (cf. Definition 6.1). Under the notation in §10 we have

$$\#\big(\mathrm{Out}(\Xi)/\mathrm{Out}_\Sigma(\Xi)\big) = \#/\#_\Xi. \tag{8.25}$$

If Σ is of the classical type, it is easy to analyze $\mathrm{Out}_\Sigma(\Xi)$ because the action of W_Σ is easy. If Ξ is irreducible, $W_\Sigma(\Xi)$ is understood well by Theorem 3.5 using Dynkin diagrams. Moreover since $N_{W_\Sigma}(\Xi) \subset N_{W_\Sigma}(\Xi + \Xi^\perp)$, we have

$$\mathrm{Out}_\Sigma(\Xi) \simeq \{g \in \mathrm{Out}_\Sigma(\Xi + \Xi^\perp)\,;\, g(\Xi) = \Xi\} \tag{8.26}$$

by (6.3) and therefore the group $\mathrm{Out}_\Sigma(\Xi)$ is described by $\mathrm{Out}_\Sigma(\Xi+\Xi^\perp)$. Hence we may assume Ξ is $\perp$-dense.

8.2.1. *Dual pairs.* If Ξ is not irreducible, $\mathrm{Out}_\Sigma(\Xi)$ may be understood as a dual pair. For example the dual pair $(D_6, 2A_1)$ in E_8 is special and the imbedding $D_6 + 2A_1 \subset E_8$ is unique up to the transformations by W_{E_8}. Hence there exists $w \in N_{W_{E_8}}(D_6 + 2A_1)$ which swaps two A_1's. Then w always defines a non-trivial element of $\mathrm{Out}(D_6)$. Namely $\mathrm{Out}_\Sigma(\Xi)$ is the diagonal subgroup of $\mathrm{Out}(D_6 + 2A_1)$ through the isomorphism $\mathrm{Out}(D_6) \simeq \mathrm{Out}(2A_1)$. The following cases are understood in this way.

$$\begin{gathered}
D_m + D_n \subset D_{m+n} \quad (m \ge 2,\, n \ge 2,\, m \ne 4,\, n \ne 4),\\
A_3 + 2A_1 \subset E_6,\\
2A_3 + A_1 \subset E_7,\ A_5 + A_2 \subset E_7,\ D_4 + 3A_1 \subset E_7,\\
E_6 + A_2 \subset E_8,\ A_5 + A_2 + A_1 \subset E_8,\ A_4 + A_4 \subset E_8,\ D_6 + 2A_1 \subset E_8,\\
D_5 + A_3 \subset E_8,\ D_4 + D_4 \subset E_8,\ A_2 + A_2 \subset F_4.
\end{gathered}$$

For the imbedding $\Xi \subset \Sigma$ in this list, a still more concrete description of $\mathrm{Out}_\Sigma(\Xi)$ is desirable if $\mathrm{Out}(\Xi) \not\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For the imbedding $D_m + D_m \simeq D_m^0 + D_m^m \subset D_{2m}$ under the notation in §10, the swapping of two D_m's under the generators given there is in $\mathrm{Out}_{D_{2m}}(D_m^0 + D_m^m)$ and therefore $\mathrm{Out}_{D_{2m}}(D_m + D_m)$ is clear. Similarly for $2A_4 \subset E_8$, if we fix $A_3 + A_4 \subset A_4 + A_4 \subset E_8$ and $A_3 + A_4 \subset A_8 \subset E_8$, we can also specify the swapping of two A_4's in $\mathrm{Out}_{E_8}(2A_4)$. Other cases in the list are described by the study of the imbedding of $7A_1 \subset E_7$ and $8A_1 \subset E_8$ through $4A_1 \subset D_4$ as is shown later.

8.2.2. *Strongly orthogonal systems of the maximal rank.* Suppose Σ is of type A_n, B_n, C_n, D_n, E_6, F_4 or G_2 and put $m = 2[\frac{n}{2}]$. Under the notation in §9 the strongly orthogonal system $\langle \Theta_\Sigma \rangle$ of Σ with the maximal rank is weakly equivalent to

$$\Theta_{A_n} := \{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4, \ldots, \epsilon_{2m-1} - \epsilon_{2m}\}, \tag{8.27}$$

$$\Theta_{D_n} := \{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4, \ldots, \epsilon_{2m-1} + \epsilon_{2m}\}, \tag{8.28}$$

$$\Theta_{B_n} := \begin{cases} \Theta_{D_n} & (n = 2m),\\ \Theta_{D_n} \cup \{\epsilon_n\} & (n = 2m+1), \end{cases} \tag{8.29}$$

$$\Theta_{C_n} := \{2\epsilon_1, \ldots, 2\epsilon_n\}, \tag{8.30}$$

$$\Theta_{E_6} := \Theta_{F_4} := \{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4\}, \tag{8.31}$$

$$\Theta_{G_2} := \{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2 - 2\epsilon_3\}. \tag{8.32}$$

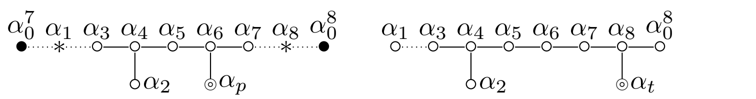
Then $\mathrm{Out}_\Sigma(\Theta_\Sigma) := \mathrm{Out}_\Sigma(\langle\Theta_\Sigma\rangle)$ is identified with the subgroup of the permutation group of Θ_Σ, which is also identified with the permutation group $\mathfrak{S}_{\#\Theta_\Sigma}$ of $\{1,\dots,\#\Theta_\Sigma\}$ according to the expression of Θ_Σ by the above ordered set. Then

$$\begin{aligned}
\mathrm{Out}_\Sigma(\Theta_\Sigma) &\simeq \mathfrak{S}_{\#\Theta_\Sigma} \qquad (\Sigma \text{ is of type } A_n,\, C_n,\, E_6,\, F_4),\\
\mathrm{Out}_{G_2}(\Theta_{G_2}) &= \{1\},\\
\mathrm{Out}_{D_n}(\Theta_{D_n}) &= \begin{cases} \big\langle (1\ 2)(3\ 4), (1\ 3)(2\ 4), \\ \quad (1\ 3\ \cdots\ 2m-1)(2\ 4\ \cdots\ 2m)\big\rangle \simeq W_{D_m} \quad (n=2m),\\ \big\langle (1\ 2), (1\ 3)(2\ 4), \\ \quad (1\ 3\ \cdots\ 2m-1)(2\ 4\ \cdots\ 2m)\big\rangle \simeq W_{B_m} \quad (n=2m+1).\end{cases}
\end{aligned}$$

Here the generators of $\mathrm{Out}_{D_n}(\Theta_{D_n})$ are expressed by products of circular permutations. Note that the group $\mathrm{Out}_{D_n}(\Theta_{D_n})$ is isomorphic to W_{D_m} or W_{B_m} if $n = 2m$ or $2m+1$, respectively.

8.2.3. $8A_1 \subset E_8$ *and* $7A_1 \subset E_7$. Since $\mathrm{Out}(8A_1)$ is isomorphic to the symmetric group $\mathfrak{S}_8$, $\mathrm{Out}_{E_8}(8A_1)$ is identified with a subgroup of $\mathfrak{S}_8$. Since $\#/\#_\Xi = 30$, $\#\,\mathrm{Out}_{E_8}(8A_1) = 8!/30 = 1344 = 2^6\cdot 3\cdot 7$. To be more precise, we fix $8A_1 \subset E_8$:

$$\begin{aligned}
&(8A_1)_o := \{\pm\alpha_2, \pm\alpha_3, \pm\alpha_5, \pm\alpha_q \pm \alpha_7, \pm\alpha_p, \pm\alpha_0^7, \pm\alpha_0^8\} \subset E_8,\\
&\alpha_2 = \epsilon_1+\epsilon_2,\ \alpha_3 = \epsilon_2-\epsilon_1,\ \alpha_5 = \epsilon_4-\epsilon_3,\ \alpha_7 = \epsilon_6-\epsilon_5,\\
&\alpha_p = -\epsilon_5-\epsilon_6,\ \alpha_q = -\epsilon_3-\epsilon_4,\ \alpha_0^8 = -\epsilon_7-\epsilon_8,\ \alpha_0^7 = \epsilon_7-\epsilon_8.
\end{aligned} \tag{8.33}$$

(8.34) 

Here we used the notation in §7.2 and §10. In particular $E_7 \subset E_8$. Note that α_0^8 and α_0^7 are negatives of maximal roots of E_8 and E_7, respectively. We identify $\mathfrak{S}_8$ with the permutation group of the set $\{1,2,3,4,5,6,7,8\}$ of numbers and this ordered set is also identified with the ordered set given as generators of $(8A_1)_o$ in (8.33).

Since $(\alpha_0^8)^\perp = E_7$, the left figure above corresponds to the first diagram in §7.2. Since $(D_8)_o := \langle\alpha_2,\alpha_3,\dots,\alpha_8,\alpha_0^8\rangle$ is of type D_8, its extended diagram is given in the right figure of (8.34) with the negative α_t of its maximal root. Here we note that $\alpha_0^7 < 0$ and $\alpha_t > 0$. Since $(D_6)_o^\perp \cap (\alpha_0^8)^\perp \simeq A_1$ by denoting $(D_6)_o := \langle\alpha_2,\dots,\alpha_7\rangle$, we have $\alpha_t = -\alpha_0^7$. Then $\mathrm{Out}_{(D_8)_o}\big((8A_1)_o\big)$ is generated by

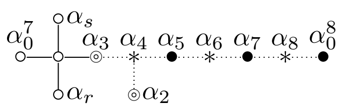
$$g_1 = (1\ 3\ 5\ 7)(2\ 4\ 6\ 8), \tag{8.35}$$

$$g_2 = (1\ 3)(2\ 4), \tag{8.36}$$

$$g_3 = (1\ 2)(3\ 4). \tag{8.37}$$

Here the generators g_j are expressed by products of cyclic permutations in $\mathfrak{S}_8$. Note that $\#\,\mathrm{Out}_{(D_8)_o}\big((8A_1)_o\big) = 2^3\cdot 4! = 2^6\cdot 3$.

Now we will consider the other diagram in §7.2.



$$\alpha_r = -\alpha_p,\ \alpha_s = -\alpha_q$$

This shows that the element

$$g_4 = (2\ 4)(6\ 7) \tag{8.38}$$

which corresponds to an element of the $W_{\langle\alpha_0^7,\alpha_1,\alpha_3,\alpha_r\rangle} \simeq W_{D_4}$ is in $\mathrm{Out}_{E_8}\big((8A_1)_o\big)$ and not in $\mathrm{Out}_{(D_8)_o}\big((8A_1)_o\big)$. Then we can conclude that $\mathrm{Out}_{E_8}\big((8A_1)_o\big)$ is generated by g_j $(j = 1,2,3,4)$, which is clear by considering the order of the groups.

Put $(7A_1)_o = (8A_1)_o \setminus \{\pm\alpha_o^8\}$. Since $E_7 = (\alpha_0^8)^\perp$, it is easy to see that $\mathrm{Out}_{E_7}((7A_1)_o)$ is generated by g_2, g_3, g_4 and

$$g_1' = (1\ 3\ 5)(2\ 4\ 6). \tag{8.39}$$

Here we naturally identify $\mathrm{Out}(7A_1)$ with $\mathfrak{S}_7$ and we have

$$\mathrm{Out}_{E_8}\big((8A_1)_o\big) = \langle g_1, g_2, g_3, g_4\rangle, \quad \#\,\mathrm{Out}_{E_8}\big((8A_1)_o\big) = 2^6\cdot 3\cdot 7 = 1344, \tag{8.40}$$

$$\mathrm{Out}_{E_7}\big((7A_1)_o\big) = \langle g_1', g_2, g_3, g_4\rangle, \quad \#\,\mathrm{Out}_{E_7}\big((7A_1)_o\big) = 2^3\cdot 3\cdot 7 = 168. \tag{8.41}$$

Put $(6A_1)_o = \{\pm\alpha_2, \pm\alpha_3, \pm\alpha_5, \pm\alpha_q, \pm\alpha_7, \pm\alpha_p\} \subset \{\alpha_0^7, \alpha_0^8\}^\perp \simeq D_6$ and $(5A_1)_o = \{\pm\alpha_2, \pm\alpha_3, \pm\alpha_5, \pm\alpha_q, \pm\alpha_7\} \subset \{\alpha_p, \alpha_o^7, \alpha_o^8\}^\perp \simeq D_4 + A_1$. Note that $(D_6, 2A_1)$ and $(D_4 + A_1, 3A_1)$ are special dual pairs in E_8 and therefore $\mathrm{Out}_{E_8}(D_6) = \mathrm{Out}(D_6)$ and $\mathrm{Out}_{E_8}(D_4 + A_1) = \mathrm{Out}(D_4 + A_1)$. Then we have easily

$$\mathrm{Out}_{E_8}\big((7A_1)_o\big) \xleftarrow{\sim} \mathrm{Out}_{E_7}\big((7A_1)_o\big), \tag{8.42}$$

$$\mathrm{Out}_{E_8}\big((6A_1)_o\big) = \big\langle g_1', g_2, (1\ 2)\big\rangle \simeq W_{B_3}, \qquad \#\,\mathrm{Out}_{E_8}\big((6A_1)_o\big) = 48, \tag{8.43}$$

$$\mathrm{Out}_{E_7}\big((6A_1)_o\big) = \big\langle g_1', g_2, g_3\big\rangle \simeq W_{D_3}, \qquad \#\,\mathrm{Out}_{E_8}\big((6A_1)_o\big) = 24, \tag{8.44}$$

$$\mathrm{Out}_{E_8}\big((5A_1)_o\big) = \big\langle (1\ 2), (2\ 3), (3\ 4)\big\rangle \simeq W_{A_3}, \quad \#\,\mathrm{Out}_{E_8}\big((5A_1)_o\big) = 24, \tag{8.45}$$

$$\mathrm{Out}_{E_7}\big((5A_1)_o\big) = \big\langle g_2, (1\ 2)\big\rangle \simeq W_{B_2}, \qquad \#\,\mathrm{Out}_{E_8}\big((5A_1)_o\big) = 8. \tag{8.46}$$

Put $(4A_1)_o = \{\pm\alpha_2, \pm\alpha_3, \pm\alpha_5, \pm\alpha_q\}$ and $(4A_1)_1 = \{\pm\alpha_2, \pm\alpha_3, \pm\alpha_5, \pm\alpha_7\}$. Then $(4A_1)_o^\perp \cap E_8 \simeq D_4$, $(4A_1)_o^{\perp\perp} \simeq D_4$ and $(4A_1)_1^\perp \cap E_8 \simeq 4A_1$ and $(4A_1)_1^{\perp\perp} = (4A_1)_1$.

8.2.4. $2D_4 \subset E_8$, $D_4 + 4A_1 \subset E_8$ *and* $D_4 + 3A_1 \subset E_7$. Retain the notation in the previous section (cf. (8.34)) and put

$$(2D_4)_o = \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_q\rangle + \langle \alpha_7, \alpha_t, \alpha_8, \alpha_0^8, \alpha_p\rangle \qquad \subset E_8, \tag{8.47}$$

$$(D_4 + 4A_1)_o = \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_q\rangle + \langle \alpha_7, \alpha_t, \alpha_0^8, \alpha_p\rangle \qquad \subset E_8, \tag{8.48}$$

$$(D_4 + 3A_1)_o = \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_q\rangle + \langle \alpha_7, \alpha_t, \alpha_p\rangle \qquad \subset E_7. \tag{8.49}$$

Then we have the natural identification

$$\begin{aligned}&\mathrm{Out}_{E_8}\big((2D_4)_o\big) \supset \mathrm{Out}_{E_7}\big((D_4 + 3A_1)_o\big)\\ &\quad \simeq \big\{g \in \mathrm{Out}_{E_8}\big((8A_1)_o\big)\,;\, g(\alpha_q) = \alpha_q \text{ and } g(\alpha_0^8) = \alpha_0^8\big\}\end{aligned}$$

together with (2.18) and therefore $\mathrm{Out}_{E_7}\big((D_4 + 3A_1)_o\big)$ is generated by

$$(1\ 2)(5\ 6):\ \alpha_2 \leftrightarrow \alpha_3,\ \alpha_7 \leftrightarrow \alpha_0^7 \ \text{ and }\ (1\ 3)(6\ 7):\ \alpha_2 \leftrightarrow \alpha_5,\ \alpha_7 \leftrightarrow \alpha_t.$$

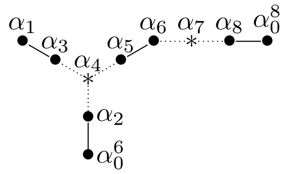
Here the first element corresponds to an element in $W_{(D_8)_o}$ and the second element equals g_2g_4. Moreover $\mathrm{Out}_{E_8}\big((2D_4)_o\big)$ contains $(1\ 5)(2\ 6)(3\ 7)(4\ 8)$ and $\mathrm{Out}_{E_8}\big((D_4 + 4A_1)_o\big)$ contains $\mathrm{Out}_{(2D_4)_o}\big((D_4 + 4A_1)_o\big)$. Hence

$$\begin{aligned}\mathrm{Out}_{E_7}\big((D_4 + 3A_1)_o\big) &= \big\langle (1\ 2)(5\ 6),\ (1\ 3)(6\ 7)\big\rangle,\\ \#\,\mathrm{Out}_{E_7}\big((D_4 + 3A_1)_o\big) &= 6,\\ \mathrm{Out}_{E_8}\big((2D_4)_o\big) &= \big\langle (1\ 2)(5\ 6),\ (1\ 3)(6\ 7),\ (1\ 5)(2\ 6)(3\ 7)(4\ 8)\big\rangle,\\ \#\,\mathrm{Out}_{E_8}\big((2D_4)_o\big) &= 12,\\ \mathrm{Out}_{E_8}\big((D_4 + 4A_1)_o\big) &= \big\langle (5\ 6)(7\ 8),\ (5\ 8)(6\ 7),\ \mathrm{Out}_{E_8}\big((2D_4)_o\big)\big\rangle,\\ \#\,\mathrm{Out}_{E_8}\big((D_4 + 4A_1)_o\big) &= 48.\end{aligned}$$

8.2.5. $4A_2 \subset E_8$, $3A_2 \subset E_7$ *and* $3A_2 \subset E_6$. First note that as groups, $\mathrm{Out}(4A_2)$ and $\mathrm{Out}(3A_2)$ are isomorphic to W_{B_4} and W_{B_3} and their orders of the groups are $2^4 \cdot 4!$ and $2^3 \cdot 3!$, respectively. Fix a representative $4A_2 \subset E_8$:

$$\begin{aligned}
\alpha_2 &= \epsilon_1 + \epsilon_2, & \alpha_0^6 &= -\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_8 + \epsilon_8),\\
\alpha_3 &= \epsilon_2 - \epsilon_1, & \alpha_1 &= \tfrac{1}{2}(\epsilon_1 + \epsilon_8) - \tfrac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7),\\
\alpha_5 &= \epsilon_4 - \epsilon_3, & \alpha_6 &= \epsilon_5 - \epsilon_4,\\
\alpha_8 &= \epsilon_7 - \epsilon_6, & \alpha_0^8 &= -\epsilon_7 - \epsilon_8,
\end{aligned}$$

$$(4A_2)_0 = \bigl\langle \{\alpha_2, \alpha_0^6, \alpha_3, \alpha_1, \alpha_5, \alpha_6, \alpha_8, \alpha_0^8\} \bigr\rangle,$$
$$(3A_2)_0 = \bigl\langle \{\alpha_2, \alpha_0^6, \alpha_3, \alpha_1, \alpha_5, \alpha_6\} \bigr\rangle.$$



Then the permutation group of the 8 generators of $(4A_2)_o$ is identified with $\mathfrak{S}_8$ as in the case of $(8A_1)_o \subset E_8$. Then $\mathrm{Out}\bigl((4A_2)_o\bigr) = \langle g_1, g_2, g_3\rangle$ and $\mathrm{Out}\bigl(3(A_2)_o\bigr) = \langle g_1', g_2, g_3\rangle$ (cf. (8.35)–(8.39)). Note that

$$\begin{aligned}
\#\,\mathrm{Out}_{E_6}\bigl((3A_2)_o\bigr) &= \#\,\mathrm{Out}\bigl((3A_2)_o\bigr)/8 = 6,\\
\#\,\mathrm{Out}_{E_7}\bigl((3A_2)_o\bigr) &= \#\,\mathrm{Out}\bigl((3A_2)_o\bigr)/4 = 12,\\
\#\,\mathrm{Out}_{E_8}\bigl((4A_2)_o\bigr) &= \#\,\mathrm{Out}\bigl((4A_2)_o\bigr)/8 = 48,\\
\mathrm{Out}_{E_6}\bigl((3A_2)_o\bigr) &\subset \mathrm{Out}_{E_8}\bigl((4A_2)_o\bigr).
\end{aligned}$$

Since $(2A_2, 2A_2)$ is a special dual pair in E_8, we have

$$\bigl(\exists 2A_2 \subset (4A_2)_o \text{ such that } w|_{2A_2} = id\bigr) \ \Rightarrow w = id \tag{8.50}$$

for $w \in \mathrm{Out}_{E_8}\bigl((4A_2)_o\bigr)$. We will choose elements in $\mathrm{Out}_{E_8}\bigl((2A_2)_o\bigr)$. The rotation of the extended Dynkin diagram of E_6 comes from W_{E_6} and therefore the element

$$h_1 = (1\ 3\ 5)(2\ 4\ 6) \tag{8.51}$$

is contained in $\mathrm{Out}_{E_6}\bigl((3A_2)_o\bigr)$. The argument in §7.1 shows that in view of the transformation of an element of $W_{\langle\alpha_0^8, \alpha_2, \alpha_4, \alpha_5, \alpha_6\rangle}$, $(1\ 6)(2\ 5)$ or $(1\ 6)(2\ 5)(3\ 4)$ should be in $W_{E_6}\bigl((3A_2)_o\bigr)$. Owing to (8.50), we can conclude that

$$h_2 = (1\ 6)(2\ 5)(3\ 4) \tag{8.52}$$

is contained in the group and $\mathrm{Out}_{E_6}\bigl((3A_2)_o\bigr) = \langle h_1, h_2\rangle$.

Since $\overline{\mathrm{Hom}}(E_6, E_7) = 1$, $\mathrm{Out}_{E_7}\bigl((3A_2)_o\bigr)$ contains

$$h_3 = (3\ 5)(4\ 6). \tag{8.53}$$

Considering in $(A_8)_o = \langle \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0^8\rangle \simeq A_8$, there is an element $w \in W_{(A_8)_o}$ such that

$$w(\alpha_1) = \alpha_1,\ w(\alpha_3) = \alpha_3\ w(\alpha_4) = \alpha_6,\ w(\alpha_5) = \alpha_7,\ w(\alpha_6) = \alpha_4,\ w(\alpha_7) = \alpha_5.$$

Then it also follows from (8.50) that

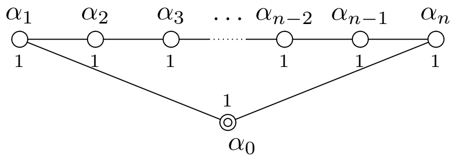
$$h_4 = (1\ 2)(5\ 7)(6\ 8) \tag{8.54}$$

is in $\mathrm{Out}_{E_8}\bigl((4A_2)_o\bigr)$. Calculating the order of the group, we have

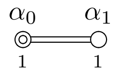
$$\mathrm{Out}_{E_6}\bigl((3A_2)_o\bigr) = \langle h_1, h_2\rangle, \qquad \#\,\mathrm{Out}_{E_6}\bigl((3A_2)_o\bigr) = 6, \tag{8.55}$$
$$\mathrm{Out}_{E_7}\bigl((3A_2)_o\bigr) = \langle h_1, h_2, h_3\rangle, \qquad \#\,\mathrm{Out}_{E_7}\bigl((3A_2)_o\bigr) = 12, \tag{8.56}$$
$$\mathrm{Out}_{E_8}\bigl((4A_2)_o\bigr) = \langle h_1, h_2, h_4\rangle, \qquad \#\,\mathrm{Out}_{E_8}\bigl((4A_2)_o\bigr) = 48. \tag{8.57}$$

9. List of irreducible root systems

A_n

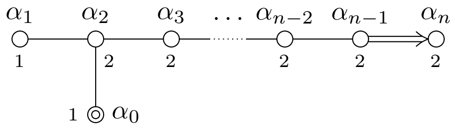


A_1

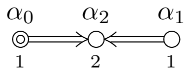


$$\Sigma = \left\{\pm(\epsilon_i - \epsilon_j)\,;\, 1 \le i < j \le n+1\right\}, \quad \#\Sigma = n(n+1),$$
$$\alpha_j = \epsilon_j - \epsilon_{j+1\ (j=1,\dots,n)},\ \alpha_0 = \epsilon_{n+1} - \epsilon_1, \quad \#W = (n+1)!.$$

B_n

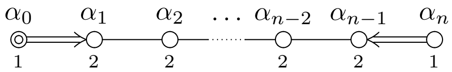


$B_2 = C_2$



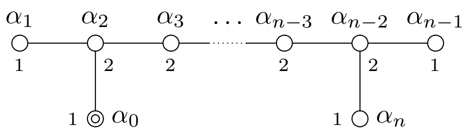
$$\Sigma = \left\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm\epsilon_k\,;\, 1 \le i < j \le n,\ 1 \le k \le n\right\}, \quad \#\Sigma = 2n^2,$$
$$\alpha_j = \epsilon_j - \epsilon_{j+1\ (j=1,\dots,n-1)},\ \alpha_n = \epsilon_n,\ \alpha_0 = -\epsilon_1 - \epsilon_2,\ \alpha_0' = -\epsilon_1, \quad \#W = 2^n \cdot n!.$$

C_n

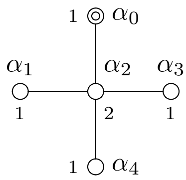


$$\Sigma = \left\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm 2\epsilon_k\,;\, 1 \le i < j \le n,\ 1 \le k \le n\right\}, \quad \#\Sigma = 2n^2,$$
$$\alpha_j = \epsilon_j - \epsilon_{j+1\ (j=1,\dots,n-1)},\ \alpha_n = 2\epsilon_n,\ \alpha_0 = -2\epsilon_1,\ \alpha_0' = -\epsilon_1 - \epsilon_2,\ \#W = 2^n \cdot n!.$$

D_n

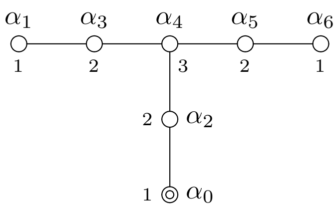


D_4



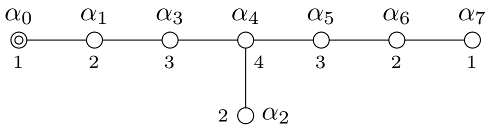
$$\Sigma = \left\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j)\,;\, 1 \le i < j \le n\right\}, \quad \#\Sigma = 2n(n-1),$$
$$\alpha_j = \epsilon_j - \epsilon_{j+1} \quad {\scriptstyle (j=1,\dots,n-1)}, \qquad \alpha_n = \epsilon_{n-1} + \epsilon_n,$$
$$\alpha_0 = -\epsilon_1 - \epsilon_2, \quad \#W = 2^{n-1} \cdot n!.$$

E_6



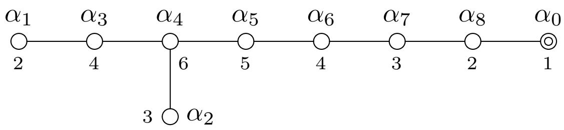
$$\Sigma = \Big\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm\tfrac{1}{2}\big(\epsilon_8 - \epsilon_7 - \epsilon_6 + \textstyle\sum_{k=1}^{5}(-1)^{\nu(k)}\epsilon_k\big)\,;$$
$$1 \le i < j \le 5,\ \textstyle\sum_{k=1}^{5}\nu(k) \text{ is even}\Big\}, \quad \#\Sigma = 72,$$
$$\alpha_1 = \tfrac{1}{2}(\epsilon_1 + \epsilon_8) - \tfrac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7),$$
$$\alpha_2 = \epsilon_1 + \epsilon_2, \quad \alpha_j = \epsilon_{j-1} - \epsilon_{j-2} \quad (3 \le j \le 6),$$
$$\alpha_0 = -\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8), \quad \#W = 2^7 \cdot 3^4 \cdot 5.$$

E_7



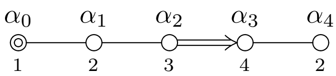
$$\Sigma = \bigl\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm(\epsilon_7 - \epsilon_8),\ \pm\tfrac{1}{2}\bigl(\epsilon_7 - \epsilon_8 + \textstyle\sum_{k=1}^{6}(-1)^{\nu(k)}\epsilon_k\bigr)\,;$$
$$1 \le i < j \le 6,\ \textstyle\sum_{k=1}^{6}\nu(k) \text{ is odd}\bigr\},\quad \#\Sigma = 126,$$
$$\alpha_1 = \tfrac{1}{2}(\epsilon_1 + \epsilon_8) - \tfrac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7),$$
$$\alpha_2 = \epsilon_1 + \epsilon_2,\quad \alpha_j = \epsilon_{j-1} - \epsilon_{j-2}\quad (3 \le j \le 7),$$
$$\alpha_0 = \epsilon_7 - \epsilon_8,\quad \#W = 2^{10}\cdot 3^4 \cdot 5 \cdot 7.$$

E_8



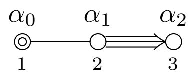
$$\Sigma = \bigl\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \tfrac{1}{2}\textstyle\sum_{k=1}^{8}(-1)^{\nu(k)}\epsilon_k\,;\ 1 \le i < j \le 8,$$
$$\textstyle\sum_{k=1}^{8}\nu(k) \text{ is even}\bigr\},\quad \#\Sigma = 240,$$
$$\alpha_1 = \tfrac{1}{2}(\epsilon_1 + \epsilon_8) - \tfrac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7),$$
$$\alpha_2 = \epsilon_1 + \epsilon_2,\quad \alpha_j = \epsilon_{j-1} - \epsilon_{j-2}\quad (3 \le j \le 8),$$
$$\alpha_0 = -\epsilon_7 - \epsilon_8,\quad \#W = 2^{14}\cdot 3^5 \cdot 5^2 \cdot 7.$$

F_4



$$\Sigma = \bigl\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm\epsilon_k,\ \pm\tfrac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\,;$$
$$1 \le i < j \le 4,\ 1 \le k \le 4\bigr\},\quad \#\Sigma = 48,$$
$$\alpha_1 = \epsilon_2 - \epsilon_3,\ \alpha_2 = \epsilon_3 - \epsilon_4,\ \alpha_3 = \epsilon_4,\ \alpha_4 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4),$$
$$\alpha_0 = -\epsilon_1 - \epsilon_2,\ \alpha_0' = -\epsilon_1,\quad \#W = 2^7 \cdot 3^2.$$

G_2



$$\Sigma = \bigl\{\pm(\epsilon_i - \epsilon_j),\ \mp(2\epsilon_1 - \epsilon_2 - \epsilon_3),\ \mp(2\epsilon_2 - \epsilon_1 - \epsilon_3),\ \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2)\,;$$
$$1 \le i < j \le 3\bigr\},\quad \#\Sigma = 12,$$
$$\alpha_1 = -2\epsilon_1 + \epsilon_2 + \epsilon_3,\ \alpha_2 = \epsilon_1 - \epsilon_2,\ \alpha_0 = \epsilon_1 + \epsilon_2 - 2\epsilon_3,\ \alpha_0' = \epsilon_2 - \epsilon_3,\quad \#W = 12.$$

Remark 9.1. i) There are natural identifications of root systems

$$D_1 = \emptyset,\quad D_2 \simeq A_1 + A_1,\quad D_3 \simeq A_3,\quad A_1 \simeq B_1 \simeq C_1,\quad B_2 \simeq C_2 \tag{9.1}$$

and Weyl groups

$$\begin{array}{ccc} \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n & \xrightarrow{\sim} & W_{B_n} = W_{C_n} \\ \cup & & \cup \\ \bigl(\sigma, (c_1, \dots, c_n)\bigr) & \mapsto & w_{\sigma,c} : \mathbb{R}^n \ni \epsilon_j \mapsto (-1)^{c_j}\epsilon_{\sigma(j)}\quad (j = 1, \dots, n), \end{array} \tag{9.2}$$
$$W_{D_n} = \{w_{\sigma,c} \in W_{B_n}\,;\ (-1)^{\sum c_j} = 1\},$$
$$W_{A_{n-1}} = \{w_{\sigma,c} \in W_{B_n}\,;\ c_1 = \cdots = c_n = 0\}.$$

ii) For the fundamental system Ψ of an irreducible root system Σ we have

$$\#W_\Psi = \#\Sigma^L \cdot \#W_{\Psi\cap\alpha_0^\perp} \tag{9.3}$$

$$= (\#\Psi)! \cdot \#\{\alpha_j \in \tilde\Psi\,;\, m_j(\alpha_{max}) = 1\} \cdot \prod_{\alpha_j\in\tilde\Psi} m_j(\alpha_{max}). \tag{9.4}$$

iii) There exist roots $\alpha_i \in \Psi$ satisfying $m_i(\alpha_{max}) = 1$, which are determined by (2.26). Moreover we have

$$m_j(\alpha_{max}) = \sum_{\alpha_\nu\in\tilde\Psi\setminus\{\alpha_j\}} m_\nu(\alpha_{max})\Bigl(-\frac{(\alpha_\nu|\alpha_j)}{(\alpha_j|\alpha_j)}\Bigr) \qquad (\alpha_j \in \tilde\Psi) \tag{9.5}$$

because of (2.22). In particular

$$2m_j(\alpha_{\max}) = \sum_{\substack{\alpha_\nu\in\tilde\Psi\setminus\{\alpha_j\}\\ (\alpha_\nu|\alpha_j)\neq 0}} m_\nu(\alpha_{max}) \qquad (\alpha_j \in \tilde\Psi\cap\Sigma^L). \tag{9.6}$$

Note that these conditions determine the extended Dynkin diagrams with the numbers $\{m_j(\alpha_{\max})\,;\,\alpha_j \in \tilde\Psi\}$ as in the following proposition, from which the classification of the root systems follows.

Definition 9.2. A diagram $\mathcal{G}$ consisting of finite vertices and lines and/or arrows is an *affine Dynkin diagram* if $\mathcal{G}$ satisfies the following conditions. Each line or arrow links a vertex to another vertex and each arrow has a stem formed by multiple lines. Moreover each vertex has an attached positive real number with the following property.

Fix any vertex P in $\mathcal{G}$ and let m be the number attached to P. Let $L_1,\dots,L_p$ be the lines and arrows linking P to other vertices. We denote the vertices and their attached numbers by $Q_1,\dots,Q_p$ and $m_1,\dots,m_p$, respectively. Then

$$2m = k_1m_1 + \cdots + k_pm_p \tag{9.7}$$

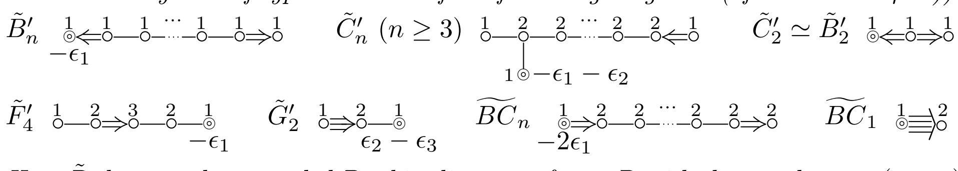
by putting

$$k_j = \begin{cases} 1 \text{ if } L_j \text{ is an arrow starting from } P \text{ or a line,}\\ \text{the number } (\geq 2) \text{ of lines in } L_j \text{ if } L_j \text{ is an arrow pointing toward } P\end{cases}$$

and the minimal number attached in any connected component of $\mathcal{G}$ equals 1.

Then we have the following proposition, which is probably known. Its proof is elementary and easy and we give it for the completeness.

Proposition 9.3. *The connected affine Dynkin diagram $\mathcal{G}$ is $\tilde R$ with an irreducible root system of type R or one of the following diagrams (cf. Remark 9.4 iii)).*



Here $\tilde R$ denotes the extended Dynkin diagram of type R with the numbers $m_j(\alpha_{max})$ attached to $\alpha_j \in \tilde\Psi$, which has been given in this section.

Proof. Fix any vertex P in $\mathcal{G}$ and retain the notation in Definition 9.2.

If $p \geq 2$ and $Q_1 = Q_2$, then $2m \geq 2m_1$ and we similarly have $2m_1 \geq 2m$ and hence $k_1 = k_2 = 1$ and $p = 2$. This only happens when $\mathcal{G} = \tilde A_1$.

Now we may assume $m_1 \geq \cdots \geq m_p$ and $Q_i \neq Q_j$ if $i \neq j$.

Claim: Let $P_1,\dots,P_\ell$ be vertices in $\mathcal{G}$ such that for any $j=2,\dots\ell-1$, P_j is linked only to both P_{j-1} and P_{j+1} and no arrow points to P_j. Then the corresponding attached numbers $m_1,\dots,m_\ell$ form an arithmetical progression series.

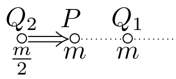
Since $2m_j\ge m$, the relation (9.7) assures $\sum k_j\le 4$.

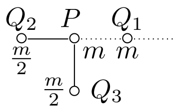
Note that if $2m_j=m$, Q_j is an end vertex to which no arrow points.

Case $\sum k_j=4$: Then $m_1=\cdots=m_p=\frac{m}{2}$, Q_j are end vertices and there is no arrow starting from P. We may assume $k_1\ge\cdots\ge k_p$. Hence $(k_1,\dots,k_p)=(1,1,1,1)$, $(2,1,1)$, $(2,2)$, $(3,1)$ or (4) and $\mathcal{G}=\tilde D_4$, $\tilde B_2'$, $\tilde B_2$, $\tilde G_2'$ or $\widetilde{BC}_1$, respectively.

Case $\sum k_j=3$ and $p=1$: We have $k_1=3$ and $m=\frac32 m_1$. Then there exists a vertex $Q'\in G\setminus\{P\}$ with the number $m'\ge\frac{m_1}{2}$ such that Q' is linked to Q. Since $2m_1\ge m+m'$, $m'=\frac12 m_1$ and Q' is an end vertex. Hence $\mathcal{G}=\tilde G_2$.

Case $\sum k_j=3$, $p\ge 2$ and $m_1\ge m$: Note that $k_1=1$ and $p=2$ or 3. When $p=2$, $k_2=2$ and $(m_1,m_2)=(m,\frac12 m)$. When $p=3$, $k_2=k_3=1$ and $(m_1,m_2,m_3)=(m,\frac12 m,\frac12 m)$. Hence if $2\le j\le p$, Q_j is an end vertex as follows.

$p=2$ 

$p=3$ 

Denoting $P_1=P$ and $P_2=Q_1$, we choose the maximal sequence of vertices $P_1,\dots,P_\ell$ given in the above claim. The numbers attached to these vertices are m. If an arrow links $P_{\ell-1}$ to P_ℓ, it has double lines and points toward the end vertex P_ℓ and hence $\mathcal{G}=\widetilde{BC}_n$ or $\tilde B_n$ according to $p=2$ or 3, respectively.

Now we may assume that P_j is linked to P_{j+1} by a line if $1\le j<\ell$. Then P_ℓ is not an end vertex and therefore P_ℓ is a branching vertex or there exists an arrow pointing toward P_ℓ. Applying the argument we have done to the vertex P_ℓ in place of P, we conclude the following. If P_ℓ is a branching vertex, P_ℓ is linked to two end vertices together with $P_{\ell-1}$ by lines and we have $\tilde C_n'$ or $\tilde D_n$ according to $p=2$ or 3, respectively. If P_ℓ is not a branching vertex, an arrow starting from an end vertex points toward P_ℓ and we have accordingly $\mathcal{G}=\tilde C_n$ or $\tilde C_n'$.

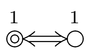
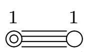
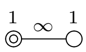
Case $\sum k_j=3$, $p\ge 2$ and $m_1<m$: Fix j with $1\le j\le p$ and let $P_1,\dots,P_{\ell_j}$ be the maximal sequence given in the claim such that $P_1=P$ and $P_2=Q_j$. The corresponding attached numbers $m_1'=m, m_2'=m_j, m_3',\dots,m_{\ell_j}'$ form a strictly decreasing arithmetical progression series and the argument in the preceding case assures that P_{ℓ_j} is not a branching vertex. Moreover P_{ℓ_j} is not linked to any arrow but it is an end vertex. Therefore $m_j=\frac{\ell_j-1}{\ell_j}m$. If $p=3$, (9.7) implies

$$\tfrac{1}{\ell_1}+\tfrac{1}{\ell_2}+\tfrac{1}{\ell_3}=1,\quad \ell_1\ge\ell_2\ge\ell_3 \tag{9.8}$$

and hence $(\ell_1,\ell_2,\ell_3)=(3,3,3)$, $(4,4,2)$ or $(6,3,2)$ and $\mathcal{G}=\tilde E_6$, $\tilde E_7$ or $\tilde E_8$, respectively. Similarly if $p=2$, we have $k_1=2$, $k_2=1$ and $(\ell_1,\ell_2)=(3,3)$ or $(4,2)$ and $\mathcal{G}=\tilde F_4'$ or $\tilde F_4$, respectively.

Other cases: Now we may assume that $\mathcal{G}$ has no branching vertex and moreover that if $\mathcal{G}$ contains an arrow, the arrow has double lines and points toward an end vertex. Hence it follows from the claim that $\mathcal{G}$ equals $\tilde B_n'$ if $\mathcal{G}$ contains an arrow and $\tilde A_n$ $(n\ge 2)$ if otherwise. $\square$

Remark 9.4. Retain the notation in Proposition 9.3.

i) $\tilde A_1$ is sometimes denoted by  or  or  .

ii) The proposition is known as the classification of the generalized Cartan matrices of affine type (cf. [5, Ch. 4]), where $\tilde R$, $\tilde B_n'$, $\tilde C_n'$, $\widetilde{BC}_n$, $\tilde F_4'$ and $\tilde G_2'$ are denoted by $R^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$, respectively.

iii) Suppose R is not simply laced and let $\alpha'_{\max}$ be the maximal root in $\Sigma \setminus \Sigma^L$ and put $\alpha'_0 = -\alpha'_{max}$. Then the root α'_0 corresponds to the vertex in $\tilde{R}'$ indicated by ⊚ and we get the Dynkin diagram of type R by deleting the vertex.

If we change the arrows in $\tilde{R}'$ by those with the opposite directions, $\tilde{R}'$ is changed into the extended Dynkin diagram of the dual root system of R.

The root system

$$\{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm\epsilon_k,\ \pm 2\epsilon_k\,;\, 1 \le i < j \le n,\ 1 \le k \le n\} \tag{9.9}$$

in $\mathbb{R}^n$ is the non-reduced root system of type BC_n. If we denote the maximal root of the non-reduced root system of type BC_n by α_{max}, the root $\alpha_0 := -\alpha_{max}$ corresponds to the vertex in $\widetilde{BC}_n$ indicated by ⊚.

Note that for an irreducible root system Σ with a fundamental system Ψ, the set $\{\beta \in \Sigma\,;\, (\alpha|\beta) \le 0\ (\forall \alpha \in \Psi)\}$ is the complete representatives of the orbits under the action of its Weyl group or equivalently the decomposition of Σ according to the length of the roots. The attached numbers in the above diagrams are the coefficients $m_j(-\alpha_0)$ in the expression $-\alpha_0 = \sum_{\alpha_j \in \Psi} m_j(-\alpha_0)\alpha_j$ for the element α_0 in the set. Here we don't assume Σ is reduced.

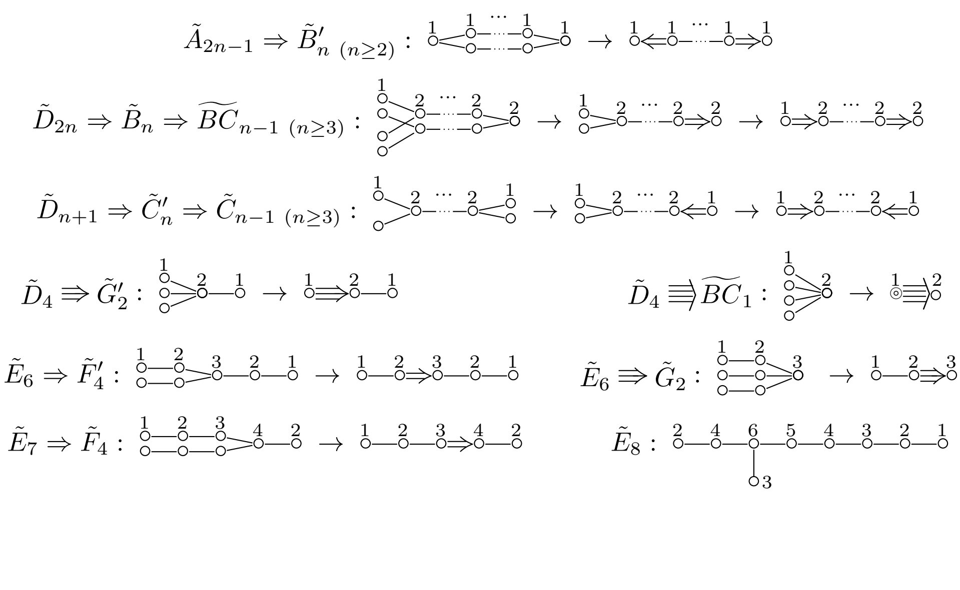
iv) We easily get a realization of $\mathcal{G}$ in an Euclidean space as follows. For example, if $\mathcal{G} = \tilde{E}_8$, we first realize $\langle \alpha_2, \ldots, \alpha_8, \alpha_0 \rangle \simeq D_8$ as in the standard way given in this section and then α_1 is determined by (2.22) and moreover $E_7 = \{\alpha_0\}^\perp$ and $E_6 = \{\alpha_0, \alpha_8\}^\perp$. If $\mathcal{G} = \tilde{F}_4$, we first realize $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \simeq B_4$ and then α_4.

v) The connected diagram $\mathcal{G}$ corresponds to an indecomposable finite subset Φ of $\mathbb{R}^n \setminus \{0\}$ with $\#\Phi = n + 1$ such that

$$\alpha \in \Phi,\ \beta \in \Phi \text{ and } \alpha \ne \beta \Rightarrow -2\frac{(\alpha|\beta)}{(\alpha|\alpha)} \in \{0, 1, 2, \ldots\}. \tag{9.10}$$

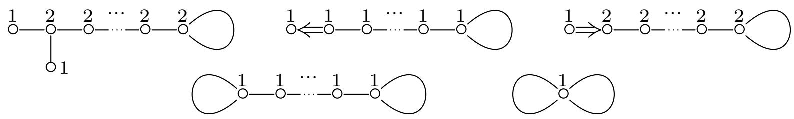
Here the subset Φ of $\mathbb{R}^n$ is indecomposable if there exists no non-trivial decomposition $\Phi = \Phi_1 \cup \Phi_2$ with $\Phi_1 \perp \Phi_2$.

vi) The diagram $\mathcal{G}$ with arrows is constructed as a quotient under the action of an automorphism of an extended Dynkin diagram of a simply laced root system. The arrow between two vertices in the quotient represents the difference of the numbers of the corresponding original vertices as follows.



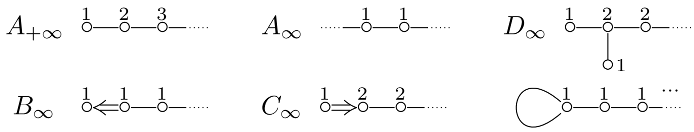
vii) Allowing a line linking a vertex to the same vertex in $\mathcal{G}$, we have the following extra ones.

(9.11)



viii) If the assumption of the finiteness of the vertices is dropped in the proposition, we moreover have the following $\mathcal{G}$, which easily follows from the proof of the proposition.

(9.12)



10. Tables

In this section we assume that Σ is an irreducible and reduced root system. We will classify the elements of $\mathrm{Hom}(\Xi,\Sigma)$ under a suitable isomorphisms for every root system Ξ.

Definition 10.1. For ι, $\iota'\in\mathrm{Hom}(\Xi,\Sigma)$ we define that ι is *weakly equivalent* to ι' if and only if there exists $g\in\mathrm{Aut}(\Sigma)=\mathrm{Hom}(\Sigma,\Sigma)$ with $\iota'(\Xi)=g\circ\iota(\Xi)$, that it, $\iota'(\Xi)\underset{\Sigma}{\overset{w}{\sim}}\Xi$. Then $\mathrm{Hom}(\Xi,\Sigma)$ is decomposed into the equivalence classes.

In many cases $\mathrm{Hom}(\Xi,\Sigma)$ itself is the equivalence class but if it is not so, we will identify every equivalence class $\mathrm{Hom}(\Xi,\Sigma)_o$ contained in $\mathrm{Hom}(\Xi,\Sigma)$ by a suitable geometric condition.

In the tables in this section we will list up all Ξ with $\mathrm{Hom}(\Xi,\Sigma)\neq\emptyset$ and classify them with some data under the following notation.

$$\mathrm{Aut}(\Xi):=\mathrm{Hom}(\Xi,\Xi),\qquad \mathrm{Aut}(\Sigma):=\mathrm{Hom}(\Sigma,\Sigma),$$

$$\mathrm{Aut}'(\Xi):=\prod_{j=1}^m\mathrm{Aut}(\Xi_j)\subset\mathrm{Aut}(\Xi)\text{ for the irreducible decomposition}$$
$$\Xi=\Xi_1+\cdots+\Xi_m,$$

$$\begin{aligned}
\#\ &:\ \#\big(W_\Sigma\backslash\mathrm{Hom}(\Xi,\Sigma)_o\big),\\
\#_\Xi\ &:\ \#\big(W_\Sigma\backslash\mathrm{Hom}(\Xi,\Sigma)_o/\mathrm{Aut}(\Xi)\big),\\
\#_{\Xi'}\ &:\ \#\big(W_\Sigma\backslash\mathrm{Hom}(\Xi,\Sigma)_o/\mathrm{Aut}'(\Xi)\big),\\
\#_\Sigma\ &:\ \#\big(\mathrm{Aut}(\Sigma)\backslash\mathrm{Hom}(\Xi,\Sigma)_o\big),\\
\Xi^{\perp\perp}\ &:\ \begin{cases}\circ & \big((\Xi^\perp)^\perp=\Xi\text{ and (6.9) is valid}\big),\\ \times & \big((\Xi^\perp)^\perp=\Xi\text{ but (6.9) is not valid}\big),\\ (\Xi^\perp)^\perp & \big((\Xi^\perp)^\perp\neq\Xi\big),\end{cases}\\
P\ &:\ \begin{cases}\#\{\Theta\subset\Psi\,;\,\langle\Theta\rangle\underset{\Sigma}{\overset{w}{\sim}}\Xi\} & \big(\text{if }\Xi\text{ is fundamental}\big),\\ \leftarrow & \big(L\text{-closure of }\Xi\text{ is given by }(\Xi^\perp)^\perp\big),\\ \rightarrow & \big(L\text{-closure of }\Xi\text{ is given in the right column}\big),\end{cases}\\
L\ &:\ L\text{-closure}\qquad\big(\text{if }\operatorname{rank}(\Xi^\perp)^\perp>\operatorname{rank}\Xi\text{ and }\Xi\text{ is not }L\text{-closed}\big),\\
S\ &:\ S\text{-closure}\qquad\big(\text{if }\Xi\text{ is not }S\text{-closed (cf. Definition 6.6)}\big),
\end{aligned}$$

$$\langle j_1,\ldots,j_m\rangle : \langle\alpha_{j_1},\ldots,\alpha_{j_m}\rangle \qquad \big(\text{under the notation in §9}\big),$$
$$\langle\backslash j\rangle : \langle\Psi\setminus\{\alpha_j\}\rangle,$$

For subsystem $\Xi\subset\Sigma$ and a subgroup G of $\mathrm{Aut}(\Sigma)$ we put

$$\mathrm{Hom}(\Xi,\Sigma)_o := \{\iota\in\mathrm{Hom}(\Xi,\Sigma)\,;\,\iota(\Xi)\underset{\Sigma}{\overset{w}{\sim}}\Xi\},$$
$$O^w_\Xi := \{\Theta\subset\Sigma\,;\,W_\Theta\Theta=\Theta \text{ and } \Theta\underset{\Sigma}{\overset{w}{\sim}}\Xi\} \qquad (\text{cf. Definition 2.8}),$$
$$O_\Xi := \{\Theta\subset\Sigma\,;\,W_\Theta\Theta=\Theta \text{ and } \Theta\underset{\Sigma}{\sim}\Xi\},$$
$$N_G(\Xi) := \{g\in G\,;\,g(\Xi)=\Xi\}.$$

Then

$$O^w_\Xi \simeq \mathrm{Hom}(\Xi,\Sigma)_o/\mathrm{Aut}(\Xi)\simeq\mathrm{Aut}(\Sigma)/N_{\mathrm{Aut}(\Sigma)}(\Xi),$$
$$O_\Xi\simeq W_\Sigma/N_{W_\Sigma}(\Xi),$$

(10.1) $$\#O^w_\Xi/\#O_\Xi = \#\big(W_\Xi\backslash\mathrm{Hom}(\Xi,\Sigma)_o/\mathrm{Aut}(\Xi)\big) = (\#_\Xi),$$

(10.2) $$\begin{aligned}(\#)/(\#_\Xi) &= \#\big(W_\Sigma\backslash\mathrm{Hom}(\Xi,\Sigma)_o\big)\ /\ \#\big(W_\Sigma\backslash\mathrm{Hom}(\Xi,\Sigma)_o/\mathrm{Aut}(\Xi)\big)\\ &= \#\big(\mathrm{Out}(\Xi)/\mathrm{Out}_\Sigma(\Xi)\big)\\ &= \#\Big(\mathrm{Out}(\Xi)\ /\ \big(N_{W_\Sigma}(\Xi)/(W_\Xi\times W_{\Xi^\perp})\big)\Big),\end{aligned}$$

(10.3) $$\#O_\Xi = (\#)\cdot\#W_\Sigma\ /\ \big((\#_\Sigma)\cdot\#\,\mathrm{Out}(\Xi)\cdot\#W_\Xi\cdot\#W_{\Xi^\perp}\big).$$

Here $(\#)$, $(\#_\Xi)$, $(\#_{\Xi'})$ and $(\#_\Sigma)$ are numbers given in the columns indicated by $\#$, $\#_\Xi$, $\#_{\Xi'}$ and $\#_\Sigma$ in the table below, respectively. Since $1\le(\#_\Xi)\le(\#_{\Xi'})$, $(\#_\Xi)$ may not be written if $(\#_{\Xi'})=1$. If $\mathrm{Out}(\Sigma)$ is trivial, $(\#_\Sigma)=(\#)$ and therefore $(\#_\Sigma)$ may not be written.

Note that (9.3) corresponds to the special case of (10.3) with $\Xi=\{\pm\alpha_0\}$.

Remark 10.2. We obtain the answers to the questions in the introduction from the table in this section as follows.

Answers to Q1 and Q2 are given by the table.

The number in Q3 is given by (10.3) with the table.

The answer to Q4 is yes if and only if $(\#_\Xi)=(\#)$ (cf. Remark 8.2 iii), Remark 10.7 iii) and §8.2).

The answer to Q5 is yes if and only if $(\#_{\Xi'})=(\#)$.

The answer to Q6 is yes if and only if $(\#_\Xi)=1$ (cf. Remark 8.1 ii)).

The answer to the first question of Q7 is given by Remark 8.3 and the number in Q7 is obtained from the column P in the table.

10.1. **Classical type.** (Σ : A_n, B_n, C_n, D_n)

Ξ : Irreducible

Σ	Ξ	$\#$	$\#_\Xi$	$\#_\Sigma$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
A_n	A_1	1	1	1	A_{n-2}	$\times$	n
$A_{n\ (1<m\le n-2)}$	A_m	2	1	1	A_{n-m-1}	$\times$	$n-m+1$
$A_{n\ (n\ge3)}$	A_{n-1}	2	1	1	$\emptyset$	Σ	2
$A_{n\ (n\ge2)}$	A_n	2	1	1	$\emptyset$	Σ	1
$\Sigma\ (n\ge5)$	Ξ	$\#$	$\#_\Xi$	$\#_\Sigma$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
D_n	A_1	1	1	1	$D_{n-2}+A_1$	$\times$	n
D_n	A_2	1	1	1	D_{n-3}	A_3	$n-1$
D_n	A_3	1	1	1	D_{n-4}	D_4	$n-2$
	(D_3)	1	1	1	$D_{n-3\ (n\ne7)}$	$\circ$	1
					$D_{4\ (n=7)}$	$\times$	

$D_{n\ (4\le k\le n-3)}$	A_k	1	1	1	D_{n-k-1}	D_{k+1}	$n-k+1$
D_n	A_{n-2}	1	1	1	$\emptyset$	Σ	3
$D_{n\ (n:\text{odd})}$	A_{n-1}	2	1	1	$\emptyset$	Σ	2
$D_{n\ (n:\text{even})}$		2	2	1			
D_n	D_4	3	1	3	$D_{n-4\ (n\ge 6)}$	2	1
					$\emptyset_{\ (n=5)}$	Σ	
$D_{n\ (4<k\le n-2)}$	D_k	1	1	1	D_{n-k}	$\circ_{\ (k\ne n-4)}$	1
						$\times_{\ (k=n-4)}$	
$D_{n\ (n\ge 6)}$	D_{n-1}	1	1	1	$\emptyset$	Σ	1
D_n	D_n	2	1	1	$\emptyset$	Σ	1

$\Sigma\ (n\ge 2)$	Ξ	$\#$	$\#_\Xi$	$\#_\Sigma$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
B_n	A_1^L	1	1	1	$B_{n-2}+A_1^L$	$\circ$	$n-1$
	A_1^S	1	1	1	B_{n-1}	$\circ$	1
$B_{n\ (n\ge 3)}$	A_2	1	1	1	B_{n-3}	B_3	$n-2$
$B_{n\ (n\ge 4)}$	A_3	1	1	1	B_{n-4}	B_4	$n-3$
$B_{n\ (n\ge 3)}$	(D_3)	1	1	1	B_{n-3}	B_3	$\leftarrow$
$B_{n\ (4\le m<n)}$	A_m	1	1	1	B_{n-m-1}	B_{m+1}	$n-m$
$B_{n\ (n\ge 4)}$	D_4	3	1	3	B_{n-4}	B_4	$\leftarrow$
$B_{n\ (4<m\le n)}$	D_m	1	1	1	B_{n-m}	B_m	$\leftarrow$
$B_{n\ (2\le m\le n)}$	B_m	1	1	1	B_{n-m}	$\circ$	1

$\Sigma\ (n\ge 2)$	Ξ	$\#$	$\#_\Xi$	$\#_{\Xi'}$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
C_n	A_1^S	1	1	1	$C_{n-2}+A_1^S$	$\circ$	$n-1$
	A_1^L	1	1	1	C_{n-1}	$\circ$	1
$C_{n\ (n\ge 3)}$	A_2	1	1	1	C_{n-3}	C_3	$n-2$
$C_{n\ (n\ge 4)}$	A_3	1	1	1	C_{n-4}	C_4	$n-3$
$C_{n\ (n\ge 3)}$	(D_3)	1	1	1	C_{n-3}	C_3	$\leftarrow\ S:C_3$
$C_{n\ (4\le m<n)}$	A_m	1	1	1	C_{n-m-1}	C_{m+1}	$n-m$
$C_{n\ (n\ge 4)}$	D_4	3	1	3	C_{n-4}	C_4	$\leftarrow\ S:C_4$
$C_{n\ (4<m\le n)}$	D_m	1	1	1	C_{n-m}	C_m	$\leftarrow\ S:C_m$
$C_{n\ (2\le m\le n)}$	C_m	1	1	1	C_{n-m}	$\circ$	1

The symbol (D_3) is the above table corresponds to D_3 in (10.4). The subsystems A_1^L and A_1^S of B_n in the above table correspond to A_1 and B_1 in (10.4), respectively.

Applying Remark 3.1 iii) to the table for $\Sigma = B_n$, we have the table for $\Sigma = C_n$.

Suppose $n > 4$. Then $\#\overline{\mathrm{Hom}}(A_{n-1}, D_n) = 2$ and the non-trivial element $g \in \mathrm{Out}(D_n)$ maps its element to the other. Let $A_{n-1} \subset D_n$ by the notation in §10. Then $h \in \mathrm{Aut}(D_n)$ defined by $h(\epsilon_j) = -\epsilon_j$ $(j = 1, \ldots, n)$ induces the non-trivial element of $\mathrm{Out}(A_{n-1})$. Here h is not an element of W_{D_n} if and only if n is odd. Hence $\#\bigl(\overline{\mathrm{Hom}}(A_{n-1}, D_n)/\mathrm{Out}(A_{n-1})\bigr) = 1$ if and only if n is odd.

$\Sigma = D_4$

Σ	Ξ	$\#$	$\#_\Xi$	$\#_{\Xi'}$	$\#_\Sigma$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P
D_4	A_1	1	1	1	1	$3A_1$	$\times$	4
D_4	A_2	1	1	1	1	$\emptyset$	Σ	3
D_4	A_3	3	3	3	1	$\emptyset$	Σ	3
D_4	D_4	6	1	1	1	$\emptyset$	Σ	1
D_4	$2A_1$	3	3	3	1	$2A_1$	$\circ$	3
D_4	$3A_1$	6	1	6	1	A_1	$\times$	1
D_4	$4A_1$	6	1	6	1	$\emptyset$	Σ	$\leftarrow$

Σ: not of type D_4

We still assume that Σ is irreducible and of classical type. We will examine $\mathrm{Hom}(\Xi,\Sigma)$ when Ξ may not be irreducible. It is not difficult because the root system and its Weyl group are easy to describe. The subsystems of Σ can be imbedded in the root space B_N with a sufficiently large N. We should distinguish two subsystems which are isomorphic as root systems but they are not equivalent by B_N.

Under the notation in §9 they are the followings:

$$\begin{aligned}
&A_1 = \{\pm(\epsilon_1-\epsilon_2)\},\\
&B_1 = \{\pm\epsilon_1\} \simeq A_1,\\
&D_2 = \langle \epsilon_1-\epsilon_2, \epsilon_1+\epsilon_2\rangle \simeq 2A_1,\\
&A_3 = \langle \epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3, \epsilon_3-\epsilon_4\rangle,\\
&D_3 = \langle \epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3, \epsilon_2+\epsilon_3\rangle \simeq A_3.
\end{aligned} \tag{10.4}$$

Let $\{\epsilon_1,\ldots,\epsilon_N\}$ be an orthonormal basis of $\mathbb{R}^N$ with a sufficiently large positive integer N. Let σ be an element of $O(N)$ defined by $\sigma(\epsilon_j)=\epsilon_{j+1}$ for $1\le j<N$ and $\sigma(\epsilon_N)=\epsilon_1$. Let A_n, B_n, C_n and D_n denote the corresponding root spaces given in §9 and we identify them with finite subsets of $\mathbb{R}^N$ and put $Q_n^i := \sigma^i(Q_n)$ for $Q=A$, B, C and D. For example

$$A_4^3 = \langle \epsilon_4-\epsilon_5, \epsilon_5-\epsilon_6, \epsilon_7-\epsilon_8, \epsilon_8-\epsilon_9\rangle \subset \mathbb{R}^N$$

For $\mathbf{m}=(m_1,m_2,\ldots)$, $\mathbf{k}=(k_1,k_2,\ldots)$, $\mathbf{n}=(n_1,n_2,\ldots)\in\mathbb{N}^{\mathbb{N}}$ with

$$k_1 = 0 \text{ and } \sum_{j=1}^{\infty}|m_j+k_j+n_j|<\infty \tag{10.5}$$

define

$$\begin{aligned}
\Xi_{\mathbf{m}} &:= \bigcup_{j\ge1}\bigcup_{\nu=0}^{k_j-1} A_j^{(j+1)\nu+\sum_{i=1}^{j-1}(i+1)m_i} \simeq \sum_{j\ge1} m_jA_j,\\
M(\mathbf{m}) &:= \sum_{j\ge1}(j+1)m_j,\\
\Xi_{\mathbf{m},\mathbf{k}} &= \Xi_{\mathbf{m}} \cup \bigcup_{j\ge2}\bigcup_{\nu=0}^{k_i-1} D_j^{M(\mathbf{m})+j\nu+\sum_{i=1}^{j-1} ik_i} \text{ with } D_2=\langle\epsilon_1-\epsilon_2,\epsilon_1+\epsilon_2\rangle,\\
M(\mathbf{m},\mathbf{k}) &:= M(\mathbf{m})+\sum_{j\ge2} jk_j,\\
p_D(\mathbf{m},\mathbf{k}) &:= \bigl((m_1+2k_2,m_2,m_3+k_3,m_4,\ldots),(0,0,0,k_4,k_5,\ldots)\bigr),\\
\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}} &:= \Xi_{\mathbf{m},\mathbf{k}} \cup \bigcup_{j\ge1}\bigcup_{\nu=0}^{k_i-1} B_j^{M(\mathbf{m},\mathbf{k})+j\nu+\sum_{i=1}^{j-1} ik_i} \text{ with } B_1=\langle\epsilon_1\rangle\\
&\underset{B_N}{\sim} \sum_{j\ge1} m_jA_1+\sum_{j\ge2}k_jD_j+\sum_{j\ge1}n_jB_j,\\
M(\mathbf{m},\mathbf{k},\mathbf{n}) &:= M(\mathbf{m},\mathbf{k})+\sum_{j\ge1} jn_j = \sum_{j\ge1}(j+1)m_j+\sum_{j\ge1}j(k_j+n_j),\\
p_B(\mathbf{m},\mathbf{k},\mathbf{n}) &:= \bigl((m_1+n_1+2k_2,m_2,m_3+k_3,m_4,\ldots),(0,0,0,k_4,k_5,\ldots),\\
&\qquad (0,n_2,n_3,\ldots)\bigr).
\end{aligned}$$

Suppose $n\ge M(\mathbf{m},\mathbf{k},\mathbf{n})$. Then $\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}}$ is naturally a subsystem of B_n and

$$\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}}^{\perp}\cap B_n \simeq k_1A_1+B_{n-M(\mathbf{m},\mathbf{k},\mathbf{n})} \tag{10.6}$$

and if there exists $w \in \mathrm{Aut}(B_n) = W_{B_n}$ such that

$$\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}} = w(\Xi_{\mathbf{m}',\mathbf{k}',\mathbf{n}'}),$$

then $(\mathbf{m},\mathbf{k},\mathbf{n}) = (\mathbf{m}',\mathbf{k}',\mathbf{n}')$.

Fix elements $\bar{\mathbf{m}} = (\bar{m}_1, \bar{m}_2, \ldots)$, $\bar{\mathbf{k}} = (\bar{k}_1, \bar{k}_2, \ldots)$ and $\bar{\mathbf{n}} = (\bar{n}_1, \bar{n}_2, \ldots)$ in $\mathbb{N}^{\mathbb{N}}$ satisfying

$$\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = \bar{n}_1 = 0. \tag{10.7}$$

Proposition 10.3 (type B_n $(n \ge 2)$)**.** *Let*

$$\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}} = \sum_{j\ge1} \bar{m}_j A_j + \sum_{j\ge4} \bar{k}_j D_j + \sum_{j\ge2} \bar{n}_j B_j. \tag{10.8}$$

Then

$$\mathrm{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}, B_n) = \coprod_{\substack{p_B(\mathbf{m},\mathbf{k},\mathbf{n})=(\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}})\\ M(\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}})\le n}} \mathrm{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}, B_n)_{(\mathbf{m},\mathbf{k},\mathbf{n})}, \tag{10.9}$$

$$\begin{aligned}\mathrm{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}, B_n)_{(\mathbf{m},\mathbf{k},\mathbf{n})} := \{\iota \in \mathrm{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}, B_n)\,;\ &\textit{there exists}\\ w \in W_{B_n} \textit{such that } &w(\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}}) = \iota(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}\}\}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}, B_n)_{(\mathbf{m},\mathbf{k},\mathbf{n})} \ne \emptyset\\ &\qquad\Leftrightarrow\ p_B(\mathbf{m},\mathbf{k},\mathbf{n}) = (\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}) \textit{ and } M(\mathbf{m},\mathbf{k},\mathbf{n}) \le n.\end{aligned} \tag{10.10}$$

Assume $\mathrm{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}, B_n)_{(\mathbf{m},\mathbf{k},\mathbf{n})} \ne \emptyset$. *Then*

$$\#\big(W_{B_n}\backslash\mathrm{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}, B_n)_{(\mathbf{m},\mathbf{k},\mathbf{n})}\big) = \frac{3^{k_4}\cdot(m_1+n_1+2k_2)!\cdot(m_3+k_3)!}{2^{k_2}\cdot m_1!\cdot n_1!\cdot k_2!\cdot m_3!\cdot k_3!}, \tag{10.11}$$

$$\#\big(W_{B_n}\backslash\mathrm{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}}, B_n)_{(\mathbf{m},\mathbf{k},\mathbf{n})}/\mathrm{Aut}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}})\big) = 1, \tag{10.12}$$

$$\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}}^{\perp} \cap B_n \simeq m_1A_1 + B_{n-M(\mathbf{m},\mathbf{k},\mathbf{n})}, \tag{10.13}$$

$$\bar{\Xi}_{\mathbf{m},\mathbf{k},\mathbf{n}} = \Xi_{\mathbf{m},\mathbf{k},\mathbf{n}}\ \Leftrightarrow\ m_2 = m_3 = \cdots = k_2 = k_3 = \cdots = 0,\ \textstyle\sum_{j\ge1} n_j \le 1, \tag{10.14}$$

$$\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}} \textit{ is fundamental}\ \Leftrightarrow\ k_2 = k_3 = \cdots = 0 \textit{ and } \textstyle\sum_{j\ge1} n_j \le 1. \tag{10.15}$$

The S-closure of $\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}}$ equals $\Xi_{\mathbf{m},\mathbf{k},(\delta_{\nu,\sum_j jn_j})_\nu}$. Here $\sum n_jB_j$ changes into $B_{\sum_j jn_j}$. The L-closure of $\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}}$ equals the fundamental subsystem $\Xi_{\mathbf{m},0,(\delta_{\nu,\sum_j j(k_j+n_j)})_\nu}$. Here $\sum k_jD_j + \sum n_jB_j$ changes into $B_{\sum_j j(k_j+n_j)}$.

Considering the dual root systems, we have the proposition for C_n:

Proposition 10.4 (type C_n $(n \ge 3)$)**.** *Let*

$$\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}},\bar{\mathbf{n}}} = \sum_{j\ge1} \bar{m}_j A_j + \sum_{j\ge4} \bar{k}_j D_j + \sum_{j\ge2} \bar{n}_j C_j. \tag{10.16}$$

Then the statements in Proposition 10.3 with replacing B_n and $B_{n-M(\mathbf{m},\mathbf{k},\mathbf{n})}$ by C_n and $C_{n-M(\mathbf{m},\mathbf{k},\mathbf{n})}$, respectively, are valid except for the last statement on S-closure.

The S-closure of this $\Xi_{\mathbf{m},\mathbf{k},\mathbf{n}}$ is $\Xi_{\mathbf{m},0,(n_1,k_2+n_2,k_3+n_3,\ldots)}$, which is obtained by replacing $\sum k_jD_j$ by $\sum k_jC_j$.

We have the following propositions when Σ is of type D_n or of type A_n.

Proposition 10.5 (type D_n $(n \ge 5)$)**.** *Let*

$$\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}} = \sum_{j\ge1} \bar{m}_j A_j + \sum_{j\ge4} \bar{k}_j D_j. \tag{10.17}$$

Then

$$\operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n) = \coprod_{\substack{p_D(\mathbf{m},\mathbf{k})=(\bar{\mathbf{m}},\bar{\mathbf{k}}) \\ M(\mathbf{m},\mathbf{k})\le n}} \operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n)_{(\mathbf{m},\mathbf{k})}, \tag{10.18}$$

$$\operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n)_{(\mathbf{m},\mathbf{k})} := \{\iota \in \operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n)\,;\ \textit{there exists } w \in W_{B_n} \\ \textit{such that } w(\Xi_{\mathbf{m},\mathbf{k}}) = \iota(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}})\},$$

$$\operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n)_{(\mathbf{m},\mathbf{k})} \ne \emptyset \ \Leftrightarrow\ p_D(\mathbf{m},\mathbf{k}) = (\bar{\mathbf{m}},\bar{\mathbf{k}}) \ \textit{and}\ M(\mathbf{m},\mathbf{k}) \le n. \tag{10.19}$$

When $\operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n)_{(\mathbf{m},\mathbf{k})} \ne \emptyset$,

$$\#\big(W_{D_n}\backslash\operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n)_{(\mathbf{m},\mathbf{k})}\big) = \varepsilon_1 \frac{3^{k_4}\cdot(m_1+2k_2)!\cdot(m_3+k_3)!}{2^{k_2}\cdot m_1!\cdot k_2!\cdot m_3!\cdot k_3!}, \tag{10.20}$$

$$\#\big(W_{D_n}\backslash\operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n)_{(\mathbf{m},\mathbf{k})}/\operatorname{Aut}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}})\big) = \varepsilon_2, \tag{10.21}$$

$$\#\big(\operatorname{Aut}(D_n)\backslash\operatorname{Hom}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}}, D_n)_{(\mathbf{m},\mathbf{k})}/\operatorname{Aut}(\Xi_{\bar{\mathbf{m}},\bar{\mathbf{k}}})\big) = 1, \tag{10.22}$$

$$\Xi_{\mathbf{m},\mathbf{k}}^{\perp} \simeq mA_1 + D_{n-M(\mathbf{m},\mathbf{k})}, \tag{10.23}$$

$$\bar{\Xi}_{\mathbf{m},\mathbf{k}} = \Xi_{\mathbf{m},\mathbf{k}} \ \Leftrightarrow\ k_2 = k_3 = \cdots = 0 \ \textit{and}\ M(\mathbf{m},\mathbf{k}) \ne n-1, \tag{10.24}$$

$$\Xi_{\mathbf{m},\mathbf{k}} \ \textit{is fundamental} \ \Leftrightarrow\ \textstyle\sum_{j\ge 2} k_j \le 1 \ \Leftrightarrow\ \Xi_{\mathbf{m},\mathbf{k}} \ \textit{is L-closed.} \tag{10.25}$$

Here

$$\varepsilon_1 = \begin{cases} 2 & \textit{if}\ \ M(\mathbf{m},\mathbf{k}) = n, \\ 1 & \textit{if}\ \ M(\mathbf{m},\mathbf{k}) < n, \end{cases}$$

$$\varepsilon_2 = \begin{cases} 2 & \textit{if}\ \ M(\mathbf{m},\mathbf{k}) = n \ \textit{and}\ m_{2\nu} = k_{\nu+1} = 0 \quad (\nu = 1,2,\dots), \\ 1 & \textit{otherwise.} \end{cases}$$

The L-closure of $\Xi_{\mathbf{m},\mathbf{k}}$ *is obtained by replacing* $\sum_{j\ge 2} k_j D_j$ *by* $D_{\sum_{j\ge 2} jk_j}$.

Proposition 10.6 (type A_n). *Let* $\Xi_{\mathbf{m}} = \sum_{j\ge 1} m_j A_j$. *Then*

$$\operatorname{Hom}(\Xi_{\mathbf{m}}, A_n) \ne \emptyset \ \Leftrightarrow\ M(\mathbf{m}) \le n+1 \tag{10.26}$$

and if $M(\mathbf{m}) \le n+1$, *we have*

$$\#\overline{\operatorname{Hom}}(\Xi_{\mathbf{m}}, A_n) = 2^{\sum_{j\ge 2} m_j}, \tag{10.27}$$

$$\#\big(\overline{\operatorname{Hom}}(\Xi_{\mathbf{m}}, A_n)/\operatorname{Out}(\Xi_{\mathbf{m}})\big) = 1, \tag{10.28}$$

$$\#\big(\operatorname{Out}(A_n)\backslash\overline{\operatorname{Hom}}(\Xi_{\mathbf{m}}, A_n)\big) = \begin{cases} 1 & (\sum_{j\ge 2} m_j = 0), \\ 2^{(\sum_{j\ge 2} m_j)-1} & (\sum_{j\ge 2} m_j > 0), \end{cases} \tag{10.29}$$

$$\Xi_{\mathbf{m}}^{\perp} \cap A_n \simeq A_{n-M(\mathbf{m})}, \tag{10.30}$$

$$\bar{\Xi}_{\mathbf{m}} = \Xi_{\mathbf{m}} \ \Leftrightarrow\ \textstyle\sum_{j\ge 1} m_j \le 1 \ \textit{and}\ M(\mathbf{m}) \ne n. \tag{10.31}$$

Any subsystem of A_n *is fundamental and hence L-closed.*

10.2. **Exceptional type.** (Σ : E_6, E_7, E_8, F_4, G_2)

Σ	Ξ	#	$\#_{\Xi'}$	$\#_{\Sigma}$	$\Xi^{\perp}$	$\Xi^{\perp\perp}$	P	
E_6	A_1	1	1	1	A_5	$\times$	6	
E_6	A_2	1	1	1	$2A_2$	$\times$	5	
E_6	A_3	1	1	1	$2A_1$	$\circ$	5	
E_6	A_4	2	1	1	A_1	A_5	4	
E_6	A_5	2	1	1	A_1	$\times$	1	$\langle\backslash 2\rangle$
E_6	D_4	1	1	1	$\emptyset$	Σ	1	
E_6	D_5	2	1	1	$\emptyset$	Σ	2	$\langle\backslash 1\rangle$, $\langle\backslash 6\rangle$
E_6	E_6	2	1	1	$\emptyset$	Σ	1	

E_6	$2A_1$	1	1	1	A_3	$\circ$	10	
E_6	$3A_1$	1	1	1	A_1	A_5	5	
E_6	$4A_1$	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L : D_4$
E_6	$A_2 + A_1$	2	1	1	A_2	$2A_2$	10	
E_6	$A_2 + 2A_1$	2	1	1	$\emptyset$	Σ	5	$\subset 3A_2$
E_6	$2A_2$	4	1	2	A_2	$\times$	1	
E_6	$2A_2 + A_1$	4	1	2	$\emptyset$	Σ	1	$\langle\backslash 4\rangle$ $\subset 3A_2$
E_6	$3A_2$	8	1	4	$\emptyset$	Σ	$\leftarrow$	§8.2.5
E_6	$A_3 + A_1$	2	1	1	A_1	A_5	4	
E_6	$A_3 + 2A_1$	2	1	1	$\emptyset$	Σ	$\rightarrow$	§8.2.1, $L : D_5$
E_6	$A_4 + A_1$	2	1	1	$\emptyset$	Σ	2	$\langle\backslash 3\rangle$, $\langle\backslash 5\rangle$
E_6	$A_5 + A_1$	2	1	1	$\emptyset$	Σ	$\leftarrow$	
Σ	Ξ	#	$\#_\Xi$	$\#_{\Xi'}$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P	
E_7	A_1	1	1	1	D_6	$\times$	7	
E_7	A_2	1	1	1	A_5	$\circ$	6	
E_7	A_3	1	1	1	$A_3 + A_1$	$\circ$	6	
E_7	A_4	1	1	1	A_2	A_5	5	
E_7	A_5]''	1	1	1	A_2	$\circ$	1	$\langle 2,4,5,6,7\rangle$
	]'	1	1	1	A_1	D_6	2	$\langle 3,4,5,6,7\rangle$
E_7	A_6	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 2\rangle$
E_7	A_7	1	1	1	$\emptyset$	Σ	$\leftarrow$	
E_7	D_4	1	1	1	$3A_1$	$\circ$	1	
E_7	D_5	1	1	1	A_1	D_6	2	
E_7	D_6	2	1	1	A_1	$\times$	1	$\langle\backslash 1\rangle$
E_7	E_6	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 7\rangle$
E_7	E_7	1	1	1	$\emptyset$	Σ	1	
E_7	$2A_1$	1	1	1	$D_4 + A_1$	$\times$	15	
E_7	$3A_1$]''	1	1	1	D_4	$\circ$	1	$\langle 2,5,7\rangle$
	]'	1	1	1	$4A_1$	$\times$	10	$\langle 3,5,7\rangle$
E_7	$4A_1$]''	4	1	4	$3A_1$	$\times$	2	$\langle 2,3,5,7\rangle$
	]'	1	1	1	$3A_1$	D_4	$\leftarrow$	
E_7	$5A_1$	15	1	15	$2A_1$	$D_4 + A_1$	$\leftarrow$	§8.2.3
E_7	$6A_1$	30	1	30	A_1	D_6	$\leftarrow$	§8.2.3
E_7	$7A_1$	30	1	30	$\emptyset$	Σ	$\leftarrow$	§8.2.3
E_7	$A_2 + A_1$	1	1	1	A_3	$A_3 + A_1$	18	
E_7	$A_2 + 2A_1$	1	1	1	A_1	D_6	12	
E_7	$A_2 + 3A_1$	1	1	1	$\emptyset$	Σ	1	
E_7	$2A_2$	2	1	1	A_2	A_5	4	
E_7	$2A_2 + A_1$	2	1	1	$\emptyset$	Σ	3	$\subset 3A_2$
E_7	$3A_2$	4	1	1	$\emptyset$	Σ	$\rightarrow$	§8.2.5, $L : E_6$
E_7	$A_3 + A_1$]''	1	1	1	A_3	$\circ$	2	$\langle 2,5,6,7\rangle$
	]'	1	1	1	$2A_1$	$D_4 + A_1$	9	$\langle 3,5,6,7\rangle$
E_7	$A_3 + 2A_1$]''	2	1	2	A_1	D_6	3	$\exists(A_3 + A_1)^\perp = A_3$
	]'	1	1	1			$\rightarrow$	$\forall(A_3 + A_1)^\perp = 2A_1$ $\subset D_3 + D_2$ $L : D_5$
E_7	$A_3 + 3A_1$	3	1	3	$\emptyset$	Σ	$\rightarrow$	$\subset 2A_3 + A_1$ $L : D_5 + A_1$
E_7	$A_3 + A_2$	2	1	1	A_1	D_6	3	$\subset 2A_3 + A_1$
E_7	$A_3 + A_2 + A_1$	2	1	1	$\emptyset$	Σ	1	$\langle\backslash 4\rangle$ $\subset 2A_3 + A_1$
E_7	$2A_3$	2	1	1	A_1	D_6	$\leftarrow$	$\subset 2A_3 + A_1$

E_7	$2A_3+A_1$	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_7	A_4+A_1	1	1	1	$\emptyset$	Σ	5	
E_7	A_4+A_2	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 5\rangle$
E_7	A_5+A_1]''	1	1	1	$\emptyset$	Σ	1	$A_5^{\perp}=A_2$, $\langle\backslash 3\rangle$
	]'	1	1	1	$\emptyset$	Σ	$\rightarrow$	$A_5^{\perp}=A_1$, $L:E_6$
E_7	A_5+A_2	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_7	D_4+A_1	3	1	1	$2A_1$	$\times$	1	
E_7	D_4+2A_1	6	1	1	A_1	D_6	$\leftarrow$	
E_7	D_4+3A_1	6	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.4
E_7	D_5+A_1	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 6\rangle$
E_7	D_6+A_1	2	1	1	$\emptyset$	Σ	$\leftarrow$	

Σ	Ξ	#	$\#_{\Xi}$	$\#_{\Xi'}$	$\Xi^{\perp}$	$\Xi^{\perp\perp}$	P	
E_8	A_1	1	1	1	E_7	$\circ$	8	
E_8	A_2	1	1	1	E_6	$\circ$	7	
E_8	A_3	1	1	1	D_5	$\circ$	7	
E_8	A_4	1	1	1	A_4	$\circ$	6	
E_8	A_5	1	1	1	A_2+A_1	$\circ$	4	
E_8	A_6	1	1	1	A_1	E_7	3	
E_8	A_7]''	1	1	1	A_1	E_7	$\leftarrow$	
	]'	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 2\rangle$
E_8	A_8	1	1	1	$\emptyset$	Σ	$\leftarrow$	
E_8	D_4	1	1	1	D_4	$\circ$	1	
E_8	D_5	1	1	1	A_3	$\circ$	2	
E_8	D_6	1	1	1	$2A_1$	$\circ$	1	
E_8	D_7	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 1\rangle$
E_8	D_8	2	1	1	$\emptyset$	Σ	$\leftarrow$	
E_8	E_6	1	1	1	A_2	$\circ$	1	
E_8	E_7	1	1	1	A_1	$\circ$	1	$\langle\backslash 8\rangle$
E_8	E_8	1	1	1	$\emptyset$	Σ	1	
E_8	$2A_1$	1	1	1	D_6	$\circ$	21	
E_8	$3A_1$	1	1	1	D_4+A_1	$\circ$	21	
E_8	$4A_1$]''	1	1	1	D_4	D_4	$\leftarrow$	
	]'	1	1	1	$4A_1$	$\circ$	7	$\langle 2,3,6,8\rangle$
E_8	$5A_1$	5	1	5	$3A_1$	D_4+A_1	$\leftarrow$	§8.2.3
E_8	$6A_1$	15	1	15	$2A_1$	D_6	$\leftarrow$	§8.2.3
E_8	$7A_1$	30	1	30	A_1	E_7	$\leftarrow$	§8.2.3
E_8	$8A_1$	30	1	30	$\emptyset$	Σ	$\leftarrow$	§8.2.3
E_8	A_2+A_1	1	1	1	A_5	$\circ$	28	
E_8	A_2+2A_1	1	1	1	A_3	D_5	28	
E_8	A_2+3A_1	1	1	1	A_1	E_7	7	
E_8	A_2+4A_1	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L:A_2+D_4$
E_8	$2A_2$	1	1	1	$2A_2$	$\circ$	8	
E_8	$2A_2+A_1$	2	1	1	A_2	E_6	9	
E_8	$2A_2+2A_1$	2	1	1	$\emptyset$	Σ	2	$\subset 4A_2$
E_8	$3A_2$	4	1	1	A_2	E_6	$\leftarrow$	
E_8	$3A_2+A_1$	4	1	1	$\emptyset$	Σ	$\rightarrow$	$\subset 4A_2$ $L:E_6+A_1$
E_8	$4A_2$	8	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.5
E_8	A_3+A_1	1	1	1	A_3+A_1	$\circ$	20	
E_8	A_3+2A_1]''	1	1	1	A_3	D_5	$\leftarrow$	
	]'	1	1	1	$2A_1$	D_6	10	$\langle 2,3,4,6,8\rangle$

E_8	A_3+3A_1	3	1	3	A_1	E_7	$\rightarrow$	$L: D_5+A_1$
E_8	A_3+4A_1	3	1	3	$\emptyset$	Σ	$\rightarrow$	$\subset A_3+D_5$ $L: D_7$
E_8	A_3+A_2	1	1	1	$2A_1$	D_6	10	
E_8	$A_3+A_2+A_1$	2	1	1	A_1	E_7	4	
E_8	$A_3+A_2+2A_1$	2	1	1	$\emptyset$	Σ	$\rightarrow$	$\subset D_6+2A_1$ $L: D_5+A_2$
E_8	$2A_3$ $]''$	1	1	1	$2A_1$	D_6	$\leftarrow$	
	$]'$	1	1	1	$\emptyset$	Σ	2	$\langle 2,3,4,6,7,8\rangle$
E_8	$2A_3+A_1$	2	1	1	A_1	E_7	$\leftarrow$	
E_8	$2A_3+2A_1$	2	1	1	$\emptyset$	Σ	$\leftarrow$	$\subset D_6+2A_1$
E_8	A_4+A_1	1	1	1	A_2	E_6	12	
E_8	A_4+2A_1	1	1	1	$\emptyset$	Σ	5	
E_8	A_4+A_2	2	1	1	A_1	E_7	4	
E_8	$A_4+A_2+A_1$	2	1	1	$\emptyset$	Σ	1	$\langle\backslash 4\rangle$ $\subset 2A_4$
E_8	A_4+A_3	2	1	1	$\emptyset$	Σ	1	$\langle\backslash 5\rangle$ $\subset 2A_4$
E_8	$2A_4$	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	A_5+A_1 $]''$	1	1	1	A_2	E_6	$\leftarrow$	
	$]'$	1	1	1	A_1	E_7	3	$\langle 1,4,5,6,7,8\rangle$
E_8	A_5+2A_1	2	1	2	$\emptyset$	Σ	$\rightarrow$	$\subset A_5+A_2+A_1$ $L: E_6+A_1$
E_8	A_5+A_2	2	1	1	A_1	E_7	$\leftarrow$	
E_8	$A_5+A_2+A_1$	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	A_6+A_1	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 3\rangle$
E_8	A_7+A_1	1	1	1	$\emptyset$	Σ	$\leftarrow$	
E_8	D_4+A_1	1	1	1	$3A_1$	$\circ$	2	
E_8	D_4+2A_1	3	1	1	$2A_1$	D_6	$\leftarrow$	
E_8	D_4+3A_1	6	1	1	A_1	E_7	$\leftarrow$	
E_8	D_4+4A_1	6	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.4
E_8	D_4+A_2	1	1	1	$\emptyset$	Σ	1	
E_8	D_4+A_3	3	1	1	$\emptyset$	Σ	$\rightarrow$	$\subset 2D_4$ $L: D_7$
E_8	$2D_4$	6	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.4
E_8	D_5+A_1	1	1	1	A_1	E_7	3	
E_8	D_5+2A_1	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L: D_7$
E_8	D_5+A_2	2	1	1	$\emptyset$	Σ	1	$\langle\backslash 6\rangle$ $\subset D_5+A_3$
E_8	D_5+A_3	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	D_6+A_1	2	1	1	A_1	E_7	$\leftarrow$	
E_8	D_6+2A_1	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	E_6+A_1	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 7\rangle$
E_8	E_6+A_2	2	1	1	$\emptyset$	Σ	$\leftarrow$	§8.2.1
E_8	E_7+A_1	1	1	1	$\emptyset$	Σ	$\leftarrow$	

Σ	Ξ	#	$\#_\Xi$	$\#_{\Xi'}$	$\Xi^\perp$	$\Xi^{\perp\perp}$	P	
F_4	A_1^L	1	1	1	C_3	$\circ$	2	
	A_1^S	1	1	1	B_3	$\circ$	2	
F_4	A_2^L	1	1	1	A_2^S	$\circ$	1	
	A_2^S	1	1	1	A_2^L	$\circ$	1	
F_4	A_3^L	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L: B_3$
	A_3^S	1	1	1	$\emptyset$	Σ	$\rightarrow$	$L, S: C_3$
F_4	D_4^L	1	1	1	$\emptyset$	Σ	$\leftarrow$	
	D_4^S	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S: F_4$
F_4	B_2	1	1	1	B_2	$\circ$	1	

F_4	B_3	1	1	1	A_1^S	$\circ$	1	$\langle\backslash 4\rangle$
F_4	C_3	1	1	1	A_1^L	$\circ$	1	$\langle\backslash 1\rangle$
F_4	B_4	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	C_4	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S : F_4$
F_4	F_4	1	1	1	$\emptyset$	Σ	1	
F_4	$2A_1^L$	1	1	1	B_2	B_2	$\leftarrow$	
	$2A_1^S$	1	1	1	B_2	B_2	$\leftarrow$	$S : B_2$
	$A_1^S + A_1^L$	1	1	1	$A_1^L + A_1^S$	$\times$	4	
F_4	$3A_1^L$	1	1	1	A_1^L	C_3	$\leftarrow$	
	$3A_1^S$	1	1	1	A_1^S	B_3	$\leftarrow$	$S : B_3$
	$A_1^S + 2A_1^L$	1	1	1	A_1^S	B_3	$\leftarrow$	
	$2A_1^S + A_1^L$	1	1	1	A_1^L	C_3	$\leftarrow$	$S : B_2 + A_1^L$
F_4	$4A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
	$4A_1^S$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S : F_4$
	$2A_1^S + 2A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S : B_2 + 2A_1^L$
F_4	$A_2^L + A_1^S$	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 3\rangle$
	$A_2^S + A_1^L$	1	1	1	$\emptyset$	Σ	1	$\langle\backslash 2\rangle$
F_4	$A_2^S + A_2^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	$B_2 + A_1^L$	1	1	1	A_1^L	C_3	$\leftarrow$	
	$B_2 + A_1^S$	1	1	1	A_1^S	B_3	$\leftarrow$	$S : B_3$
F_4	$B_2 + 2A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
	$B_2 + 2A_1^S$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S : B_4$
F_4	$2B_2$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S : B_4$
F_4	$A_3^S + A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S : C_3 + A_1^L$
F_4	$A_3^L + A_1^S$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	$C_3 + A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	
F_4	$B_3 + A_1^S$	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S : B_4$
G_2	A_1^L	1	1	1	A_1^S	$\circ$	1	$\langle\backslash 2\rangle$
	A_1^S	1	1	1	A_1^L	$\circ$	1	$\langle\backslash 1\rangle$
G_2	A_2^L	1	1	1	$\emptyset$	Σ	$\leftarrow$	
	A_2^S	1	1	1	$\emptyset$	Σ	$\leftarrow$	$S : G_2$
G_2	G_2	1	1	1	$\emptyset$	Σ	1	
G_2	$A_1^S + A_1^L$	1	1	1	$\emptyset$	Σ	$\leftarrow$	

We explain some symbols used in the above table.

Remark 10.7. i) In the table we use following notation.

$$\begin{gathered}\Sigma^L := \{\alpha \in \Sigma\,;\, |\beta| \le |\alpha| \quad (\forall \beta \in \Sigma)\},\\ A_m^S \simeq A_m^L \simeq A_m, \quad A_m^L \subset \Sigma^L, \quad A_m^S \cap \Sigma^L = \emptyset,\\ D_m^S \simeq D_m^L \simeq D_m, \quad D_m^L \subset \Sigma^L, \quad D_m^S \cap \Sigma^L = \emptyset.\end{gathered}$$

ii) The symbols $]'$ and $]''$ in the column Σ.
Suppose Σ is irreducible and of exceptional type. Then $\#\overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}(\Xi) \le 2$. When $\#\overline{\mathrm{Hom}}(\Xi,\Sigma)/\mathrm{Out}(\Xi) = 2$, Σ is of type E_7 or E_8 and then the symbols $[\Xi]'$ and $[\Xi]''$ are used in [4] to distinguish the equivalence classes of the imbeddings $\Xi \subset \Sigma$. Then $[\Xi]'$ means that there is a representative Ξ in the equivalence class such that

$$\Xi \subset A_n \subset \Sigma = E_n \tag{10.32}$$

with $n = 7$ or 8. For example, $\#\overline{\mathrm{Hom}}(4A_1, E_7)/\mathrm{Out}(4A_1) = 2$ and the symbols $[4A_1]'$ and $[4A_1]''$ are used in [4], which are expressed by $]'$ and $]''$ respectively in the column Σ in our table (cf. (7.6)).

In [4] the distinction of the elements of $\mathrm{Out}(\Sigma)\backslash\overline{\mathrm{Hom}}(\Xi, \Sigma)/\mathrm{Out}(\Xi)$ such as $]'$ and $]''$ is not discussed but it is stated there that the distinction is due to actual calculation.

iii) The structure of $\mathrm{Out}_\Sigma(\Xi)$.
If $(\#) = \#\,\mathrm{Out}(\Xi)$ or $(\#) = 1$ in the table, it follows from (10.2) that $\#\,\mathrm{Out}_\Sigma(\Xi) = 1$ or $\mathrm{Out}_\Sigma(\Xi) \xrightarrow{\sim} \mathrm{Out}(\Xi)$, respectively. In the column P in the table, a reference such as §8.2.3 gives the description of $\mathrm{Out}_\Sigma(\Xi)$ for other non-trivial cases.

If $\Xi = \Xi_1 + \Xi_2 \subset \Xi' = \Xi_1 + \Xi_1^\perp \subset \Sigma$ and $\mathrm{Out}(\Xi) \xleftarrow{\sim} \mathrm{Out}(\Xi_1) \times \mathrm{Out}(\Xi_2)$ and $\Xi^\perp = \emptyset$, we have

$$\mathrm{Out}_\Sigma(\Xi) \simeq N_{\mathrm{Aut}_\Sigma(\Xi')}(\Xi_2)/W_\Xi. \tag{10.33}$$

The symbol "$\subset \Xi'$" is indicated in the column P if $\mathrm{Out}_\Sigma(\Xi)$ is easily obtained by this relation. For example, $\mathrm{Out}_{E_8}(D_5 + A_2)$ is isomorphic to $\mathrm{Out}_{E_8}(D_5 + A_3)$ through the imbedding $D_5 + A_2 \subset D_5 + A_3 \subset E_8$.

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