Classification of Fuchsian systems and their connection problem

By

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§ 1.   Introduction

Middle convolutions introduced by Katz [Kz] and extensions and restrictions introduced by Yokoyama [Yo] give interesting operations on Fuchsian systems on the Riemann sphere. They are invertible and under them the solutions of the systems are transformed by integral transformations and the correspondence of their monodromy groups is concretely described (cf. [Ko4], [Ha], [HY], [DR2], [HF], [O2] etc.).

In this note we review the Deligne-Simpson problem, a combinatorial structure of middle convolutions and their relation to a Kac-Moody root system discovered by Crawley-Boevey [CB]. We show with examples that middle convolutions transform the Fuchsian systems with a fixed number of accessory parameters into fundamental systems whose spectral type is in a finite set. In §9 we give an explicit connection formula for solutions of Fuchsian differential equations without moduli.

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§ 2.   Tuples of partitions

Let $\mathbf{m} = \bigl(m_{j,\nu}\bigr)_{\substack{j=0,1,\dots \\ \nu=1,2,\dots}}$ be an ordered set of infinite number of non-negative integers indexed by non-negative integers j and positive integers ν. Then $\mathbf{m}$ is called a $(k+1)$-*tuple of partitions of* n if the following two conditions are satisfied.

$$(2.1) \qquad \sum_{\nu=1}^{\infty} m_{j,\nu} = n \qquad (j = 0, 1, \dots),$$

(2.2) $$m_{j,1} = n \qquad (j = k+1, k+2, \ldots).$$

The totality of $(k+1)$-tuples of partitions of n are denoted by $\mathcal{P}_{k+1}^{(n)}$ and we put

(2.3) $$\mathcal{P}_{k+1} := \bigcup_{n=0}^{\infty} \mathcal{P}_{k+1}^{(n)}, \quad \mathcal{P}^{(n)} := \bigcup_{k=0}^{\infty} \mathcal{P}_{k+1}^{(n)}, \quad \mathcal{P} := \bigcup_{k=0}^{\infty} \mathcal{P}_{k+1},$$

(2.4) $$\operatorname{ord} \mathbf{m} := n \quad \text{if} \quad \mathbf{m} \in \mathcal{P}^{(n)},$$

(2.5) $$\mathbf{1} := \big(m_{j,\nu} = \delta_{\nu,1}\big)_{\substack{j=0,1,\ldots \\ \nu=1,2,\ldots}} \in \mathcal{P}^{(1)},$$

(2.6) $$\operatorname{idx}(\mathbf{m}, \mathbf{m}') := \sum_{j=0}^{k} \sum_{\nu=1}^{\infty} m_{j,\nu} m'_{j,\nu} - (k-1) \operatorname{ord} \mathbf{m} \cdot \operatorname{ord} \mathbf{m}' \quad (\mathbf{m}, \ \mathbf{m}' \in \mathcal{P}_{k+1}).$$

Here $\operatorname{ord} \mathbf{m}$ is called the order of $\mathbf{m}$. For $\mathbf{m}, \mathbf{m}' \in \mathcal{P}$ and a non-negative integer p, the tuples $p\mathbf{m}$ and $\mathbf{m} + \mathbf{m}' \in \mathcal{P}$ are naturally defined. For $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ we choose integers $n_0, \ldots, n_k$ so that $m_{j,\nu} = 0$ for $\nu > n_j$ and $j = 0, \ldots, k$ and we will express $\mathbf{m}$ by

$$\begin{aligned}
\mathbf{m} &= (\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_k) \\
&= m_{0,1}, \ldots, m_{0,n_0}; \ldots; m_{k,1}, \ldots, m_{k,n_k} \\
&= m_{0,1} \cdots m_{0,n_0}, m_{1,1} \cdots m_{1,n_1}, \ldots, m_{k,1} \cdots m_{k,n_k}
\end{aligned}$$

if there is no confusion. Similarly $\mathbf{m} = (m_{0,1}, \ldots, m_{0,n_0})$ if $\mathbf{m} \in \mathcal{P}_1$. Here

$$\mathbf{m}_j = (m_{j,1}, \ldots, m_{j,n_j}) \quad \text{and} \quad \operatorname{ord} \mathbf{m} = m_{j,1} + \cdots + m_{j,n_j} \quad (0 \le j \le k).$$

For example $\mathbf{m} = (m_{j,\nu}) \in \mathcal{P}_3^{(4)}$ with $m_{1,1} = 3$ and $m_{0,\nu} = m_{2,\nu} = m_{1,2} = 1$ for $\nu = 1, \ldots, 4$ will be expressed by

(2.7) $$\mathbf{m} = 1,1,1,1; 3,1; 1,1,1,1 = 1111, 31, 1111 = 1^4, 31, 1^4.$$

Definition 2.1. A tuple of partition $\mathbf{m} \in \mathcal{P}$ is called *monotone* if

(2.8) $$m_{j,\nu} \ge m_{j,\nu+1} \quad (j = 0, 1, \ldots, \ \nu = 1, 2, \ldots)$$

and $\mathbf{m}$ is called *indivisible* if the greatest common divisor of $\{m_{j,\nu}\}$ equals 1.

Let $\mathfrak{S}_\infty$ be the restricted permutation group of the set of indices $\{0, 1, 2, 3, \ldots\} = \mathbb{Z}_{\ge 0}$, which is generated by the transpositions $(j, j+1)$ with $j \in \mathbb{Z}_{\ge 0}$. Put $\mathfrak{S}'_\infty := \{\sigma \in \mathfrak{S}_\infty \, ; \, \sigma(0) = 0\}$, which is isomorphic to $\mathfrak{S}_\infty$.

Definition 2.2. Transformation groups S_∞ and S'_∞ of $\mathcal{P}$ are defined by

(2.9)
$$S_\infty := H \ltimes S'_\infty, \quad S'_\infty := \prod_{j=0}^{\infty} G_j, \quad G_j \simeq \mathfrak{S}'_\infty, \quad H \simeq \mathfrak{S}_\infty,$$

$$m'_{j,\nu} = m_{\sigma(j), \sigma_j(\nu)} \qquad (j = 0, 1, \ldots, \ \nu = 1, 2, \ldots)$$

for $g = (\sigma, \sigma_1, \ldots) \in S_\infty$, $\mathbf{m} = (m_{j,\nu}) \in \mathcal{P}$ and $\mathbf{m}' = g\mathbf{m}$.

§3. Conjugacy classes of matrices

For $\mathbf{m} = (m_1, \ldots, m_N) \in \mathcal{P}_1^{(n)}$ and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ we define a matrix $L(\mathbf{m}; \lambda) \in M(n, \mathbb{C})$ as follows, which is introduced and effectively used by [Os]:

If $\mathbf{m}$ is monotone, then

$$L(\mathbf{m}; \lambda) := \Big(A_{ij} \Big)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}}, \quad A_{ij} \in M(m_i, m_j, \mathbb{C}),$$

(3.1)
$$A_{ij} = \begin{cases} \lambda_i I_{m_i} & (i = j) \\ I_{m_i, m_j} := \Big(\delta_{\mu\nu} \Big)_{\substack{1 \leq \mu \leq m_i \\ 1 \leq \nu \leq m_j}} = \begin{pmatrix} I_{m_j} \\ 0 \end{pmatrix} & (i = j - 1) \\ 0 & (i \neq j,\ j - 1) \end{cases}.$$

Here I_{m_i} denote the identity matrix of size m_i and $M(m_i, m_j, \mathbb{C})$ means the set of matrices of size $m_i \times m_j$ with components in $\mathbb{C}$ and $M(m, \mathbb{C}) := M(m, m, \mathbb{C})$.

For example

(3.2)
$$L(2, 1, 1; \lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}.$$

If $\mathbf{m}$ is not monotone, fix a permutation σ of $\{1, \ldots, N\}$ so that $(m_{\sigma(1)}, \ldots, m_{\sigma(N)})$ is monotone and put $L(\mathbf{m}; \lambda) = L(m_{\sigma(1)}, \ldots, m_{\sigma(N)}; \lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)})$.

When $\lambda_1 = \cdots = \lambda_N = \mu$, $L(\mathbf{m}; \lambda)$ will be simply denoted by $L(\mathbf{m}, \mu)$.

We denote $A \sim B$ for $A,\ B \in M(n, \mathbb{C})$ if and only if there exists $g \in GL(n, \mathbb{C})$ with $B = gAg^{-1}$. If $A \sim L(\mathbf{m}; \lambda)$, $\mathbf{m}$ is called the *spectral type* of A and denoted by $\mathrm{spt}\, A$.

Remark 1.   i) If $\mathbf{m} = (m_1, \ldots, m_N) \in \mathcal{P}_1^{(n)}$ is monotone, we have

(3.3) $A \sim L(\mathbf{m}; \lambda) \iff \mathrm{rank} \prod_{\nu=1}^{k} (A - \lambda_\nu) = n - (m_1 + \cdots + m_k) \quad (k = 0, 1, \ldots, N).$

ii) For $\mu \in \mathbb{C}$ put

(3.4) $(\mathbf{m}; \lambda)_\mu = (m_{i_1}, \ldots, m_{i_K}, \mu)$ with $\{i_1, \ldots, i_K\} = \{i\, ;\, \lambda_i = \mu\}.$

Then we have

(3.5)
$$L(\mathbf{m}; \lambda) \sim \bigoplus_{\mu \in \mathbb{C}} L\big((\mathbf{m}; \lambda)_\mu \big).$$

iii) Suppose $\mathbf{m}$ is monotone. Then for $\mu \in \mathbb{C}$

(3.6)
$$L(\mathbf{m}, \mu) \sim \bigoplus_{j=1}^{m_1} J\big(\max\{\nu\, ;\, m_\nu \geq j\}, \mu \big),$$

$$J(k, \mu) := L(1^k, \mu) \in M(k, \mathbb{C}). \qquad \text{(Jordan cell)}$$

iv) For $A \in M(n, \mathbb{C})$ we put $Z_{M(n,\mathbb{C})}(A) := \{X \in M(n, \mathbb{C})\,;\, AX = XA\}$. Then

$$(3.7) \qquad \dim Z_{M(n,\mathbb{C})}\big(L(\mathbf{m};\lambda)\big) = m_1^2 + m_2^2 + \cdots.$$

Note that the Jordan canonical form of $L(\mathbf{m};\lambda)$ is easily obtained by (3.5) and (3.6). For example $L(2,1,1,\mu) \sim J(3,\mu) \oplus J(1,\mu)$.

Lemma 3.1. *Let $A(t)$ be a continuous map of $[0,1)$ to $M(n,\mathbb{C})$. Suppose there exists a partition $\mathbf{m} = (m_1, \ldots, m_N)$ of n and continuous function $\lambda(t)$ of $(0,1)$ to $\mathbb{C}^N$ so that $A(t) \sim L\big(\mathbf{m};\lambda(t)\big)$ for any $t \in (0,1)$. If $\dim Z_{M(n,\mathbb{C})}\big(A(t)\big)$ is constant for $t \in [0,1)$, then $A(0) \sim L\big(\mathbf{m};\lim_{t\to 0}\lambda(t)\big)$.*

Proof. The proof is reduced to the result (cf. Remark 20) in [Os] but a more elementary proof will be given. First note that $\lim_{t\to 0}\lambda(t)$ exists.

We may assume that $\mathbf{m}$ is monotone. Fix $\mu \in \mathbb{C}$ and put $\{i_1, \ldots, i_K\} = \{i;\, \lambda_i(0) = \mu\}$ with $1 \le i_1 < i_2 < \cdots < i_K \le N$. Then

$$\operatorname{rank}\big(A(0) - \mu\big)^k \le \operatorname{rank}\prod_{\nu=1}^{k}\big(A(t) - \lambda_{i_\nu}(t)\big) = n - (m_{i_1} + \cdots + m_{i_k}).$$

Putting $m'_{i_k} = \operatorname{rank}\big(A(0) - \mu\big)^{k-1} - \operatorname{rank}\big(A(0) - \mu\big)^{k}$, we have

$$m_{i_1} \ge m_{i_2} \ge \cdots \ge m_{i_K} > 0, \quad m'_{i_1} \ge m'_{i_2} \ge \cdots \ge m'_{i_K} \ge 0,$$
$$m_{i_1} + \cdots + m_{i_k} \le m'_{i_1} + \cdots + m'_{i_k} \quad (k = 1, \ldots, K).$$

Then the following lemma and the equality $\sum m_i^2 = \sum (m'_i)^2$ imply $m_i = m'_i$. $\square$

Lemma 3.2. *Let $\mathbf{m}$ and $\mathbf{m}' \in \mathcal{P}_1$ be monotone partitions satisfying*

$$(3.8) \qquad m_1 + \cdots + m_j \le m'_1 + \cdots + m'_j \quad (j = 1, 2, \ldots).$$

If $\mathbf{m} \ne \mathbf{m}'$, then

$$\sum_{j=1}^{\infty} m_j^2 < \sum_{j=1}^{\infty} (m'_j)^2.$$

Proof. Let K be the largest integer with $m_K \ne 0$ and p be the smallest integer j such that the inequality in (3.8) holds. Note that the lemma is clear if $p \ge K$.

Suppose $p < K$. Then $m'_p > 1$. Let q and r be the smallest integers satisfying $m'_p > m'_{q+1}$ and $m'_p - 1 > m'_r$. Then $m_p < m'_q$ and the inequality in (3.8) holds for $k = p, \ldots, r-1$ because $m_k \le m_p \le m'_{r-1}$.

$$
\begin{array}{ccccccc}
m'_1, \ldots, m'_{p-1}, & m'_p, \ldots, & m'_q, & m'_{q+1}, \ldots, & m'_{r-1}, & m'_r \\
\| & \| & \vee & \vee & |\vee & |\vee \\
m_1, \ldots, m_{p-1}, & m_p, \ldots, & m_q, & m_{q+1}, \ldots, & m_{r-1}, & m_r
\end{array}
$$

Here $p \leq q < r \leq K+1$, $m'_p = \cdots = m'_q > m'_{q+1} = \cdots = m'_{r-1}$ and $m'_{r-1} > m'_r$. Put

$$m''_j = m'_j - \delta_{j,q} + \delta_{j,r}.$$

Then $\mathbf{m}''$ is monotone, $\sum(m''_j)^2 < (\sum m'_j)^2$ and $m_1 + \cdots + m_j \leq m''_1 + \cdots + m''_j$ $(j = 1, 2, \ldots)$. Thus we have the lemma by the induction on the lexicographic order of the triplet $(K - p, m'_p, q)$ for a fixed $\mathbf{m}$. $\qquad\square$

Proposition 3.3. *Let $A(t)$ be a real analytic map of $(-1, 1)$ to $M(n, \mathbb{C})$ such that $\dim Z_{\mathfrak{g}}\big(A(t)\big)$ doesn't depend on t. Then there exist a partition $\mathbf{m} = (m_1, \ldots, m_N)$ of n and a continuous function $\lambda(t) = (\lambda_1(t), \ldots, \lambda_N(t))$ of $(-1, 1)$ satisfying*

$$(3.9) \qquad\qquad A(t) \sim L\big(\mathbf{m}; \lambda(t)\big).$$

Proof. We find $c_j \in (-1, 1)$, monotone partitions $\mathbf{m}^{(j)} \in \mathcal{P}_1^{(n)}$ and real analytic functions $\lambda^{(j)}(t) = (\lambda_1^{(j)}(t), \ldots)$ on $I_j := (c_j, c_{j+1})$ such that

$$c_{j-1} < c_j < c_{j+1}, \ \lim_{\pm j \to \infty} c_j = \pm 1, \ A(t) \sim L\big(\mathbf{m}^{(j)}; \lambda^{(j)}(t)\big) \quad (\forall t \in I_j).$$

Lemma 3.1 assures that we may assume $\lambda^{(j)}(t)$ is continuous on the closure $\bar{I}_j$ of I_j and $A(t) \sim L\big(\mathbf{m}^{(j)}; \lambda^{(j)}(t)\big)$ for $t \in \bar{I}_j$. Hence $\mathbf{m}^{(j)}$ doesn't depend on j, which we denoted by $\mathbf{m}$. We can inductively define permutations $\sigma_{\pm j}$ of the indices $\{1, \ldots, N\}$ for $j = 1, 2, \ldots$ so that $\sigma_0 = id$, $m_{\sigma_{\pm j}(p)} = m_p$ for $p = 1, \ldots, N$ and moreover that $\big(\lambda_{\sigma_\nu(1)}^{(\nu)}(t), \ldots, \lambda_{\sigma_\nu(N)}^{(\nu)}(t)\big)$ for $-j \leq \nu \leq j$ define a continuous function on (c_{-j}, c_{j+1}). $\qquad\square$

Remark 2. i) Suppose that $\dim Z_{M(n,\mathbb{C})}\big(A(t)\big)$ is constant for a continuous map $A(t)$ of $(-1, 1)$ to $M(n, \mathbb{C})$. For $c \in (-1, 1)$ we can find $t_j \in (-1, 1)$ and $\mathbf{m} \in \mathcal{P}^{(1)}$ such that $\lim_{j \to \infty} t_j = c$ and $\operatorname{spt} A(t_j) = \mathbf{m}$. The proof of Lemma 3.1 shows $\operatorname{spt} A(c) = \mathbf{m}$. Hence

$$(3.10) \qquad \operatorname{spt} A(t) \text{ doesn't depend on } t \ \Leftrightarrow \ \dim Z_{M(n,\mathbb{C})}(A) \text{ doesn't depend on } t.$$

ii) It is easy to show that Proposition 3.3 is valid even if we replace "real analytic" by "continuous" but it is not true if we replace "real analytic" and "$(-1, 1)$" by "holomorphic" and "$\{t \in \mathbb{C} \,;\, |t| < 1\}$", respectively. The matrix $A(t) = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$ is a counter example.

§4. Deligne-Simpson problem

For simplicity we put $\mathfrak{g} = M(n, \mathbb{C})$ and $G = GL(n, \mathbb{C})$ only in this section.

Let $\mathbf{A} = (A_0, \ldots, A_k) \in \mathfrak{g}^{k+1}$. Put

(4.1) $$M(n, \mathbb{C})_0^{k+1} := \{(C_0, \ldots, C_k) \in \mathfrak{g}^{k+1}\,;\, C_0 + \cdots + C_k = 0\},$$

(4.2) $$Z_{\mathfrak{g}}(\mathbf{A}) := \{X \in \mathfrak{g}\,;\, [A_j, X] = 0 \quad (j = 0, \ldots, k)\}.$$

A tuple of matrices $\mathbf{A} \in \mathfrak{g}^{k+1}$ is called *irreducible* if any subspace $V \subset \mathbb{C}^n$ satisfying $A_j V \subset V$ for $j = 0, \ldots, k$ is $\{0\}$ or $\mathbb{C}^n$.

Suppose $\operatorname{trace} A_0 + \cdots + \operatorname{trace} A_k = 0$. The *additive Deligne-Simpson problem* presented by Kostov [Ko] is to determine the condition to $\mathbf{A}$ for the existence of an irreducible tuple $\mathbf{B} = (B_0, \ldots, B_k) \in M(n, \mathbb{C})_0^{k+1}$ satisfying $A_j \sim B_j$ for $j = 0, \ldots, k$. The condition is concretely given by Crawley-Boevey [CB] (cf. Theorem 10.1 and [Ko4]).

Suppose $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$. Then $\mathbf{A}$ is called *rigid* if $\mathbf{A} \sim \mathbf{B}$ for any element $\mathbf{B} = (B_0, \ldots, B_k) \in M(n, \mathbb{C})_0^{k+1}$ satisfying $B_j \sim A_j$ for $j = 0, \ldots, k$. Here we denote $\mathbf{A} \sim \mathbf{B}$ if there exists $g \in G$ with $(B_0, \ldots, B_k) = (gA_0g^{-1}, \ldots, gA_kg^{-1})$.

Remark 3.   Note that the local monodromy at ∞ of the Fuchsian system

(4.3) $$\frac{du}{dz} = \sum_{j=1}^{k} \frac{A_j}{z - z_j} u$$

on a Riemann sphere corresponds to A_0 with $\mathbf{A} = (A_0, \ldots, A_k) \in M(n, \mathbb{C})_0^{k+1}$. Then the quotient $M(n, \mathbb{C})_0^{k+1}/\sim$ classifies the Fuchsian systems.

Under the identification of $\mathfrak{g}$ with its dual space by the symmetric bilinear form $\langle X, Y \rangle = \operatorname{trace} XY$ for $(X, Y) \in \mathfrak{g}^2$, the dual map of $\operatorname{ad}_A : X \mapsto [A, X]$ of $\mathfrak{g}$ equals $-\operatorname{ad}_A$ and therefore $\operatorname{ad}_A(\mathfrak{g})$ is the orthogonal compliment of $\ker \operatorname{ad}_A$ under the bilinear form:

(4.4) $$\operatorname{ad}_A(\mathfrak{g}) := \{[A, X]\,;\, X \in \mathfrak{g}\} = \{X \in \mathfrak{g}\,;\, \operatorname{trace} XY = 0 \quad (\forall Y \in Z_{\mathfrak{g}}(A))\}.$$

For $\mathbf{A} = (A_0, \ldots, A_k) \in \mathfrak{g}^{k+1}$ we put

$$\begin{array}{ccc} \pi_{\mathbf{A}} : & G^{k+1} & \to & \mathfrak{g} \\ & \cup & & \cup \\ & (g_0, \ldots, g_k) & \mapsto & \sum_{j=0}^{k} g_j A_j g_j^{-1} \end{array}$$

The image of $\pi_{\mathbf{A}}$ is a homogeneous space G^{k+1}/H of G^{k+1} with

$$H := \{(g_0, \ldots, g_k) \in G^{k+1}\,;\, \sum_{j=0}^{k} g_j A_j g_j^{-1} = \sum_{j=0}^{k} A_j\}$$

and the tangent space of the image at $A_0 + \cdots + A_k$ is isomorphic to

$$\sum_{j=0}^{k} \operatorname{ad}_{A_j}(\mathfrak{g}) = \{X \in \mathfrak{g}\,;\, \operatorname{trace} XY = 0 \quad \big(\forall Y \in Z_{\mathfrak{g}}(\mathbf{A}) = \bigcap_{j=0}^{k} Z_{\mathfrak{g}}(A_j)\big)\}.$$

Hence the dimension of the manifold G^{k+1}/H equals $n^2 - \dim Z_{\mathfrak{g}}(\mathbf{A})$ and therefore the dimension of H equals $kn^2 + \dim Z_{\mathfrak{g}}(\mathbf{A})$. Since the manifold

$$(4.5) \qquad \widetilde{O}_{\mathbf{A}} := \{(C_0, \ldots, C_k) \in \mathfrak{g}^{k+1} \,;\, C_j \sim A_j \text{ and } \sum_{j=0}^{k} C_j = \sum_{j=0}^{k} A_j\}$$

is naturally isomorphic to $H/Z_G(A_0) \times \cdots \times Z_G(A_k)$ with $Z_G(A_j) := \{g \in G \,;\, gA_jg^{-1} = A_j\}$, the dimension of $\widetilde{O}_{\mathbf{A}}$ equals $kn^2 + \dim Z_{\mathfrak{g}}(\mathbf{A}) - \sum_{j=0}^{k} \dim Z_{\mathfrak{g}}(\mathbf{A}_j)$.

Note that the dimension of the manifold

$$(4.6) \qquad O_{\mathbf{A}} := \bigcup_{g \in G} (gA_0g^{-1}, \ldots, gA_kg^{-1}) \subset \mathfrak{g}^{k+1}$$

equals $n^2 - \dim Z_{\mathfrak{g}}(\mathbf{A})$.

Suppose $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$. Then $\widetilde{O}_{\mathbf{A}} \supset O_{\mathbf{A}}$ and we have the followings.

Proposition 4.1. $\dim \widetilde{O}_{\mathbf{A}} - \dim O_{\mathbf{A}} = (k-1)n^2 - \sum_{j=0}^{k} \dim Z_{\mathfrak{g}}(A_j) + 2\dim Z_{\mathfrak{g}}(\mathbf{A})$.

Definition 4.2. The index of rigidity $\mathrm{idx}\,\mathbf{A}$ of $\mathbf{A}$ is introduced by [Kz]:

$$\mathrm{idx}\,\mathbf{A} := \sum_{j=0}^{k} \dim Z_{\mathfrak{g}}(A_j) - (k-1)n^2 = 2n^2 - \sum_{j=0}^{k} \dim\{gA_jg^{-1} \,;\, g \in G\},$$

$$\mathrm{Pidx}\,\mathbf{A} := \dim Z_{\mathfrak{g}}(\mathbf{A}) + \tfrac{1}{2}(k-1)n^2 - \tfrac{1}{2}\sum_{j=0}^{k} \dim Z_{\mathfrak{g}}(A_j) = \dim Z_{\mathfrak{g}}(\mathbf{A}) - \tfrac{1}{2}\mathrm{idx}\,\mathbf{m}.$$

Note that $\mathrm{Pidx}\,\mathbf{A} \geq 0$ and $\dim\{gA_jg^{-1} \,;\, g \in G\}$ are even.

Corollary 4.3. $\dim \widetilde{O}_{\mathbf{A}} - \dim O_{\mathbf{A}}$ *and* $\mathrm{idx}\,\mathbf{A}$ *are even and* $\mathrm{idx}\,\mathbf{A} \leq 2\dim Z_{\mathfrak{g}}(\mathbf{A})$.

Note that if $\mathbf{A}$ is irreducible, $\dim Z_{\mathfrak{g}}(\mathbf{A}) = 1$.
The following result by Katz is fundamental.

Theorem 4.4 ([Kz]). *Suppose* $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ *is irreducible. Then* $\mathrm{idx}\,\mathbf{A} = 2$ *if and only if* $\mathbf{A}$ *is rigid, namely,* $\widetilde{O}_{\mathbf{A}} = O_{\mathbf{A}}$.

§ 5. Middle convolutions

We will review the additive middle convolutions in the way interpreted by Dettweiler and Reiter [DR, DR2].

Definition 5.1 ([DR]).   Fix $\mathbf{A} = (A_0, \ldots, A_k) \in M(n, \mathbb{C})_0^{k+1}$. The *addition* $M_{\mu'}(\mathbf{A}) \in \mathfrak{g}^{k+1}$ of $\mathbf{A}$ with respect to $\mu' = (\mu'_1, \ldots, \mu'_k) \in \mathbb{C}^k$ is $(A_0 - \mu'_1 - \cdots - \mu'_k, A_1 + \mu'_1, \ldots, A_k + \mu'_k)$. The *convolution* $(G_0, \ldots, G_k) \in M(kn, \mathbb{C})_0^{k+1}$ of $\mathbf{A}$ with respect to $\lambda \in \mathbb{C}$ is defined by

$$(5.1) \qquad G_j = \Big(\delta_{p,j}(A_q + \delta_{p,q}\lambda) \Big)_{\substack{1 \le p \le k \\ 1 \le q \le k}} \qquad (j = 1, \ldots, k)$$

$$= \begin{array}{c} {}^{\overset{j}{\smile}} \\ j\,\big) \left(\begin{matrix} A_1 & A_2 & \cdots & A_j + \lambda & A_{j+1} & \cdots & A_k \end{matrix} \right), \end{array}$$

$$(5.2) \qquad G_0 = -(G_1 + \cdots + G_k).$$

Put $\mathcal{K} = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} ; v_j \in \ker A_j \quad (j = 1, \ldots, k) \right\}$ and $\mathcal{L} = \ker G_0$. Then $\mathcal{K}$ and $\mathcal{L}$ are G_j-invariant subspaces of $\mathbb{C}^{kn}$ and we define $\bar{G}_j := G_j|_{\mathbb{C}^{kn}/(\mathcal{K}+\mathcal{L})} \in \mathrm{End}(\mathbb{C}^{n'}) \simeq M(n', \mathbb{C})$ with $n' = kn - \dim(\mathcal{K} + \mathcal{L})$. The *middle convolution* $mc_\lambda(\mathbf{A}) \in M(n', \mathbb{C})_0^{k+1}$ of $\mathbf{A}$ with respect to λ is defined by $mc_\lambda(\mathbf{A}) := (\bar{G}_0, \ldots, \bar{G}_k)$. Note that $\mathcal{K} \cap \mathcal{L} = \{0\}$ if $\lambda \ne 0$.

The conjugacy classes of $\bar{G}_j$ in the above definition are given by [DR2], which is simply described using the normal form in §3 (cf. Proposition 3.3):

Theorem 5.2 ([DR, DR2]).   *Fix* $\mathbf{A} = (A_0, A_1, \ldots, A_k) \in M(n, \mathbb{C})_0^{k+1}$ *and* $\mu = (\mu_0, \ldots, \mu_k) \in \mathbb{C}^{k+1}$ *and put*

$$(5.3) \qquad \begin{aligned} mc_\mu &:= M_{-\mu'} \circ mc_{|\mu|} \circ M_{-\mu'}, \\ \mu' &:= (\mu_1, \ldots, \mu_k), \quad |\mu| := \mu_0 + \mu_1 + \cdots + \mu_k. \end{aligned}$$

Assume the following conditions (which are satisfied if $n > 1$ *and* $\mathbf{A}$ *is irreducible):*

$$(5.4) \qquad \bigcap_{\substack{1 \le j \le k \\ j \ne i}} \ker(A_j - \mu_j) \cap \ker(A_0 - \tau) = \{0\} \qquad (i = 1, \ldots, k, \ \forall \tau \in \mathbb{C})$$

$$(5.5) \qquad \sum_{\substack{1 \le j \le k \\ j \ne i}} \mathrm{Im}(A_j - \mu_j) + \mathrm{Im}(A_0 - \tau) = \mathbb{C}^n \qquad (i = 1, \ldots, k, \ \forall \tau \in \mathbb{C})$$

Then $\mathbf{A}' := mc_\mu(\mathbf{A})$ *satisfies (5.4) and (5.5) with replacing* $-\mu_j$ *by* $+\mu_j$ *and*

$$(5.6) \qquad\qquad\qquad \mathrm{idx}\,\mathbf{A}' = \mathrm{idx}\,\mathbf{A}.$$

If $\mathbf{A}$ *is irreducible, so is* $\mathbf{A}'$. *If* $\mu = 0$, *then* $\mathbf{A}' \sim \mathbf{A}$. *If* $\mathbf{A} \sim \mathbf{B}$, *then* $mc_\mu(\mathbf{A}) \sim mc_\mu(\mathbf{B})$.

Moreover for any $\tau_0 \in \mathbb{C}$ we have

$$(5.7) \qquad mc_{(-\tau_0, -\mu')} \circ mc_{(\mu_0, \mu')}(\mathbf{A}) \sim M_{2\mu'} \circ mc_{(2\mu_0 - \tau_0 - |\mu|, \mu')}(\mathbf{A}),$$

$$(5.8) \qquad mc_{-\mu} \circ mc_\mu(\mathbf{A}) \sim \mathbf{A}.$$

Choose $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ and $\lambda_{j,\nu} \in \mathbb{C}$ so that

$$(5.9) \qquad A_j \sim L(\mathbf{m}_j; \lambda_j) \text{ with } \mathbf{m}_j := (m_{j,1}, \ldots, m_{j,n_j}) \text{ and } \lambda_j := (\lambda_{j,1}, \ldots, \lambda_{j,n_j}).$$

Denoting $I_j := \{\nu \,;\, \lambda_{j,\nu} = \mu_j\}$ and putting

$$(5.10) \qquad \ell_j = \begin{cases} \min\{p \in I_j \,;\, m_p = \max\{m_\nu \,;\, \nu \in I_j\}\} & (I_j \neq \emptyset) \\ n_j + 1 & (I_j = \emptyset) \end{cases},$$

$$(5.11) \qquad d_\ell(\mathbf{m}) := m_{0,\ell_0} + m_{1,\ell_1} + \cdots + m_{k,\ell_k} - (k-1)n,$$

$$(5.12) \qquad m'_{j,\nu} := m_{j,\nu} - \delta_{\ell_j,\nu} \cdot d_\ell(\mathbf{m}),$$

$$(5.13) \qquad \lambda'_{j,\nu} := \begin{cases} \lambda_{j,\nu} + |\mu| - 2\mu_j & (\nu \neq \ell_j) \\ -\mu_j & (\nu = \ell_j) \end{cases},$$

we have $A'_j \sim L(\mathbf{m}'_j; \lambda'_j)$ $(j = 0, \ldots, k)$ if $|\mu| \neq 0$.

Example 5.3. Suppose λ_i, μ_j and τ_ℓ are generic. Starting from $\mathbf{A} = (-\lambda_1 - \lambda_2, \lambda_1, \lambda_2) \in M(1, \mathbb{C})_0^3$, we have the following list of eigenvalues of the matrices under the application of middle convolutions to $\mathbf{A}$ (cf. hypergeometric family in Example 6.1):

$$1, 1, 1 \ (H_1) \ \longleftrightarrow \ 11, 11, 11 \ (H_2 : {}_2F_1) \ \longleftrightarrow \ 111, 111, 12 \ (H_3 : {}_3F_2)$$

$$\left\{ -\lambda_1 - \lambda_2 \quad \lambda_1 \quad \lambda_2 \right\} \xrightarrow{mc_{\mu_0,\mu_1,\mu_2}}$$

$$\begin{cases} -\lambda_1 - \lambda_2 - \mu_0 + \mu_1 + \mu_2 & \lambda_1 + \mu_0 - \mu_1 + \mu_2 & \lambda_2 + \mu_0 + \mu_1 - \mu_2 \\ -\mu_0 & -\mu_1 & -\mu_2 \end{cases} \xrightarrow{mc_{\tau_0,\tau_1,-\mu_2}}$$

$$\begin{cases} -\lambda_1 - \lambda_2 - \mu_0 + \mu_1 - \tau_0 + \tau_1 & \lambda_1 + \mu_0 - \mu_1 + \tau_0 - \tau_1 & \lambda_2 + \mu_0 + \mu_1 + \tau_0 + \tau_1 \\ -\mu_0 - \tau_0 + \tau_1 - \mu_2 & -\mu_1 + \tau_0 - \tau_1 - \mu_2 & \mu_2 \\ -\tau_0 & -\tau_1 & \mu_2 \end{cases}$$

Here the eigenvalues are vertically written. Note that the matrices are semisimple if the parameters are generic. Denoting $\mathbf{A}' = (A'_0, A'_1, A'_2) = mc_{\mu_0,\mu_1,\mu_2}(\mathbf{A})$ and $\mathbf{A}'' = (A''_0, A''_1, A''_2) = mc_{\tau_0,\tau_1,-\mu_2}(\mathbf{A}')$, we have

$$(5.14) \qquad \begin{aligned} A'_0 &\sim L(1, 1; -\lambda_1 - \lambda_2 - \mu_0 + \mu_1 + \mu_2, -\mu_0), \\ A'_j &\sim L(1, 1; \lambda_j + \mu_0 + \mu_1 + \mu_2 - 2\mu_j, -\mu_j) \quad (j = 1, 2), \end{aligned}$$

$$(5.15) \qquad A''_2 \sim L(1, 2; \lambda_2 + \mu_0 + \mu_1 + \tau_0 + \tau_1, \mu_2), \text{ etc.}$$

Then Theorem 5.2 implies that the irreducible rigid tuple $\mathbf{A} = (A'_0, A'_1, A'_2) \in M(2, \mathbb{C})_0^3$ satisfying (5.14) exists if and only if $\lambda_1 \neq \mu_1$, $\lambda_2 \neq \mu_2$, $\lambda_1 + \lambda_2 + \mu_0 \neq 0$ and $\mu_0 + \mu_1 + \mu_2 \neq 0$, which corresponds to $\mathcal{K} = \mathcal{L} = \{0\}$ and $|\mu| \neq 0$. Moreover all the irreducible rigid tuples $\mathbf{A} \in M(2, \mathbb{C})_0^3$ are obtained in this way.

Definition 5.4.   i) Under the notation in Theorem 5.2 the tuple of partitions $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ is called the *spectral type* of $\mathbf{A}$ and denoted by spt $\mathbf{A}$.

ii) Let $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ and $\lambda_{j,\nu}$ be generic complex numbers satisfying

$$(5.16) \qquad \sum_{j=0}^{k} \sum_{\nu=1}^{n_j} m_{j,\nu} \lambda_{j,\nu} = 0.$$

Then $\mathbf{m}$ is *realizable* if there exists a tuple $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ satisfying (5.9). Moreover $\mathbf{m}$ is *irreducibly realizable* if there exists an irreducible tuple $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ satisfying (5.9). An irreducibly realizable tuple $\mathbf{m}$ is *rigid* if $\operatorname{idx} \mathbf{m} := \operatorname{idx}(\mathbf{m}, \mathbf{m}) = 2$, namely, the corresponding irreducible tuple $\mathbf{A}$ is rigid.

For $\ell = (\ell_0, \dots, \ell_k) \in \mathbb{Z}_{\geq 1}^{k+1}$ we define $\partial_\ell(\mathbf{m}) = \mathbf{m}'$ by (5.11) and (5.12) and denote the unique monotone element in $S'_\infty \mathbf{m}$ by $s(\mathbf{m})$. Moreover we define

$$(5.17) \qquad \partial(\mathbf{m}) := \partial_{(1,1,\dots)}(\mathbf{m}) = \partial_{\mathbf{1}}(\mathbf{m}),$$

$$(5.18) \qquad \partial_{max}(\mathbf{m}) := \partial_\ell(\mathbf{m}) \quad \text{with} \quad \ell_j = \min\{\nu \,;\, m_{j,\nu} = \max\{m_{j,1}, m_{j,2}, \dots\}\}$$

and $\mathbf{m}$ is *basic* if $\mathbf{m}$ is indivisible and $\sum_{j=0}^{k} \max\{m_{j,1}, m_{j,2}, \dots\} \leq (k-1)\operatorname{ord} \mathbf{m}$ which means $\operatorname{ord} \partial_{max}(\mathbf{m}) \geq \operatorname{ord} \mathbf{m}$. Under the notation (5.18) and (5.9) we put

$$(5.19) \qquad mc_{max}(\mathbf{A}) := mc_{\lambda_{\ell_0}, \lambda_{\ell_1}, \dots}(\mathbf{A}).$$

Remark 4.   i) Suppose $\mathbf{m} \in \mathcal{P}_{k+1}$ is irreducibly realizable. Then $mc_\ell(\mathbf{m}) \in \mathcal{P}_{k+1}$ if $\#\{(j, \nu) \,;\, m_{j,\nu} > 0 \text{ and } \nu \neq \ell_j\} > 1$. Moreover if $\mathbf{A}$ is a generic element of $M(n, \mathbb{C})_0^{k+1}$ satisfying spt $\mathbf{A} = \mathbf{m}$ and moreover $\mu = (\mu_0, \dots, \mu_k) \in \mathbb{C}^{k+1}$ is generic under the condition that $\mu_j = \lambda_{j,\ell_j}$ for any ℓ_j satisfying $m_{j,\ell_j} > 0$, then $mc_\mu(\mathbf{A})$ is a generic element of $M(n, \mathbb{C})_0^{k+1}$ with the spectral type $\partial_\ell(\mathbf{m})$.

ii) Let $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ with a spectral type $\mathbf{m}$. Let $\ell = (\ell_0, \ell_1, \dots)$ with $\ell_j \in \mathbb{Z}_{>0}$ and $\ell_\nu = 1$ for $\nu > k$. Define $\mathbf{1}_\ell = (m'_{j,\nu}) \in \mathcal{P}^{(1)}$ by $m'_{j,\nu} = \delta_{\ell_j, \nu}$. Then

$$(5.20) \qquad \operatorname{idx} \mathbf{A} = \operatorname{idx} \mathbf{m} := \operatorname{idx}(\mathbf{m}, \mathbf{m}),$$

$$(5.21) \qquad d_\ell(\mathbf{m}) = \operatorname{idx}(\mathbf{m}, \mathbf{1}_\ell).$$

Theorem 5.5.   i) ([Kz], [DR]) *Let* $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ *and put* $\mathbf{m} = \operatorname{spt} \mathbf{A}$. *Then* $\mathbf{A}$ *is irreducible and rigid if and only if* $n = 1$ *or* $mc_{max}(\mathbf{A})$ *is irreducible and rigid and*

$\operatorname{ord} \partial_{max}(\mathbf{m}) < n$. *Hence if* $\mathbf{A}$ *is irreducible and rigid,* $\mathbf{A}$ *is constructed from an element of* $M(1, \mathbb{C})_0^{k+1}$ *by a finite iteration of suitable middle convolutions* mc_μ *in Theorem 5.2.*

ii) ([Ko4], [CB]) *An indivisible tuple* $\mathbf{m} \in \mathcal{P}$ *is irreducibly realizable if and only if one of the following three conditions holds.*

$$(5.22) \qquad \operatorname{ord} \mathbf{m} = 1$$

$$(5.23) \qquad \mathbf{m} \text{ *is basic, namely,* } \mathbf{m} \text{ *is indivisible and* } \operatorname{ord} \partial_{max}(\mathbf{m}) \geq \operatorname{ord} \mathbf{m}$$

$$(5.24) \qquad \partial_{max}(\mathbf{m}) \in \mathcal{P} \text{ *is well-defined and irreducibly realizable.*}$$

Note that $\partial_\ell(\mathbf{m}) \in \mathcal{P}$ *is well-defined if and only if* $m_{j,\ell_j} \geq d_\ell(\mathbf{m})$ *for* $j = 0, 1, \ldots$.

iii) (Theorem 10.2 in §10) *Suppose a tuple* $\mathbf{m} \in \mathcal{P}$ *is not indivisible. Put* $\mathbf{m} = d\overline{\mathbf{m}}$ *with an integer* $d > 1$ *and an indivisible tuple* $\overline{\mathbf{m}} \in \mathcal{P}$. *Then* $\mathbf{m}$ *is irreducibly realizable if and only if* $\overline{\mathbf{m}}$ *is irreducibly realizable and* $\operatorname{idx} \mathbf{m} < 0$.

Example 5.6.   Successive applications of $s \circ \partial$ to monotone elements of $\mathcal{P}$:

$\underline{4}11, \underline{4}11, \underline{4}2, \underline{3}3 \overset{15-2\cdot6=3}{\longrightarrow} 111, 111, 21, 3 = \underline{1}11, \underline{1}11, \underline{2}1 \overset{4-3=1}{\longrightarrow} \underline{1}1, \underline{1}1, \underline{1}1 \overset{3-2=1}{\longrightarrow} 1, 1, 1$ (rigid)

$\underline{2}11, \underline{2}11, \underline{1}111 \overset{5-4=1}{\longrightarrow} \underline{1}11, \underline{1}11, \underline{1}11 \overset{3-3=0}{\longrightarrow} 111, 111, 111$ (realizable, not rigid)

$\underline{2}11, \underline{2}11, \underline{2}11, \underline{3}1 \overset{9-8=1}{\longrightarrow} \underline{1}11, \underline{1}11, \underline{1}11, \underline{2}1 \overset{5-6=-1}{\longrightarrow} 211, 211, 211, 31$ (realizable, not rigid)

$\underline{2}2, \underline{2}2, \underline{1}111 \overset{5-4=1}{\longrightarrow} \underline{2}1, \underline{2}1, \underline{1}11 \overset{5-3=2}{\longrightarrow} \times$ (not realizable)

The numbers on the above arrows are $d_{(1,1,\ldots)}(\mathbf{m}) = m_{0,1} + \cdots + m_{k,1} - (k-1) \cdot \operatorname{ord} \mathbf{m}$.

§6.   Rigid tuples

Let $\mathcal{R}_k^{(n)}$ denote the totality of rigid tuples in $\mathcal{P}_k^{(n)}$ (cf. Definition 5.4). Put $\mathcal{R}_k = \bigcup_{n=1}^{\infty} \mathcal{R}_k^{(n)}$, $\mathcal{R}^{(n)} = \bigcup_{k=1}^{\infty} \mathcal{R}_k^{(n)}$ and $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_k$. We identify elements of $\mathcal{R}$ if they are in the same S_∞-orbit (cf. Definition 2.2) and then $\bar{\mathcal{R}}$ denotes the set of elements of $\mathcal{R}$ under this identification. Similarly we denote $\bar{\mathcal{R}}_k$ and $\bar{\mathcal{R}}^{(n)}$ for $\mathcal{R}_k$ and $\mathcal{R}^{(n)}$, respectively, with this identification.

Example 6.1.   i) The list of $\mathbf{m} \in \bar{\mathcal{R}}^{(n)}$ with $\mathbf{m}_0 = 1^n$ is given by Simpson [Si]:

$1^n, 1^n, n-11$ (H_n: hypergeometric family) $1^{2m}, mm, mm-11$ (EO_{2m}: even family)

$1^{2m+1}, m+1m, mm1$ (EO_{2m+1}: odd family) $111111, 222, 42$ (X_6: extra case)

ii) We show examples and the numbers of elements of $\bar{\mathcal{R}}^{(n)}$.

Table $\bar{\mathcal{R}}^{(n)}$ $(2 \leq n \leq 7)$

2:11,11,11	3:111,111,21	3:21,21,21,21
4:1111,1111,31	4:1111,211,22	4:211,211,211
4:211,22,31,31	4:22,22,22,31	4:31,31,31,31,31
5:11111,11111,41	5:11111,221,32	5:2111,2111,32

```
5:2111,221,311            5:221,221,221              5:221,221,41,41
5:221,32,32,41            5:311,311,32,41            5:32,32,32,32
5:32,32,41,41,41          5:41,41,41,41,41,41        6:111111,111111,51
6:111111,222,42           6:111111,321,33            6:21111,2211,42
6:21111,222,33            6:21111,222,411            6:21111,3111,33
6:2211,2211,33            6:2211,2211,411            6:2211,222,51,51
6:2211,321,321            6:2211,33,42,51            6:222,222,321
6:222,3111,321            6:222,33,33,51             6:222,33,411,51
6:3111,3111,321           6:3111,33,411,51           6:321,321,42,51
6:321,33,51,51,51         6:321,42,42,42             6:33,33,33,42
6:33,33,411,42            6:33,411,411,42            6:33,42,42,51,51
6:411,411,411,42          6:411,42,42,51,51          6:51,51,51,51,51,51,51
7:1111111,1111111,61      7:1111111,331,43           7:211111,2221,52
7:211111,322,43           7:22111,22111,52           7:22111,2221,511
7:22111,3211,43           7:22111,331,421            7:2221,2221,43
7:2221,2221,61,61         7:2221,31111,43            7:2221,322,421
7:2221,331,331            7:2221,331,4111            7:2221,43,43,61
7:31111,31111,43          7:31111,322,421            7:31111,331,4111
7:3211,3211,421           7:3211,322,331             7:3211,322,4111
7:3211,331,52,61          7:322,322,322              7:322,322,52,61
7:322,331,511,61          7:322,421,43,61            7:322,43,52,52
7:331,331,43,61           7:331,331,61,61,61         7:331,43,511,52
7:4111,4111,43,61         7:4111,43,511,52           7:421,421,421,61
7:421,421,52,52           7:421,43,43,52             7:421,43,511,511
7:421,43,52,61,61         7:43,43,43,43              7:43,43,43,61,61
7:43,43,61,61,61,61       7:43,52,52,52,61           7:511,511,52,52,61
7:52,52,52,61,61,61       7:61,61,61,61,61,61,61,61
```

$$\mathcal{R}_k^{(n)}: \text{rigid } k\text{-tuples of partitions with order } n$$

ord	$\#\bar{\mathcal{R}}_3^{(n)}$	$\#\bar{\mathcal{R}}^{(n)}$	ord	$\#\bar{\mathcal{R}}_3^{(n)}$	$\#\bar{\mathcal{R}}^{(n)}$	ord	$\#\bar{\mathcal{R}}_3^{(n)}$	$\#\bar{\mathcal{R}}^{(n)}$
2	1	1	15	1481	2841	28	114600	190465
3	1	2	16	2388	4644	29	143075	230110
4	3	6	17	3276	6128	30	190766	310804
5	5	11	18	5186	9790	31	235543	371773
6	13	28	19	6954	12595	32	309156	493620
7	20	44	20	10517	19269	33	378063	588359
8	45	96	21	14040	24748	34	487081	763126
9	74	157	22	20210	36078	35	591733	903597
10	142	306	23	26432	45391	36	756752	1170966
11	212	441	24	37815	65814	37	907150	1365027
12	421	857	25	48103	80690	38	1143180	1734857
13	588	1177	26	66409	112636	39	1365511	2031018
14	1004	2032	27	84644	139350	40	1704287	2554015

§7.   A Kac-Moody root system

We will review the relation between a Kac-Moody root system and the middle convolution which is clarified by Clawley-Boevey [CB].

Let $\mathfrak{h}$ be an infinite dimensional real vector space with the set of basis Π, where

$$(7.1) \qquad \Pi = \{\alpha_0, \alpha_{j,\nu}\,;\, j = 0, 1, 2, \ldots,\ \nu = 1, 2, \ldots\}.$$

Put

$$(7.2) \qquad Q := \sum_{\alpha \in \Pi} \mathbb{Z}\alpha \ \supset\ Q_+ := \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0}\alpha.$$

We define an indefinite inner product on $\mathfrak{h}$ by

$$(7.3) \qquad \begin{aligned} (\alpha|\alpha) &= 2 \qquad (\alpha \in \Pi), \\ (\alpha_0|\alpha_{j,\nu}) &= -\delta_{\nu,1} \quad (j = 0, 1, \ldots,\ \nu = 1, 2, \ldots), \\ (\alpha_{i,\mu}|\alpha_{j,\nu}) &= \begin{cases} 0 & (i \neq j \ \text{ or } \ |\mu - \nu| > 1) \\ -1 & (i = j \ \text{ and } \ |\mu - \nu| = 1) \end{cases}. \end{aligned}$$

Let $\mathfrak{g}_\infty$ denote the Kac-Moody Lie algebra associated to the Cartan matrix

$$(7.4) \qquad A := \left(\frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} \right)_{i,j \in I},$$

$$(7.5) \qquad I := \{0,\, (j, \nu)\,;\, j = 0, 1, \ldots,\ \nu = 1, 2, \ldots\}.$$

We introduce linearly independent vectors e_0 and $e_{j,\nu}$ $(j = 0, 1, \ldots,\ \nu = 1, 2, \ldots)$ with

$$(7.6) \qquad (e_0|e_0) = 2,\ (e_0|e_{j,\nu}) = -\delta_{\nu,1} \ \text{ and } \ (e_{j,\nu}|e_{j',\nu'}) = \delta_{j,j'}\delta_{\nu,\nu'}.$$

For a sufficiently large positive integer k let $\mathfrak{h}^k$ be a subspace of $\mathfrak{h}$ spanned by $\{\alpha_0,\, \alpha_{j,\nu}\,;\, j = 0, 1, \ldots, k,\ \nu = 0, 1, \ldots\}$. Putting $e_0^k = e_0 + e_{0,1} + \cdots + e_{k,1}$, we have $(e_0^k|e_0^k) = 2 + (k+1) - 2(k+1) = 1 - k$. For a sufficiently large k we have an orthogonal basis $\{e_0^k,\, e_{j,\nu}\,;\, j = 0, \ldots, k,\ \nu = 1, 2, \ldots\}$ with

$$(7.7) \qquad \begin{aligned} (e_0^k|e_0^k) &= 1 - k, \quad (e_{j,\nu}|e_{j',\nu'}) = \delta_{j,j'}\delta_{\nu,\nu'}, \\ (e_0^k|e_{j,\nu}) &= 0 \qquad (j = 0, \ldots, k,\ \nu = 1, 2, \ldots) \end{aligned}$$

and therefore we may put

$$(7.8) \qquad \begin{aligned} \alpha_0 &= e_0 = e_0^k - e_{0,1} - e_{1,1} - \cdots - e_{k,1}, \\ \alpha_{j,\nu} &= e_{j,\nu} - e_{j,\nu+1} \qquad (j = 0, \ldots, k,\ \nu = 1, 2, \ldots). \end{aligned}$$

The element

$$(7.9) \qquad \alpha_0(\ell_0,\ldots,\ell_k) := e_0^k - \sum_{j=0}^{k} \sum_{\nu=1}^{\ell_j+1} \frac{e_{j,\nu}}{\ell_j+1}$$

is in the space spanned by α_0 and $\alpha_{j,\nu}$ $(j=0,\ldots,k,\ \nu=1,\ldots,\ell_j)$ and it is orthogonal to any $\alpha_{j,\nu}$ for $\nu=1,\ldots,\ell_j$ and $j=0,\ldots,k$.

Remark 5.   We may assume $\ell_0 \geq \ell_1 \geq \cdots \geq \ell_k \geq 1$. It is easy to have

$$\bigl(\alpha_0(\ell_0,\ldots,\ell_k)|\alpha_0(\ell_0,\ldots,\ell_k)\bigr) = 1 - k + \sum_{j=0}^{k} \frac{1}{\ell_j+1}$$

$$\begin{cases} > 0 & (k=1) \\ > 0 & (k=2 : \ell_1 = \ell_2 = 1 \text{ or } (\ell_0,\ell_1,\ell_2) = (2,2,1),\ (3,2,1) \text{ or } (4,2,1)) \\ = 0 & (k=2 : (\ell_0,\ell_1,\ell_2) = (2,2,2),\ (3,3,1) \text{ or } (5,2,1)) \\ < 0 & (k=2 : \ell_1 \geq 2 \text{ and } \ell_0 + 2\ell_1 + 3\ell_2 > 12) \\ = 0 & (k=3 : \ell_0 = \ell_1 = \ell_2 = \ell_3 = 1) \\ < 0 & (k=3 : \ell_0 > 1) \\ < 0 & (k \geq 4) \end{cases}$$

The Weyl group W_∞ of $\mathfrak{g}_\infty$ is the subgroup of $O(\mathfrak{h}) \subset GL(\mathfrak{h})$ generated by the simple reflections

$$(7.10) \qquad r_i(x) := x - 2\frac{(x|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i = x - (x|\alpha_i)\alpha_i \qquad (x \in \mathfrak{h},\ i \in I).$$

The subgroup of W_∞ generated by r_i for $i \in I \setminus \{0\}$ is denoted by W'_∞. Putting $\sigma(\alpha_0) = \alpha_0$ and $\sigma(\alpha_{j,\nu}) = \alpha_{\sigma(j),\nu}$ for $\sigma \in \mathfrak{S}_\infty$, we define a subgroup of $O(\mathfrak{h})$:

$$(7.11) \qquad \widetilde{W}_\infty := \mathfrak{S}_\infty \ltimes W_\infty.$$

For a tuple of partitions $\mathbf{m} = \bigl(m_{j,\nu}\bigr)_{j \geq 0,\ \nu \geq 1} \in \mathcal{P}_{k+1}^{(n)}$ of n, we define

$$n_{j,\nu} := m_{j,\nu+1} + m_{j,\nu+2} + \cdots,$$
$$(7.12) \qquad \alpha_{\mathbf{m}} := n\alpha_0 + \sum_{j=0}^{\infty} \sum_{\nu=1}^{\infty} n_{j,\nu}\alpha_{j,\nu} = ne_0^k - \sum_{j=0}^{\infty} \sum_{\nu=1}^{\infty} m_{j,\nu}e_{j,\nu} \in Q_+.$$

Proposition 7.1.   i) $\mathrm{idx}(\mathbf{m},\mathbf{m}') = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'})$.

ii)   *Given $i \in I$, we have $\alpha_{\mathbf{m}'} = r_i(\alpha_{\mathbf{m}})$ with*

$$\mathbf{m}' = \begin{cases} \partial\mathbf{m} & (i = 0), \\ (m_{0,1}\ldots,\overset{1}{\underset{\smile}{m_{j,1}}}\ldots \overset{\nu}{\underset{\smile}{m_{j,\nu+1}}}\overset{\nu+1}{\underset{\smile}{m_{j,\nu}}}\ldots,\ldots) & (i = (j,\nu)). \end{cases}$$

Moreover for $\ell = (\ell_0, \ell_1, \ldots) \in \mathbb{Z}_{>0}^\infty$ satisfying $\ell_\nu = 1$ for $\nu \gg 1$ we have

$$(7.13) \qquad \alpha_\ell := \alpha_{\mathbf{1}_\ell} = \alpha_0 + \sum_{j=0}^\infty \sum_{\nu=1}^{\ell_j-1} \alpha_{j,\nu} = \left(\prod_{j \geq 0} r_{j,\ell_j-1} \cdots r_{j,2} r_{j,1} \right)(\alpha_0),$$

$$(7.14) \qquad \alpha_{\partial_\ell(\mathbf{m})} = \alpha_{\mathbf{m}} - 2\frac{(\alpha_{\mathbf{m}}|\alpha_\ell)}{(\alpha_\ell|\alpha_\ell)}\alpha_\ell = \alpha_{\mathbf{m}} - (\alpha_m|\alpha_\ell)\alpha_\ell.$$

Proof. i) For a sufficiently large positive integer k we have

$$\mathrm{idx}(\mathbf{m}, \mathbf{m}') = \sum_{j=0}^\infty \sum_{\nu=1}^\infty m_{j,\nu} m'_{j,\nu} - (k-1)\, \mathrm{ord}\, \mathbf{m} \cdot \mathrm{ord}\, \mathbf{m}'$$

$$= \sum_{j=0}^k (n - n_{j,1})(n' - n'_{j,1}) + \sum_{j=0}^k \sum_{\nu=1}^\infty (n_{j,\nu} - n_{j,\nu+1})(n'_{j,\nu} - n'_{j,\nu+1}) - (k-1)nn'$$

$$= 2nn' + 2\sum_{j=0}^k n_{j,\nu} n'_{j,\nu} - \sum_{j=0}^k (nn'_{j,1} + n'n_{j,1}) - \sum_{j=0}^k \sum_{\nu=1}^\infty (n_{j,\nu} n'_{j,\nu+1} + n'_{j,\nu} n_{j,\nu+1})$$

$$= (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}).$$

The claim ii) easily follows from i). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6 ([Kc]).   The set Δ^{re} of *real roots* of the Kac-Moody Lie algebra equals $W_\infty \Pi$. Denoting $K := \{\beta \in Q_+ \,;\, \mathrm{supp}\,\beta$ is connected and $(\beta, \alpha) \leq 0 \quad (\forall \alpha \in \Pi)\}$, the set of *positive imaginary roots* Δ_+^{im} equals $W_\infty K$. The set Δ of roots equals $\Delta^{re} \cup \Delta^{im}$ by denoting $\Delta_-^{im} = -\Delta_+^{im}$ and $\Delta^{im} = \Delta_+^{im} \cup \Delta_-^{im}$. Put $\Delta_+ = \Delta \cap Q_+$, $\Delta_- = -\Delta_+$. Then $\Delta = \Delta_+ \cup \Delta_-$ and the root in Δ_+ is called *positive*. Here $\mathrm{supp}\,\beta = \{\alpha \in \Pi \,;\, n_\alpha \neq 0\}$ if $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$. A subset $L \subset \Pi$ is called *connected* if the decomposition $L_1 \cup L_2 = L$ with $L_1 \neq \emptyset$ and $L_2 \neq \emptyset$ always implies the existence of $v_j \in L_j$ for $j = 1$ and 2 satisfying $(v_1|v_2) \neq 0$.

Lemma 7.2.   i) *Let* $\alpha = n\alpha_0 + \sum_{j=0}^\infty \sum_{\nu=1}^\infty n_{j,\nu}\alpha_{j,\nu} \in \Delta_+$ *with* $\mathrm{supp}\,\alpha \supsetneq \{\alpha_0\}$. *Then*

$$(7.15) \qquad n \geq n_{j,1} \geq n_{j,2} \geq n_{j,3} \geq \cdots \qquad (j = 0, 1, \ldots),$$

$$(7.16) \qquad n \leq \sum n_{j,1} - \max\{n_{j,1}, n_{j,2}, \ldots\}.$$

ii) *Let* $\alpha = n\alpha_0 + \sum_{j=0}^\infty \sum_{\nu=1}^\infty n_{j,\nu}\alpha_{j,\nu} \in Q_+$. *Suppose* α *is indivisible, that is,* $\frac{1}{k}\alpha \notin Q$ *for* $k = 2, 3, \ldots$. *Then* α *corresponds to a basic tuple if and only if*

$$(7.17) \qquad \begin{cases} 2n_{j,\nu} \leq n_{j,\nu-1} + n_{j,\nu+1} \quad (n_{j,0} = n,\ j = 0, 1, \ldots,\ \nu = 1, 2, \ldots), \\ \quad 2n \leq n_{0,1} + n_{1,1} + n_{2,1} + \cdots. \end{cases}$$

Proof. The lemma is clear from the following for $\alpha = n\alpha_0 + \sum n_{j,\nu}\alpha_{j,\nu} \in \Delta_+$:

$$(7.18) \qquad r_{i,\mu}(\alpha) = n\alpha_0 + \sum \big(n_{j,\nu} - \delta_{i,j}\delta_{\mu,\nu}(2n_{j,\mu} - n_{j,\mu-1} - n_{j,\mu+1})\big)\alpha_{j,\nu} \in \Delta,$$

$$(7.19) \qquad r_0(\alpha) = \big(\sum n_{j,1} - n\big)\alpha_0 + \sum n_{j,\nu}\alpha_{j,\nu} \in \Delta.$$

For example, putting $n_{j,0} = n > 0$ and $r_{i,N} \cdots r_{i,\mu+1} r_{i,\mu}\alpha = n\alpha_0 + \sum n'_{j,\nu}\alpha_{j,\nu} \in \Delta_+$ for a sufficiently large N, we have $n'_{j,N} = n_{j,N} + n_{j,\mu-1} - n_{j,\mu} = n_{j,\mu-1} - n_{j,\mu} \geq 0$ for $\mu = 1, 2, \ldots$ and moreover (7.16) by $r_0\alpha \in \Delta_+$. □

Remark 7.   i) It follows from (7.14) that Katz's middle convolution corresponds to the reflection with respect to the root α_ℓ under the identification $\mathcal{P} \subset Q_+$ with (7.12).

Moreover there is a natural correspondence between the set of irreducibly realizable tuples of partitions and the set of positive roots α of $\mathfrak{g}_\infty$ with $\operatorname{supp}\alpha \ni \alpha_0$ such that α is indivisible or $(\alpha|\alpha) < 0$. Then the rigid tuple of partitions corresponds to the positive real root whose support contains α_0.

$\mathcal{P}$	Kac-Moody root system
$\mathbf{m}$	$\alpha_{\mathbf{m}}$ (cf. (7.12))
$\mathbf{m}$: rigid	$\alpha \in \Delta_+^{re}$: $\operatorname{supp}\alpha \ni \alpha_0$
$\mathbf{m}$: basic (cf. (5.23))	$\alpha \in Q_+$: $(\alpha\|\beta) \leq 0$ $(\forall\beta \in \Pi)$ indivisible and $\operatorname{supp}\alpha$ is connected
$\mathbf{m}$: irreducibly realizable	$\alpha \in \Delta_+$: $\operatorname{supp}\alpha \ni \alpha_0$ indivisible or $(\alpha\|\alpha) < 0$
$\operatorname{ord}\mathbf{m}$	n : $\alpha = n\alpha_0 + \sum_{j,\nu} n_{j,\nu}\alpha_{j,\nu}$
$\operatorname{idx}(\mathbf{m}, \mathbf{m}')$	$(\alpha_{\mathbf{m}}\|\alpha_{\mathbf{m}'})$
$\operatorname{Pidx}\mathbf{m} + \operatorname{Pidx}\mathbf{m}' = \operatorname{Pidx}(\mathbf{m} + \mathbf{m}')$	$(\alpha_{\mathbf{m}}\|\alpha_{\mathbf{m}'}) = -1$
$(\nu, \nu+1) \in G_j \subset S'_\infty$ (cf. (2.9))	$s_{j,\nu} \in W'_\infty$ (cf. (7.10))
∂ in (5.17)	r_0 in (7.19)
$H \simeq \mathfrak{S}_\infty$ (cf. (2.9))	$\mathfrak{S}_\infty$ in (7.11)
$\langle\partial, S_\infty\rangle$ (cf. Definition 2.2)	$\widetilde{W}_\infty$ in (7.11)

Here we define $\operatorname{Pidx}\mathbf{m} := 1 - \frac{1}{2}\operatorname{idx}\mathbf{m}$ as in Definition 4.2 and $\langle\partial, S_\infty\rangle$ denotes the group generated by ∂ and S_∞.

ii) For an irreducibly realizable tuple $\mathbf{m} \in \mathcal{P}$, $\partial(\mathbf{m})$ is well-defined if and only if $\operatorname{ord}\mathbf{m} > 1$ or $\sum_{j=0}^\infty m_{j,2} > 1$, which corresponds to the condition (5.4).

iii) Suppose a tuple $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ is basic. The subgroup of W_∞ generated by reflections with respect to α_ℓ (cf. (7.13)) satisfying $(\alpha_{\mathbf{m}}|\alpha_\ell) = 0$ and $\operatorname{supp}\alpha_\ell \subset \operatorname{supp}\alpha_{\mathbf{m}}$ is infinite if and only if $\operatorname{idx}\mathbf{m} = 0$.

Note that the condition $(\alpha_{\mathbf{m}}|\alpha_\ell) = 0$ means that the corresponding middle convolution of $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ with $\operatorname{spt}\mathbf{A} = \mathbf{m}$ keeps the partition type invariant.

Proposition 7.3.   *For irreducibly realizable* $\mathbf{m} \in \mathcal{P}$ *and* $\mathbf{m}' \in \mathcal{R}$ *satisfying*

$$(7.20) \qquad \operatorname{ord}\mathbf{m} > \operatorname{idx}(\mathbf{m}, \mathbf{m}') \cdot \operatorname{ord}\mathbf{m}',$$

we have

$$(7.21) \qquad \mathbf{m}'' := \mathbf{m} - \operatorname{idx}(\mathbf{m}, \mathbf{m}')\mathbf{m}' \text{ is irreducibly realizable,}$$

$$(7.22) \qquad \operatorname{idx}\mathbf{m}'' = \operatorname{idx}\mathbf{m}.$$

Here (7.20) *is always valid if* $\mathbf{m}$ *is not rigid.*

Proof.   The claim follows from the fact that $\alpha_{m''}$ is the reflection of the root $\alpha_{\mathbf{m}}$ with respect to the real root $\alpha_{\mathbf{m}'}$. $\qquad\square$

§8.   A classification of tuples of partitions

In this section we assume that a $(k+1)$-tuple $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{0 \le j \le k \\ 1 \le \nu \le n_j}}$ of partitions of a positive integer satisfies

$$(8.1) \qquad m_{j,1} \ge m_{j,2} \ge \cdots \ge m_{j,n_j} \ge 1 \quad\text{and}\quad n_j \ge 2 \quad (j = 0, 1, \ldots, k).$$

Note that

$$m_{j,1} + m_{j,2} + \cdots + m_{j,n_j} = \operatorname{ord}\mathbf{m} \ge 2 \quad (j = 0, 1, \ldots, k).$$

Proposition 8.1.   *Let* $\mathcal{K}$ *denote the totality of basic elements of* $\mathcal{P}$ *defined in* (5.23) *and for an even integer* p *put*

$$\mathcal{K}(p) := \{\mathbf{m} \in \mathcal{K}\,;\, \operatorname{idx}\mathbf{m} = p\}.$$

Then $\#\mathcal{K}(p) < \infty$. *In particular* $\mathcal{K}(p) = \emptyset$ *if* $p > 0$ *and*

$$(8.2) \qquad \mathcal{K} \cap \langle \partial, S'_\infty \rangle \mathbf{m} = \{\mathbf{m}\} \quad (\mathbf{m} \in \mathcal{K}),$$

$$(8.3) \qquad \bar{\mathcal{K}}(0) = \{11, 11, 11, 11 \quad 111, 111, 111 \quad 22, 1111, 1111 \quad 33, 222, 111111\}.$$

Here we use the notation in Remark 7 i), $\bar{\mathcal{K}}(p)$ *denotes the quotient of* $\mathcal{K}(p)$ *under the action of the group* S_∞ *and the element of* $\bar{\mathcal{K}}(p)$ *is denoted by its representative.*

Proof.   It follows from Remark 7 i) that $\mathcal{K}$ corresponds to the set of indivisible roots in K in Remark 6 and we have (8.2) because $K \cap W_\infty \alpha = \{\alpha\}$ for $\alpha \in K$.

Let $\mathbf{m} \in \mathcal{K} \cap \mathcal{P}_{k+1}$. We may assume that $\mathbf{m}$ is monotone and indivisible. Since

$$(8.4) \qquad \operatorname{idx}\mathbf{m} + \sum_{j=0}^{k} \sum_{\nu=2}^{n_j} (m_{j,1} - m_{j,\nu}) \cdot m_{j,\nu} = \left(\sum_{j=0}^{k} m_{j,1} - (k-1)\operatorname{ord}\mathbf{m}\right) \cdot \operatorname{ord}\mathbf{m},$$

the assumption $\mathbf{m} \in \mathcal{K}$ is equivalent to

$$(8.5) \qquad \sum_{j=0}^{k} \sum_{\nu=2}^{n_j} (m_{j,1} - m_{j,\nu}) \cdot m_{j,\nu} \leq -\operatorname{idx} \mathbf{m}.$$

Hence $\operatorname{idx} \mathbf{m} \leq 0$.

First suppose $\operatorname{idx} \mathbf{m} = 0$. Then $m_{j,1} = m_{j,2} = \cdots = m_{j,n_j}$ and the identity

$$(8.6) \qquad \sum_{j=0}^{k} \frac{m_{j,1}}{\operatorname{ord} \mathbf{m}} = k - 1 + \frac{\operatorname{idx} \mathbf{m}}{(\operatorname{ord} \mathbf{m})^2} + \sum_{j=0}^{k} \sum_{\nu=1}^{n_j} \frac{(m_{j,1} - m_{j,\nu}) m_{j,\nu}}{(\operatorname{ord} \mathbf{m})^2}$$

implies $\sum_{j=0}^{k} \frac{1}{n_j} = k - 1$. Since $\sum_{j=0}^{k} \frac{1}{n_j} \leq \frac{k+1}{2}$, we have $k \leq 3$. When $k = 3$, we have $n_0 = n_1 = n_2 = n_3 = 2$. When $k = 2$, $\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} = 1$ and we easily conclude that $\{n_0, n_1, n_2\}$ equals $\{3, 3, 3\}$ or $\{2, 4, 4\}$ or $\{2, 3, 6\}$, which means (8.3).

Since $\operatorname{idx} \mathbf{m} = 2(\operatorname{ord} \mathbf{m})^2 - \sum_{j=0}^{k} N_j$ with $N_j = (\operatorname{ord} \mathbf{m})^2 - \sum_{\nu=0}^{n_j} m_{j,\nu}^2 > 0$, there exist a finite number of $\mathbf{m} \in \mathcal{P}$ such that the numbers $\operatorname{ord} \mathbf{m}$ and $\operatorname{idx} \mathbf{A}$ are fixed because k is bounded. Therefore to prove the remaining part of the lemma we may assume

$$(8.7) \qquad \operatorname{idx} \mathbf{m} \leq -2 \quad \text{and} \quad \operatorname{ord} \mathbf{m} \geq -7 \operatorname{idx} \mathbf{m} + 7.$$

Then

$$(8.8) \qquad \operatorname{ord} \mathbf{m} \geq 21 \quad \text{and} \quad (\operatorname{ord} \mathbf{m})^2 > -147 \operatorname{idx} \mathbf{m}.$$

If $m_{j,1} > m_{j,n_j} > 0$, (8.5) implies $m_{j,1} - 1 \leq -\operatorname{idx} \mathbf{m} \leq \frac{1}{7} \operatorname{ord} \mathbf{m} - 1$ and therefore

$$(8.9) \qquad m_{j,1} \leq \frac{1}{7} \operatorname{ord} \mathbf{m},$$

$$(8.10) \qquad \sum_{\nu=1}^{n_j} m_{j,\nu}^2 \leq m_{j,1} \cdot \operatorname{ord} \mathbf{m} \leq \frac{1}{7} (\operatorname{ord} \mathbf{m})^2.$$

Hence $2 m_{j,1} \leq \operatorname{ord} \mathbf{m}$ for $j = 0, \ldots, k$,

$$\operatorname{idx} \mathbf{m} + (k-1) \cdot (\operatorname{ord} \mathbf{m})^2 = \sum_{j=0}^{k} \sum_{\nu=1}^{n_j} m_{j,\nu}^2 \leq \sum_{j=0}^{k} \frac{1}{2} (\operatorname{ord} \mathbf{m})^2 = \frac{k+1}{2} (\operatorname{ord} \mathbf{m})^2$$

and $\frac{k-3}{2} (\operatorname{ord} \mathbf{m})^2 \leq -\operatorname{idx} \mathbf{m} < \frac{1}{7} \operatorname{ord} \mathbf{m}$, which proves $k \leq 3$.

Suppose $k = 3$. Since $\mathbf{m} \neq 11, 11, 11, 11$, we have $m_{j,1} \leq \frac{1}{3} \operatorname{ord} \mathbf{m}$ with a suitable j,

$$\operatorname{idx} \mathbf{m} = \sum_{j=0}^{3} \sum_{\nu=1}^{n_j} m_{j,\nu}^2 - 2 \cdot (\operatorname{ord} \mathbf{m})^2 \leq \sum_{j=0}^{3} m_{j,1} \operatorname{ord} \mathbf{m} - 2(\operatorname{ord} \mathbf{m})^2$$

$$\leq (\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{3} - 2)(\operatorname{ord} \mathbf{m})^2 = -\tfrac{1}{6} (\operatorname{ord} \mathbf{m})^2$$

and $\operatorname{ord}\mathbf{m} \leq -\frac{6\operatorname{idx}\mathbf{m}}{\operatorname{ord}\mathbf{m}} \leq -\frac{2}{7}\operatorname{idx}\mathbf{m}$, which contradicts to (8.7).

Suppose $k = 2$ and put $J = \{j\,;\, m_{j,1} \neq m_{j,n_j} \quad (j = 0,1,2)\}$. Then

$$1 + \frac{\operatorname{idx}\mathbf{m}}{(\operatorname{ord}\mathbf{m})^2} = \frac{\sum_{\nu=1}^{n_0} m_{0,\nu}^2}{(\operatorname{ord}\mathbf{m})^2} + \frac{\sum_{\nu=1}^{n_1} m_{1,\nu}^2}{(\operatorname{ord}\mathbf{m})^2} + \frac{\sum_{\nu=1}^{n_2} m_{2,\nu}^2}{(\operatorname{ord}\mathbf{m})^2}$$

and therefore

$$1 - \frac{1}{147} - \frac{\#J}{7} < \sum_{j \in \{0,1,2\} \setminus J} \frac{1}{n_j} < 1$$

because of (8.7), (8.8) and (8.10) for $j \in J$. Lemma 8.2 assures that this never holds because $1 - \frac{1}{147} - \frac{3}{7} > 0$, $1 - \frac{1}{147} - \frac{2}{7} > \frac{1}{2}$, $1 - \frac{1}{147} - \frac{1}{7} > \frac{5}{6}$ and $1 - \frac{1}{147} > \frac{41}{42}$ according to $\#J = 3, 2, 1$ and 0, respectively. □

Lemma 8.2. *Put $I_{k+1} = \left\{\sum_{j=0}^{k} \frac{1}{n_j}\,;\, n_j \in \{2,3,4,\dots\}\right\} \cap [0,1)$. Then*

$$I_1 \subset (0,\tfrac{1}{2}],\ I_2 \subset (0,\tfrac{5}{6}]\ \ and\ \ I_3 \subset (0,\tfrac{41}{42}].$$

Proof. Let $r \in I_{k+1}$. It is clear that $r \leq \frac{1}{2}$ for $r \in I_1$.

Let $r = \frac{1}{n_0} + \frac{1}{n_1} \in I_2$. If $n_0 = 2$, then $n_1 \geq 3$ and $r \leq \frac{5}{6}$. If $n_0 \geq 3$, then $r \leq \frac{2}{3}$.

Let $r = \frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} \in I_3$. We may assume $n_0 \leq n_1 \leq n_2$.

If $n_0 \leq 4$, then $r \leq \frac{3}{4}$.

Suppose $n_0 = 3$. If $n_1 \geq 4$, $r \leq \frac{5}{6}$. If $n_1 = 3$, then $n_2 \geq 4$ and $r \leq \frac{11}{12}$.

Suppose $n_0 = 2$. Then $n_1 \geq 3$. If $n_1 = 3$, then $n_2 > 6$ and $r \leq \frac{41}{42}$. If $n_1 \geq 4$, then $n_2 > 4$ and $r \leq \frac{19}{20}$. □

Remark 8. i) $\bar{\mathcal{K}}(0)$ is given by Kostov [Ko2] and its elements correspond to the indivisible positive null-roots α of the affine root systems $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ (cf. Remark 5, Proposition 7.1 and Table $\bar{\mathcal{K}}(0)$).

ii) In the proof we obtained $\operatorname{ord}\mathbf{m} + 7\operatorname{idx}\mathbf{m} \leq 6$ for $\mathbf{m} \in \mathcal{K}$ but we can prove

(8.11) $$\operatorname{ord}\mathbf{m} + 3\operatorname{idx}\mathbf{m} \leq 6\ \ \text{for}\ \ \mathbf{m} \in \mathcal{K},$$

(8.12) $$\operatorname{ord}\mathbf{m} + \operatorname{idx}\mathbf{m} \leq 2\ \ \text{for}\ \ \mathbf{m} \in \mathcal{K} \setminus \mathcal{P}_3.$$

Example 8.3. For a positive integer m we have special 4 elements

(8.13)
$$D_4^{(m)} : mm-11, m^2, m^2, m^2 \qquad E_6^{(m)} : m^2 m-11, m^3, m^3$$
$$E_7^{(m)} : m^3 m-11, m^4, (2m)^2 \qquad E_8^{(m)} : m^5 m-11, (2m)^3, (3m)^2$$

in $\bar{\mathcal{K}}(2-2m)$ with orders $2m$, $3m$, $4m$ and $6m$, respectively.

Proposition 8.4. *We have*

$$\bar{\mathcal{K}}(-2) = \big\{11,11,11,11,11 \quad 21,21,111,111 \quad 31,22,22,1111 \quad 22,22,22,211$$
$$211,1111,1111 \quad 221,221,11111 \quad 32,11111,11111 \quad 222,222,2211$$
$$33,2211,111111 \quad 44,2222,22211 \quad 44,332,11111111 \quad 55,3331,22222$$
$$66,444,2222211\big\}.$$

Proof. Let $\mathbf{m} \in \mathcal{K}(-2) \cap \mathcal{P}_{k+1}$ be monotone. Then (8.5) and (8.4) with $\mathrm{idx}\,\mathbf{m} = -2$ implies $\sum(m_{j,1} - m_{j,\nu})m_{j,\nu} = 0$ or 2 and we have the following 5 possibilities.

(A) $m_{0,1} \cdots m_{0,n_0} = 2 \cdots 211$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le k$.

(B) $m_{0,1} \cdots m_{0,n_0} = 3 \cdots 31$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le k$.

(C) $m_{0,1} \cdots m_{0,n_0} = 3 \cdots 32$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le k$.

(D) $m_{i,1} \cdots m_{i,n_0} = 2 \cdots 21$ and $m_{j,1} = m_{j,n_j}$ for $0 \le i \le 1 < j \le k$.

(E) $m_{j,1} = m_{j,n_j}$ for $0 \le j \le k$ and $\mathrm{ord}\,\mathbf{m} = 2$.

Case (A). If $2 \cdots 211$ is replaced by $2 \cdots 22$, $\mathbf{m}$ is transformed into $\mathbf{m}'$ with $\mathrm{idx}\,\mathbf{m}' = 0$. If $\mathbf{m}'$ is indivisible, $\mathbf{m}' \in \mathcal{K}(0)$ and $\mathbf{m}$ is $211, 1^4, 1^4$ or $33, 2211, 1^6$. If $\mathbf{m}'$ is not indivisible, $\frac{1}{2}\mathbf{m}' \in \mathcal{K}(0)$ and $\mathbf{m}$ is one of the tuples given in (8.13) with $m = 2$.

Put $m = n_0 - 1$ and examine the identity (8.6).

Case (B). $\frac{3}{3m+1} + \frac{1}{n_1} + \cdots + \frac{1}{n_k} = k - 1$. Since $n_j \ge 2$, we have $\frac{1}{2}k - 1 \le \frac{3}{3m+1} < 1$ and $k \le 3$.

If $k = 3$, we have $m = 1$, $\mathrm{ord}\,\mathbf{m} = 4$, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{5}{4}$, $\{n_1, n_2, n_3\} = \{2, 2, 4\}$ and $\mathbf{m} = 31, 22, 22, 1111$.

Assume $k = 2$. Then $\frac{1}{n_1} + \frac{1}{n_2} = 1 - \frac{3}{3m+1}$ and Lemma 8.2 implies $m \le 5$. We have $\frac{1}{n_1} + \frac{1}{n_2} = \frac{13}{16}, \frac{10}{13}, \frac{7}{10}, \frac{4}{7}$ and $\frac{1}{4}$ according to $m = 5, 4, 3, 2$ and 1, respectively. Hence we have $m = 3$, $\{n_1, n_2\} = \{2, 5\}$ and $\mathbf{m} = 3331, 55, 22222$.

Case (C). $\frac{3}{3m+2} + \frac{1}{n_1} + \cdots + \frac{1}{n_k} = k - 1$. Since $n_j \ge 2$, $\frac{1}{2}k - 1 \le \frac{3}{3m+2} < 1$ and $k \le 3$. If $k = 3$, then $m = 1$, $\mathrm{ord}\,\mathbf{m} = 5$ and $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{7}{5}$, which never occurs.

Thus we have $k = 2$, $\frac{1}{n_1} + \frac{1}{n_2} = 1 - \frac{3}{3m+2}$ and Lemma 8.2 implies $m \le 5$. We have $\frac{1}{n_1} + \frac{1}{n_2} = \frac{14}{17}, \frac{11}{14}, \frac{8}{11}, \frac{5}{8}$ and $\frac{2}{5}$ according to $m = 5, 4, 3, 2$ and 1, respectively. Hence we have $m = 1$ and $n_1 = n_2 = 5$ and $\mathbf{m} = 32, 11111, 11111$ or $m = 2$ and $n_1 = 2$ and $n_2 = 8$ and $\mathbf{m} = 332, 44, 11111111$.

Case (D). $\frac{2}{2m+1} + \frac{2}{2m+1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} = k - 1$. Since $n_j \ge 3$ for $j \ge 2$, we have $k - 1 \le \frac{3}{2}\frac{4}{2m+1} = \frac{6}{2m+1}$ and $m \le 2$. If $m = 1$, then $k \le 3$ and $\frac{1}{n_2} + \frac{1}{n_3} = 2 - \frac{4}{3} = \frac{2}{3}$ and we have $\mathbf{m} = 21, 21, 111, 111$. If $m = 2$, then $k = 2$, $\frac{1}{n_2} = 1 - \frac{4}{5}$ and $\mathbf{m} = 221, 221, 11111$.

Case (E). Since $m_{j,1} = 1$ and (8.4) means $-2 = \sum_{j=0}^{k} 2m_{j,1} - 4(k - 1)$, we have $k = 4$ and $\mathbf{m} = 11, 11, 11, 11, 11$. $\square$

By the aid of a computer we have the following tables.

Table of $\#\bar{\mathcal{K}}(p)$.

index	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20
$\#\bar{\mathcal{K}}(p)$	4	13	36	67	90	162	243	305	420	565	720
# triplets	3	9	24	44	56	97	144	163	223	291	342
# 4-tuples	1	3	9	17	24	45	68	95	128	169	239

Table of (ord $\mathbf{m}$: $\mathbf{m}$) of $\bar{\mathcal{K}}(-4)$ ($*$ corresponds to (8.13) and $+$ means $\partial_{max}(\mathbf{m}) \neq \mathbf{m}$))

```
 +2:11,11,11,11,11,11      3:111,21,21,21,21         4:22,22,22,31,31
 +3:111,111,111,21        +4:1111,22,22,22           4:1111,1111,31,31
  4:211,211,22,22          4:1111,211,22,31         *6:321,33,33,33
  6:222,222,33,51         +4:1111,1111,1111          5:11111,11111,311
  5:11111,2111,221         6:111111,222,321          6:111111,21111,33
  6:21111,222,222          6:111111,111111,42        6:222,33,33,42
  6:111111,33,33,51        6:2211,2211,222           7:1111111,2221,43
  7:1111111,331,331        7:2221,2221,331           8:11111111,3311,44
  8:221111,2222,44         8:22211,22211,44         *9:3321,333,333
  9:111111111,333,54       9:22221,333,441          10:1111111111,442,55
 10:22222,3322,55         10:222211,3331,55         12:22221111,444,66
*12:33321,3333,66         14:2222222,554,77        *18:3333321,666,99
```

We express the root $\alpha_{\mathbf{m}}$ for $\mathbf{m} \in \bar{\mathcal{K}}(0)$ and $\bar{\mathcal{K}}(-2)$ using Dynkin diagram. The circles in the diagram represent the simple roots in $\operatorname{supp} \alpha_{\mathbf{m}}$ and two circles are connected by a line if the inner product of the corresponding simple roots is not zero. The number attached to a circle is the corresponding coefficient n or $n_{j,\nu}$ in the expression (7.12).

For example, if $\mathbf{m} = 11, 11, 11, 11$, then $\alpha_{\mathbf{m}} = 2\alpha_0 + \alpha_{0,1} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{3,2}$, which corresponds to the first diagram in the following.
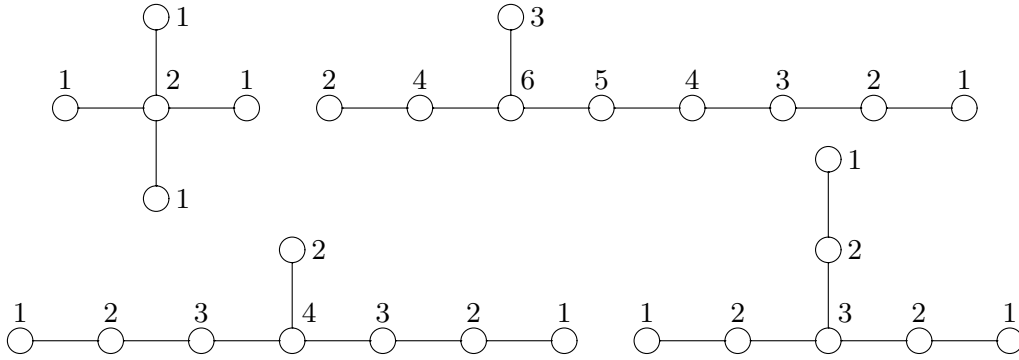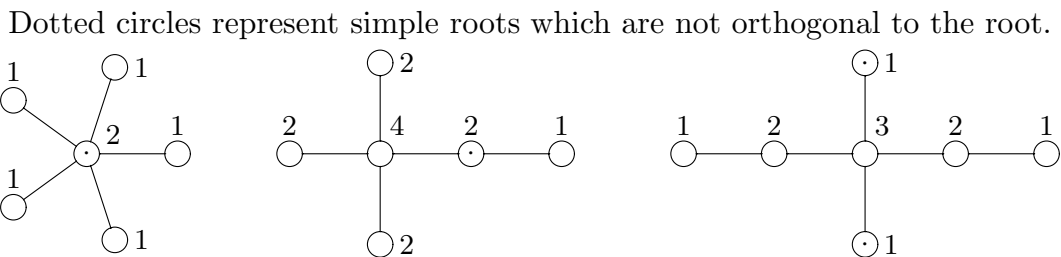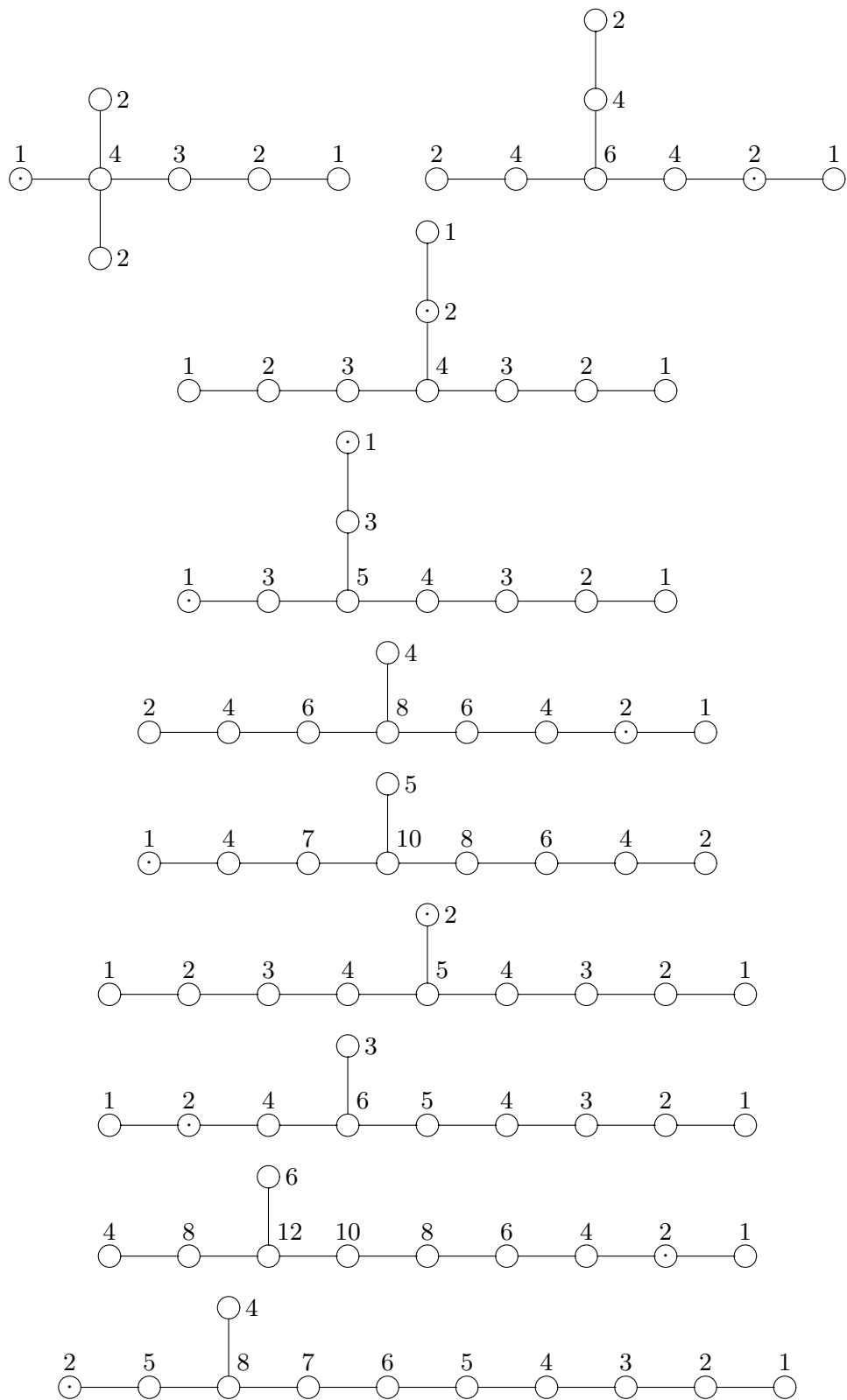
Table $\bar{\mathcal{K}}(0)$

Table $\bar{\mathcal{K}}(-2)$

Dotted circles represent simple roots which are not orthogonal to the root.

E_6 and related affine Dynkin diagrams with marks:

Diagram 1 (affine D_4-type):
```
        2
        |
1 — 4 — 3 — 2 — 1
    |
    2
```

Diagram 2:
```
              2
              |
              4
              |
2 — 4 — 6 — 4 — 2 — 1
```

Diagram 3:
```
              1
              |
              2
              |
1 — 2 — 3 — 4 — 3 — 2 — 1
```

Diagram 4:
```
          1
          |
          3
          |
1 — 3 — 5 — 4 — 3 — 2 — 1
```

Diagram 5:
```
              4
              |
2 — 4 — 6 — 8 — 6 — 4 — 2 — 1
```

Diagram 6:
```
              5
              |
1 — 4 — 7 — 10 — 8 — 6 — 4 — 2
```

Diagram 7:
```
                  2
                  |
1 — 2 — 3 — 4 — 5 — 4 — 3 — 2 — 1
```

Diagram 8:
```
              3
              |
1 — 2 — 4 — 6 — 5 — 4 — 3 — 2 — 1
```

Diagram 9:
```
              6
              |
4 — 8 — 12 — 10 — 8 — 6 — 4 — 2 — 1
```

Diagram 10:
```
          4
          |
2 — 5 — 8 — 7 — 6 — 5 — 4 — 3 — 2 — 1
```

§9.   Connection problem

Fix a tuple $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\ldots,k \\ \nu=1,\ldots,n_j}} \in \mathcal{P}_{k+1}^{(n)}$ in this section. For complex numbers $\lambda_{j,\nu} \in \mathbb{C}$ and $\mu \in \mathbb{C}$ we put

$$
\{\lambda_{\mathbf{m}}\} := \left\{ \begin{array}{ccc} [\lambda_{0,1}]_{(m_{0,1})} & \cdots & [\lambda_{k,1}]_{(m_{k,1})} \\ \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & \cdots & [\lambda_{k,n_k}]_{(m_{k,n_k})} \end{array} \right\}, \quad [\mu]_{(p)} := \begin{pmatrix} \mu \\ \mu+1 \\ \vdots \\ \mu+p-1 \end{pmatrix}.
$$

We may identify $\{\lambda_{\mathbf{m}}\}$ with an element of $M(n, k+1, \mathbb{C})$.

Definition 9.1.   A rigid tuple $\mathbf{m} \in \mathcal{R}_{k+1}$ is a *rigid sum* of $\mathbf{m}'$ and $\mathbf{m}''$ if

(9.1) $$\mathbf{m} = \mathbf{m}' + \mathbf{m}'' \quad\text{and}\quad \mathbf{m}',\ \mathbf{m}'' \in \mathcal{R}_{k+1}$$

and we express this by $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$, which we call a *rigid decomposition* of $\mathbf{m}$.

Theorem 9.2.   *Fix $k+1$ points $\{z_0, \ldots, z_k\} \subset \mathbb{C} \cup \{\infty\}$ and a rigid tuple $\mathbf{m} \in \mathcal{R}_{k+1}$. Assume $\lambda_{j,\nu} \in \mathbb{C}$ are generic under the Fuchs relation $|\{\lambda_{\mathbf{m}}\}| = 0$ with*

(9.2) $$|\{\lambda_{\mathbf{m}}\}| := \sum_{j=0}^{k} \sum_{\nu=0}^{n_j} m_{j,\nu}\lambda_{j,\nu} - \operatorname{ord}\mathbf{m} + 1.$$

i) *There uniquely exists a single Fuchsian differential equation $Pu = 0$ of order n with regular singularities at $\{z_0, \ldots, z_k\} \subset \mathbb{C} \cup \{\infty\}$ such that the set of exponents at $z_j \subset \mathbb{C} \cup \{\infty\}$ is equal to that of components of the $(j+1)$-th column of $\{\lambda_{\mathbf{m}}\}$ and moreover that the local monodromies are semisimple at z_j for $j = 0, \ldots, k$.*

ii) *Assume $k = 2$, $m_{0,n_0} = m_{1,n_1} = 1$ and $m_{j,\nu} > 0$ for $\nu = 1, \ldots, n_j$ and $j = 0,1,2$. Let $c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1})$ denote the connection coefficient from the normalized local solution of $Pu = 0$ in i) corresponding to the exponent λ_{0,n_0} at z_0 to the normalized local solution corresponding to the exponent λ_{1,n_1} at z_1. Then*

(9.3) $$c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) = \frac{\displaystyle\prod_{\nu=1}^{n_0-1} \Gamma\big(\lambda_{0,n_0} - \lambda_{0,\nu} + 1\big) \cdot \prod_{\nu=1}^{n_1-1} \Gamma\big(\lambda_{1,\nu} - \lambda_{1,n_1}\big)}{\displaystyle\prod_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m} \\ m'_{0,n_0}=m''_{1,n_1}=1}} \Gamma\big(|\{\lambda_{\mathbf{m}'}\}|\big)},$$

(9.4) $$\sum_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m} \\ m'_{0,n_0}=m''_{1,n_1}=1}} m'_{j,\nu} = (n_1-1)m_{j,\nu} - \delta_{j,0}(1 - n_0\delta_{\nu,n_0}) + \delta_{j,1}(1 - n_1\delta_{\nu,n_1})$$
$$(0 \le j \le 2,\ 1 \le \nu \le n_j).$$

Remark 9.   i) Putting $(j, \nu) = (0, n_0)$ in (9.4) or considering the sum $\sum_\nu$ for (9.4) with $j = 1$, we have

(9.5) $\#\{\mathbf{m}' \in \mathcal{R}_3 \,;\, \mathbf{m}' \oplus \mathbf{m}'' = \mathbf{m} \text{ with } m'_{0,n_0} = m''_{0,n_1} = 1\} = n_0 + n_1 - 2,$

(9.6) $$\sum_{\substack{\mathbf{m}' \oplus \mathbf{m}'' = \mathbf{m} \\ m'_{0,n_0} = m''_{1,n_1} = 1}} \operatorname{ord} \mathbf{m}' = (n_1 - 1) \operatorname{ord} \mathbf{m}.$$

ii) We may regard $\{\lambda_{\mathbf{m}}\}$ as a Riemann scheme of the Fuchsian equation with the condition that the local monodromies at the singular points are semisimple for generic $\lambda_{j,\nu}$ under the Fuchs condition. The equation for general $\lambda_{j,\nu}$ is defined by the analytic continuation. The corresponding Riemann scheme will be denoted by $P\{\lambda_{\mathbf{m}}\}$.

iii) A proof of this theorem and related results will be given in another paper. The proof is a generalization of that of Gauss summation formula for Gauss hypergeometric series due to Gauss, which doesn't use integral representations of the solutions.

iv) In the theorem the condition $k = 2$ means that there exists no geometric moduli in the Fuchsian equation and we may assume $(z_0, z_1, z_2) = (0, 1, \infty)$. By the transformation of the solutions $u \mapsto z^{-\lambda_{0,n_0}}(1 - z)^{-\lambda_{1,n_1}} u$ we may moreover assume $\lambda_{0,n_0} = \lambda_{1,n_1} = 0$. Then the meaning of "normalized local solution" is clear under the condition $m_{0,n_0} = m_{1,n_1} = 1$.

v) By the aid of a computer the author obtained the table of the concrete connection coefficients (9.3) for $\mathbf{m} \in \mathcal{R}_3$ satisfying $\operatorname{ord} \mathbf{m} \le 40$ together with checking (9.4), which contains 4,111,704 independent cases.

Example 9.3 (H_n : hypergeometric family).   The Fuchsian differential equation of hypergeometric family of order n has the spectral type $\mathbf{m} = (1^n, n - 11, 1^n)$. Its Riemann scheme is

(9.7) $$P\left\{\begin{array}{ccc} \lambda_{0,1} & [\lambda_{1,1}]_{(n-1)} & \lambda_{2,1} \\ \vdots & & \vdots \\ \lambda_{0,n-1} & & \lambda_{2,n-1} \\ \lambda_{0,n} & \lambda_{1,2} & \lambda_{2,n} \end{array}\right\}$$

with complex numbers $\lambda_{j,\nu}$ satisfying the Fuchs relation

(9.8) $$\sum_\nu (\lambda_{0,\nu} + \lambda_{2,\nu}) + (n - 1)\lambda_{1,1} + \lambda_{1,2} = n - 1.$$

It follows from (9.5) that there are n rigid decompositions $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ of $\mathbf{m}$ with $m'_{0,n} = m''_{1,2} = 1$ and they are

$$1 \cdots 1\overline{1}, \, n - 1\underline{1}, \, 1 \cdots 1 = 0 \cdots 0\overline{1}, \quad 1 \quad \underline{0}, \, 0 \cdots 0\overset{\overset{i}{\smile}}{1}0 \cdots 0$$
$$\oplus \, 1 \cdots 1\overline{0}, \, n - 2\underline{1}, \, 1 \cdots 101 \cdots 1 \qquad (i = 1, \ldots, n),$$

which are symbolically expressed by $H_n = H_1 \oplus H_{n-1}$. Then the formula (9.3) implies

$$c(\lambda_{0,n} \rightsquigarrow \lambda_{1,2}) = \frac{\prod_{i=1}^{n-1} \Gamma(\lambda_{0,n} - \lambda_{0,i} + 1) \cdot \Gamma(\lambda_{1,1} - \lambda_{1,2})}{\prod_{i=1}^{n} \Gamma(\lambda_{0,n} + \lambda_{1,1} + \lambda_{2,i})}$$

and

$$c(\lambda_{1,2} \rightsquigarrow \lambda_{0,n}) = \frac{\Gamma(\lambda_{1,2} - \lambda_{1,1} + 1) \cdot \prod_{i=1}^{n-1} \Gamma(\lambda_{0,i} - \lambda_{0,n})}{\prod_{i=1}^{n} \Gamma\left(\left|\left\{\begin{array}{ccc} (\lambda_{0,\nu})_{1 \leq \nu \leq n-1} & [\lambda_{1,1}]_{(n-2)} & (\lambda_{2,\nu})_{\substack{1 \leq \nu \leq n \\ \nu \neq i}} \\ \lambda_{1,2} \end{array}\right\}\right|\right)}.$$

Here we denote

$$(\mu_\nu)_{1 \leq \nu \leq n} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \in \mathbb{C}^n \quad \text{and} \quad (\mu_\nu)_{\substack{1 \leq \nu \leq n \\ \nu \neq i}} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{i-1} \\ \mu_{i+1} \\ \vdots \\ \mu_n \end{pmatrix} \in \mathbb{C}^{n-1}$$

for complex numbers $\mu_1, \ldots, \mu_n$. In the same way as above we have

$$c(\lambda_{0,n} \rightsquigarrow \lambda_{2,n}) = \prod_{i=1}^{n-1} \frac{\Gamma(\lambda_{2,i} - \lambda_{2,n})}{\Gamma\left(\left|\left\{\lambda_{0,n} \ \lambda_{1,1} \ \lambda_{2,i}\right\}\right|\right)}$$

$$\cdot \prod_{i=1}^{n-1} \frac{\Gamma(\lambda_{0,n} - \lambda_{0,i} + 1)}{\Gamma\left(\left|\left\{\begin{array}{ccc} (\lambda_{0,\nu})_{\substack{1 \leq \nu \leq n \\ \nu \neq i}} & [\lambda_{1,1}]_{(n-2)} & (\lambda_{2,\nu})_{1 \leq \nu \leq n-1} \\ \lambda_{1,2} \end{array}\right\}\right|\right)}$$

by the rigid decompositions

$$1 \cdots 1\bar{1}, \, n-11, \, 1 \cdots 1\underline{1} = \begin{array}{c} 0 \cdots 0\bar{\underline{1}}, \quad 1 \ \ 0, 0 \ldots 0\overset{\overset{i}{\smile}}{1}0 \cdots 00 \\ \oplus 1 \cdots 1\underline{0}, \, n-21, \, 1 \cdots 101 \cdots 1\underline{\bar{1}} \end{array}$$

$$= \begin{array}{c} 1 \cdots 10\overset{\overset{i}{\smile}}{1} \cdots 1\bar{1}, \, n-21, \, 1 \cdots 1\underline{0} \\ \oplus 0 \ldots 010 \cdots 0\underline{0}, \quad 1 \ \ 0, 0 \cdots 0\underline{\bar{1}} \end{array} \quad (i = 1, \ldots, n-1).$$

The generalized hypergeometric series

$${}_nF_{n-1}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_{n-1})_k (1)_k} z^k$$

is a solution of the differential equation

$$(9.9) \qquad \left(\prod_{j=1}^{n-1} (z\frac{d}{dz} + \beta_j) \cdot \frac{d}{dz} - \prod_{j=1}^{n} (z\frac{d}{dz} + \alpha_j)\right) u = 0$$

with the Riemann scheme

(9.10)
$$\left\{ \begin{array}{ccc} z = 0 & 1 & \infty \\ 1 - \beta_1 & [0]_{(n-1)} & \alpha_1 \\ \vdots & & \vdots \\ 1 - \beta_{n-1} & & \alpha_{n-1} \\ 0 & -\beta_n & \alpha_n \end{array} \right\} \quad ; \ z \quad \text{with} \quad \sum_{\nu=1}^{n} \alpha_\nu = \sum_{\nu=1}^{n} \beta_\nu.$$

This is the Gauss hypergeometric series when $n = 2$. Here we denote

$$(\gamma)_k = \gamma(\gamma + 1) \cdots (\gamma + k - 1)$$

for $\gamma \in \mathbb{C}$ and $k = 1, 2, \ldots$ and $(\gamma)_0 = 1$. Hence by putting

$$\lambda_{0,\nu} = 1 - \beta_\nu \ \ (1 \le \nu \le n - 1), \ \ \lambda_{0,n} = 0, \ \ \lambda_{1,1} = 0,$$
$$\lambda_{1,2} = -\beta_n \ \ \text{and} \ \ \lambda_{2,i} = \alpha_i \ \ (1 \le i \le n)$$

we have

$$c(\lambda_{0,n} \rightsquigarrow \lambda_{1,2}) = \prod_{i=1}^{n} \frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)}$$
$$= \lim_{x \to 1-0} (1 - x)^{\beta_n} {}_n F_{n-1}(\alpha, \beta; x) \qquad (\operatorname{Re} \beta_n > 0),$$
$$c(\lambda_{1,2} \rightsquigarrow \lambda_{0,n}) = \prod_{i=1}^{n} \frac{\Gamma(1 - \beta_i)}{\Gamma(1 - \alpha_i)}, \quad c(\lambda_{0,n} \rightsquigarrow \lambda_{2,n}) = \prod_{i=1}^{n-1} \frac{\Gamma(\beta_i)\Gamma(\alpha_i - \alpha_n)}{\Gamma(\alpha_i)\Gamma(\beta_i - \alpha_n)}.$$

These connection coefficients are calculated by Levelt [Le] and Okubo et al [OTY].

Example 9.4 (EO_{2m} : even family). Let m be a positive integer. The single Fuchsian differential equation whose Riemann scheme is

(9.11)
$$P \left\{ \begin{array}{ccc} \lambda_{0,1} & [\lambda_{1,1}]_{(m)} & [\lambda_{2,1}]_{(m)} \\ \vdots & [\lambda_{1,2}]_{(m-1)} & [\lambda_{2,2}]_{(m)} \\ \lambda_{0,2m} & \lambda_{1,3} & \end{array} \right\}$$

with the Fuchs relation

(9.12)
$$\sum_{\nu=1}^{2m} \lambda_{0,\nu} + m\lambda_{1,1} + (m-1)\lambda_{1,2} + \lambda_{1,3} + m\lambda_{2,1} + m\lambda_{2,2} = 2m - 1$$

is of even family of order $2m$. Then Theorem 9.2 and the rigid decompositions

$$1 \cdots 1\overline{1}, mm - 1\underline{1}, mm = 0 \cdots 0\overline{1}, 10\underline{0}, \overset{i}{\smile}{10} \oplus 1 \cdots 1\overline{0}, m - 1m - 1\underline{1}, \overset{i}{\smile}{01}$$
$$= 0 \cdots 1\overline{1}, 11\underline{0}, \overset{j}{\smile}{11} \oplus 1 \cdots 0\overline{0}, m - 1m - 2\underline{1}, m - 1m - 1,$$

which are symbolically expressed by $EO_{2m} = H_1 \oplus EO_{2m-1} = H_2 \oplus EO_{2m-2}$, imply

$$c(\lambda_{0,2m} \rightsquigarrow \lambda_{1,3}) = \prod_{i=1}^{2} \frac{\Gamma(\lambda_{1,i} - \lambda_{1,3})}{\Gamma(|\{\lambda_{0,2m} \ \lambda_{1,1} \ \lambda_{2,i}\}|)} \cdot \prod_{j=1}^{2m-1} \frac{\Gamma(\lambda_{0,2m} - \lambda_{0,j} + 1)}{\Gamma(\left|\left\{\begin{matrix} \lambda_{0,j} & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2m} & \lambda_{1,2} & \lambda_{2,2} \end{matrix}\right\}\right|)},$$

$$c(\lambda_{1,3} \rightsquigarrow \lambda_{0,2m}) = \prod_{i=1}^{2} \frac{\Gamma(\lambda_{1,3} - \lambda_{1,i} + 1)}{\Gamma(\left|\left\{\begin{matrix} & [\lambda_{1,1}]_{(m-1)} & [\lambda_{2,\nu}]_{(m)} \\ (\lambda_{0,\nu})_{1 \le \nu \le 2m-1} & [\lambda_{1,2}]_{(m-1)} & [\lambda_{2,3-i}]_{(m-1)} \\ & \lambda_{1,3} & \end{matrix}\right\}\right|)}$$

$$\cdot \prod_{j=1}^{2m-1} \frac{\Gamma(\lambda_{0,j} - \lambda_{0,2m})}{\Gamma(\left|\left\{\begin{matrix} & [\lambda_{1,1}]_{(m-1)} & [\lambda_{2,1}]_{(m-1)} \\ (\lambda_{0,\nu})_{\substack{1 \le \nu \le 2m-1 \\ \nu \ne j}} & [\lambda_{1,2}]_{(m-2)} & [\lambda_{2,2}]_{(m-1)} \\ & \lambda_{1,3} & \end{matrix}\right\}\right|)}.$$

§10.   Appendix

Crawley-Boevey [CB] gives the following complete answer to the additive Deligne-Simpson problem.

Theorem 10.1 ([CB]).   *Let k and n be positive integers, $\mathbf{m}_j = (m_{j,1}, \ldots, m_{j,n_j})$ be partitions of n and $\lambda_j = (\lambda_{j,1}, \ldots, \lambda_{j,n_j}) \in \mathbb{C}^{n_j}$ for $j = 0, \ldots, k$. Put $\mathbf{m} = (\mathbf{m}_0, \ldots, \mathbf{m}_k) \in \mathcal{P}_{k+1}^{(n)}$ and assume the condition (5.16). Then there exists an irreducible tuple of matrices $\mathbf{A} = (A_0, \ldots, A_k) \in M(n, \mathbb{C})^{k+1}$ satisfying*

(10.1) $\qquad A_j \sim L(\mathbf{m}_j; \lambda_j) \quad (j = 0, \ldots, k) \quad and \quad A_0 + \cdots + A_k = 0$

if and only if $\alpha_{\mathbf{m}}$ is a positive root and moreover

(10.2) $\qquad \left(\sum_{j,\nu} m_{j,\nu}^{(1)} \lambda_{j,\nu}, \ldots, \sum_{j,\nu} m_{j,\nu}^{(N)} \lambda_{j,\nu} \right) \ne (0, \ldots, 0) \in \mathbb{C}^N$

for any decomposition

(10.3) $\qquad \mathbf{m} = \mathbf{m}^{(1)} + \cdots + \mathbf{m}^{(N)}$

with $N \ge 2$ and $\mathbf{m}^{(i)} \in \mathcal{P}_{k+1}$ such that

(10.4) $\qquad \begin{cases} \alpha_{\mathbf{m}^{(i)}} \text{ defined by (7.12) are positive roots } (i = 1, \ldots, N), \\ \mathrm{Pidx}\,\mathbf{m} \le \mathrm{Pidx}\,\mathbf{m}^{(1)} + \cdots + \mathrm{Pidx}\,\mathbf{m}^{(N)} \end{cases}$

under the notation and the correspondence in Remark 7 i).

K. Takemura indicated to the author that the following result follows from Theorem 10.1 and kindly allows the author to include the proof in this note.

Theorem 10.2.   *Retain the notation and the assumption in Theorem 10.1. If there exists an irreducible tuple of matrices* $\mathbf{A} = (A_0, \ldots, A_k) \in M(n, \mathbb{C})^{k+1}$ *satisfying* (10.1), *then* $\alpha_{\mathbf{m}}$ *defined by* (7.12) *is a positive root such that* $\mathbf{m}$ *is indivisible or* $\mathrm{idx}\,\mathbf{m} < 0$. *Conversely if a tuple* $\mathbf{m} \in \mathcal{P}$ *is indivisible or* $\mathbf{m}$ *satisfies* $\mathrm{idx}\,\mathbf{m} < 0$ *and moreover* $\alpha_{\mathbf{m}}$ *is a positive root, then* $\mathbf{m}$ *is irreducibly realizable.*

Proof.   Note that this theorem follows from Theorem 10.1 if $\mathbf{m}$ is indivisible because (10.2) always holds when $\lambda_{j,\nu}$ are generic under the condition (5.16).

Suppose $\mathbf{m} = d\overline{\mathbf{m}}$ with an integer $d > 1$ and an indivisible tuple $\overline{\mathbf{m}} \in \mathcal{P}_{k+1}$. Since $\mathrm{Pidx}\,\mathbf{m} = 1 - \frac{1}{2}\,\mathrm{idx}\,\mathbf{m} = 1 - \frac{1}{2}(\alpha_{\mathbf{m}}, \alpha_{\mathbf{m}})$, we have

$$(10.5) \qquad\qquad \mathrm{Pidx}\, d\overline{\mathbf{m}} = 1 + d^2(\mathrm{Pidx}\,\overline{\mathbf{m}} - 1).$$

If $\mathrm{Pidx}\,\overline{\mathbf{m}} = 1$, we have $\mathrm{Pidx}\,\mathbf{m} = \mathrm{Pidx}\,(d-1)\overline{\mathbf{m}} = 1$ and this theorem also follows from Theorem 10.1 with the decomposition $\mathbf{m} = \overline{\mathbf{m}} + (d-1)\overline{\mathbf{m}}$ corresponding to (10.3).

Hence we may moreover suppose $\mathrm{Pidx}\,\overline{\mathbf{m}} > 1$. Assume the existence of the decomposition (10.3) such that $\sum_{j,\nu} m_{j,\nu}^{(i)} \lambda_{j,\nu} = 0$ in Theorem 10.1. If $\lambda_{j,\nu}$ are generic, we have $\mathbf{m}^{(i)} = d_i \overline{\mathbf{m}}$ with positive integers d_i satisfying $d = d_1 + \cdots + d_N$. Then

$$\mathrm{Pidx}\,\mathbf{m} - \sum_{i=1}^{N} \mathrm{Pidx}\, d_i \overline{\mathbf{m}} = 1 + d^2(\mathrm{Pidx}\,\overline{\mathbf{m}} - 1) - \sum_{i=1}^{N}\big(1 + d_i^2(\mathrm{Pidx}\,\overline{\mathbf{m}} - 1)\big)$$

$$= 2 \sum_{1 \le i < j \le N} d_i d_j (\mathrm{Pidx}\,\overline{\mathbf{m}} - 1) - (N - 1) > 0$$

when $\mathrm{Pidx}\,\overline{\mathbf{m}} \ge 2$ and $N \ge 2$. Hence Theorem 10.1 completes the proof. $\square$

Remark 10.   i) Kostov [Ko2] studies the above result when $\mathrm{idx}\,\mathbf{m} = 0$.

ii) It follows from Theorem 10.2 that the spectral type of any irreducible tuple $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ is irreducibly realizable.

iii) We define that a tuple $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ and the corresponding Fuchsian system (4.3) are *fundamental* if $\mathbf{A}$ is irreducible and cannot be transformed into a tuple of matrices with a lower rank by any successive applications of additions and middle convolutions. We also define that a tuple $\mathbf{m} \in \mathcal{P}$ is *fundamental* if it corresponds to a suitable fundamental tuple $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$.

Then a tuple $\mathbf{m} \in \mathcal{P}$ is fundamental if and only if $\mathbf{m}$ is basic or there exist a positive number d and a basic tuple $\overline{\mathbf{m}} \in \mathcal{P}$ satisfying $\mathbf{m} = d\overline{\mathbf{m}}$ and $\mathrm{idx}\,\overline{\mathbf{m}} < 0$.

Hence it follows from Proposition 8.1 and the equality (10.5) that there exist only a finite number of fundamental tuples $\mathbf{m} \in \mathcal{P}$ such that $\mathrm{idx}\,\mathbf{m}$ equal to a fixed number.

iv) (Nilpotent case : [CB], [Ko3]) Under the notation in Theorem 10.1 there exists an irreducible tuple $\mathbf{A} \in M(n, \mathbb{C})^{k+1}$ satisfying (10.1) with $\lambda_{j,\nu} = 0$ for any j and ν if and only if ord $\mathbf{m} = 1$ or $\mathbf{m}$ is fundamental and moreover $\mathbf{m}$ is not the special element in Example 8.3 with $m \geq 2$. Here we have the decompositions $D_4^{(m+1)} = D_4^{(m)} + D_4^{(1)}$ and $E_j^{(m+1)} = E_j^{(m)} + E_j^{(1)}$ for $j = 6$, 7 and 8 which satisfy (10.3) and (10.4).

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