Middle convolution of KZ-type equations and Single-elimination tournaments

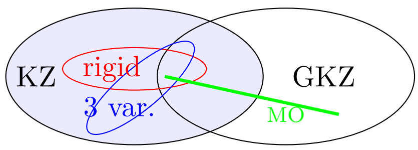
By

Toshio Oshima

Abstract

We introduce an extension of the generalized Riemann scheme for Fuchsian ordinary differential equations in the case of a KZ-type equation and denote it by $\mathrm{Sp}\,\mathcal{M}$. This extension describes the local structure of the equation obtained by resolving the singularities of the KZ-type equation. We present the transformation of this extension under middle convolutions. The transformation was determined by [Ok] when $n = 4$ and in general, we interpret it in terms of the combinatorics of single-elimination tournaments. Here n is the number of variables of $\mathcal{M}$. Since many hypergeometric functions with several variables including Lauricella's F_D and its generalizations are solutions to KZ-type equations, the results in this paper can be applied to the analysis of these functions (cf. [Oi, MO]).

1　KZ 型方程式



定義 1.

$$\boxed{\mathcal{M} : \frac{\partial u}{\partial x_i} = \sum_{\substack{0\le\nu\le n-1\\ \nu\ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u \quad (i \in L_n)}$$

$L_n := \{0,\dots,n-1\},\ \ A_{ij} = A_{ji} \in M(N,\mathbb{C}),\ A_{ii} = 0,\ u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$

積分可能条件：$[A_{ij}, A_{k\ell}] = [A_{ij}, A_{ik} + A_{jk}] = 0$　($i,\ j,\ k,\ \ell$：相異なる) を仮定する．
この方程式 $\mathcal{M}$ を KZ（Knizhnik-Zamolodchikov）型方程式という．

KZ 型方程式 $\hookleftarrow$ rigid フックス型常微分 (原岡 [H])　$\Leftarrow$ $\frac{du}{dx} = \frac{A_{01}}{x-x_1}u + \cdots + \frac{A_{0n-1}}{x-x_{n-1}}u$

$\hookleftarrow$ 3 点の特異点を持つ $\mathbb{P}^1$ 上の Fuchs 型方程式

middle convolution と addition $\rightsquigarrow$ アクセサリー・パラメーターと局所構造

KZ 型方程式：$\mathrm{Sp}\,\mathcal{M}$ $\leftarrow$ 常微分のスペクトル型 / (generalized) Riemann scheme

KZ 型方程式 $\mathcal{M}$ が **rigid** $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{M}$ が $\mathrm{Sp}\,\mathcal{M}$ によって一意に定まる

アクセサリー・パラメーター：$\mathrm{Sp}\,\mathcal{M}$ で定まらない KZ 型方程式のパラメーター

定義 2. $A_{i\infty} := -(A_{i0} + A_{i1} + \cdots + A_{in-1})$　　$(i \in L_n)$

$\widetilde{L}_n := L_n \cup \{\infty\},\quad A_i = A_\emptyset = 0$

$$A_{i_1\cdots i_k} := \sum_{1\le p<q\le k} A_{i_p i_q} \quad (\{i_1,\dots,i_k\} \subset \widetilde{L}_n) \quad \text{(generalized) 留数行列}$$

定理 1. $\boxed{[A_I, A_J] = 0 \quad (I \subset J \text{ or } I \supset J \text{ or } I \cap J = \emptyset \ \text{ for } \ I, J \subset \widetilde{L}_n)}$

- $[A_I, A_{L_n}] = 0 \;\Rightarrow\; A_{L_n} = \kappa$: scalar ($\Leftarrow \mathcal{M}$：既約)　($\kappa = 0$：斉次という)
- $I \subset L_n \;\;\Rightarrow\;\; A_I = A_{\tilde{L}_n \setminus I} + \kappa$

記号. $[A, B] = 0 \;\Rightarrow\; \exists$ 同時 (一般) 固有空間分解：

$$A = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 3 \end{pmatrix}$$

$$\Rightarrow\;\; [A : B] = \{[0:1]_1, [0:2]_2, [4:3]_1\} = \{[0:1], [0:1]_2, [4:3]\}$$

- $[B_i, B_j] = 0 \;\; (i, j = 1, .., r) \;\Rightarrow\; [B_1 : \cdots : B_r] = \{[\lambda_{1,1} : \cdots : \lambda_{r,1}]_{m_1}, \cdots\}$

定義 3. 有限集合 L の可換部分集合合族 $\mathcal{I} = \{I_\nu \mid \nu = 1, \ldots, r\}$

$\overset{\text{def.}}{\Longleftrightarrow}$ $I_\nu \subset L,\; |I_\nu| > 1$ and ($I_\nu \subset I_{\nu'}$ or $I_\nu \supset I_{\nu'}$ or $I_\nu \cap I_{\nu'} = \emptyset$)

$\mathcal{I}$ が極大 $\overset{\text{def.}}{\Longleftrightarrow}$ ($\mathcal{I}, \mathcal{I}'$：可換部分集合族で $\mathcal{I} \subset \mathcal{I}' \;\Rightarrow \mathcal{I} = \mathcal{I}'$)

定義 4. $\mathcal{L}_n := \{L_n = \{0, \ldots, n-1\}$ の極大可換部分集合族 $\}$

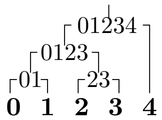
$$\mathrm{Sp}\ \mathcal{M} := \big\{[A_{I_1} : \cdots : A_{I_{n-1}}] \mid \mathcal{I} = \{I_1, \ldots, I_{n-2}, I_{n-1}\}\big\}_{\mathcal{I} \in \mathcal{L}_n}$$

$$\mathrm{Sp}'\ \mathcal{M} := \big\{[A_{I_1} : \cdots : A_{I_{n-2}}] \mid \mathcal{I} = \{I_1, \ldots, I_{n-2}, L_n\}\big\}_{\mathcal{I} \in \mathcal{L}_n}$$

Fact 1. $\mathcal{L}_n \simeq \{L_n$ の元でラベルづけられた n チームのトーナメント戦 $\}$

$|\mathcal{L}_n| = (2n-3)!!$　　$|\mathcal{I}| = n - 1 \;\; (\mathcal{I} \in \mathcal{L}_n)$

例 1. $L_5 = \{0, 1, 2, 3, 4\},\; |\mathcal{L}_5| = 105$



$\mathcal{I} = \big\{\{0, 1\}, \{0, 1, 2, 3\}, \{2, 3\}, \{0, 1, 2, 3, 4\}\big\}$

$\big\{[A_{ij} : A_{k\ell} : A_{ijk\ell}],\, [A_{ij} : A_{ijk} : A_{\ell m}],\, [A_{ij} : A_{ijk} : A_{ijk\ell}] \mid \{i, j, k, \ell, m\} = L_5\big\}$

2　特異点解消

KZ 型方程式（パッフ系）：　$du = \Omega u, \quad \Omega = \displaystyle\sum_{0 \le i < j < n} A_{ij}\, d\log(x_i - x_j), \quad \Omega \wedge \Omega = 0$

$\mathcal{I} = \{I_1, \ldots, I_{n-2}, L_n\} \in \mathcal{L}$：$L_n = \{0, 1, \ldots, n-1\}$ の極大可換部分集合族

$J,\, J' \in \tilde{\mathcal{I}} := \mathcal{I} \cup \bigcup_{\nu \in L_n} \{\{\nu\}\}$　with $L_n = J \sqcup J'$：$\mathcal{I}$ の準決勝の試合

$J = \{j_0, \ldots, j_k\},\;\; J' = \{j'_0, \ldots, j'_{k'}\} \quad (k + k' = n - 2)$

特異点　　：$x_{j_0} = \cdots = x_{j_k}$ かつ $x_{j'_0} = \cdots = x_{j'_{k'}} \quad (k,\, k' \ge 0)$

局所座標系：$(x_{j_1}, \ldots, x_{j_k}, x_{j'_1}, \ldots, x_{j'_{k'}})$, ただし $x_{j_0} = 0$ かつ $x_{j'_0} = 1$.

$\{n_i, n'_i\}$　　：試合 I_i で対戦したチーム

定義 5. 次式で定まる局所座標系 $(X_1, \ldots, X_{n-2})$：$x_{n_i} - x_{n'_i} = \displaystyle\prod_{I_i \subset I_j \ne L_n} X_j$.

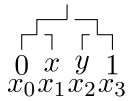
定理 2 [Om]. $\Omega - \displaystyle\sum_{i=1}^{n-2} A_{I_i}\, d\log X_i$　は $(X_1, \ldots, X_{n-2})$ の原点の近傍で特異点を持たない.

例 2. 定理の記述を用いて $\Omega' := \displaystyle\sum_{i=1}^{n-2} A_{I_i}\, d\log X_i$ とおく.

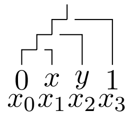
$\mathbf{n = 4}$:

$$\Omega = A_{x0}\frac{dx}{x} + A_{y0}\frac{dy}{y} + A_{x1}\frac{d(x-1)}{x-1} + A_{y1}\frac{d(y-1)}{y-1} + A_{xy}\frac{d(x-y)}{x-y}.$$

$$(x, y) = (0, 1) : \begin{cases} x_1 - x_0 = x = X, \\ x_3 - x_2 = 1 - y = Y, \end{cases} \quad \Omega' = A_{x0}\frac{dX}{X} + A_{y1}\frac{dY}{Y} \quad (|x|,\, |y - 1| \ll 1)$$



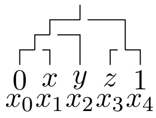
$$(x, y) = (0, 0) : \begin{cases} x_2 - x_0 = y = Y, \\ x_1 - x_0 = x = XY, \\ x_2 - x_1 = y - x = (1 - X)Y, \end{cases} \quad \begin{cases} \frac{dx}{x} = \frac{dX}{X} + \frac{dY}{Y}, \\ \frac{d(x-y)}{x-y} = \frac{dY}{Y} + \frac{d(X-1)}{X-1}, \\ (X, Y) = (\frac{x}{y}, y), \end{cases}$$



$$\Omega' = A_{x0}\tfrac{dX}{X} + A_{xy0}\tfrac{dY}{Y}, \quad A_{xy0} := A_{x0} + A_{y0} + A_{xy} \quad (|x| \ll |y| \ll 1).$$

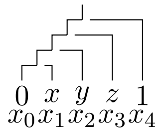
$\mathbf{n=5}$:
$$\Omega = A_{x0}\tfrac{dx}{x} + A_{y0}\tfrac{dy}{y} + A_{z0}\tfrac{dz}{z} + A_{x1}\tfrac{d(x-1)}{x-1} + A_{y1}\tfrac{d(y-1)}{y-1} + A_{z1}\tfrac{d(z-1)}{z-1}$$
$$+ A_{xy}\tfrac{d(x-y)}{x-y} + A_{yz}\tfrac{d(y-z)}{y-z} + A_{xz}\tfrac{d(x-z)}{x-z}.$$

$$(x,y,z)=(0,0,1):\ \begin{cases} x_2-x_0=y=Y, \\ x_1-x_0=x=XY, \\ x_4-x_3=z-1=Z, \end{cases} \qquad \begin{cases} x-y=(X-1)Y, \\ z=1-Z, \\ (X,Y,Z)=(\frac{x}{y},y,1-z), \end{cases}$$



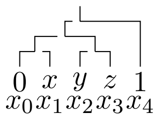
$$\Omega' = A_{x0}\tfrac{dX}{X} + A_{xy0}\tfrac{dY}{Y} + A_{z1}\tfrac{dZ}{Z} \quad (|x| \ll |y| \ll 1,\ |z-1| \ll 1),$$

$$(x,y,z)=(0,0,0):\ \begin{cases} x_3-x_0=z=Z, \\ x_2-x_0=y=YZ, \\ x_1-x_0=x=XYZ, \end{cases} \qquad \begin{cases} y-z=(Y-1)Z, \\ x-y=(X-1)YZ, \\ x-z=(XY-1)Z, \\ (X,Y,Z)=(\frac{x}{y},\frac{y}{z},z), \end{cases}$$



$$\Omega' = A_{x0}\tfrac{dX}{X} + A_{xy0}\tfrac{dY}{Y} + A_{xyz0}\tfrac{dZ}{Z} \quad (|x| \ll |y| \ll |z| \ll 1),$$

$$(x,y,z)=(0,0,0):\ \begin{cases} x_3-x_0=z=Z, \\ x_1-x_0=x=XZ, \\ x_3-x_2=z-y=YZ, \end{cases} \qquad \begin{cases} y=(1-Y)Z, \\ x-y=(X+Y-1)Z, \\ x-z=(X-1)Z, \\ (X,Y,Z)=(\frac{x}{z},\frac{z-y}{z},z), \end{cases}$$



$$\Omega' = A_{x0}\tfrac{dX}{X} + A_{yz}\tfrac{dY}{Y} + A_{xyz0}\tfrac{dZ}{Z} \quad (|x|,\ |y-z| \ll |z| \ll 1).$$

3 変数 x_0 に対する middle convolution

定義 6. $\mu \in \mathbb{C}$ で定まる $\mathcal{M}$ の convolution $\tilde{\mathcal{M}} = \widetilde{\mathrm{mc}}_{x_0,\mu}\mathcal{M}$ は次のように定義される.

$$\tilde{\mathcal{M}}\ :\ \frac{\partial \tilde{u}}{\partial x_i} = \sum_{0\le\nu<n} \frac{\tilde{A}_{i\nu}}{x_i - x_\nu}\tilde{u} \quad (0 \le i < n)$$

$$\tilde{A}_{0k} = \begin{array}{c} \\ k \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{01} & \cdots & A_{0k}+\mu & \cdots & A_{0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \\ \begin{array}{ccccc} \phantom{A_{01}} & & i & & j \end{array} \end{array} \in M((n-1)N,\mathbb{C}) \qquad \text{(Dettwiler-Reiter [DR]} \Leftarrow \text{Katz)},$$

$$\tilde{A}_{ij} = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \begin{pmatrix} A_{ij} & & & & & \\ & \ddots & & & & \\ & & A_{ij}+A_{0j} & & -A_{0j} & \\ & & & \ddots & & \\ & & -A_{0i} & & A_{ij}+A_{0i} & \\ & & & & & \ddots \\ & & & & & & A_{ij} \end{pmatrix} \in M((n-1)N,\mathbb{C}) \qquad \text{(Haraoka [H])}.$$

さらに
$$\iota_j(v) := (v)_j := j\begin{pmatrix} 0 \\ \vdots \\ v \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{(n-1)N} \quad (v \in \mathbb{C}^N), \quad \iota_j : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N}$$

$$\iota_I := \sum_{i\in I}\iota_i : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N},\ (v)_I := \iota_I(v) \in \mathbb{C}^{(n-1)N} \quad (v \in \mathbb{C}^N,\ I \subset L_n^0)$$

$$\tilde{\mathcal{K}}_j := \iota_j(\mathrm{Ker}\,A_{0j}) = j\begin{pmatrix} 0 \\ \vdots \\ \mathrm{Ker}\,A_{0j} \\ \vdots \\ 0 \end{pmatrix} \subset \mathbb{C}^{(n-1)N}, \quad L_n^0 := \{1,\ldots,n-1\}$$

$$\tilde{\mathcal{K}}_\infty := \ker \tilde{A}_{0,\infty} \overset{\mu\neq 0}{=} \iota_{L_n}\left(\mathrm{Ker}\,(A_{0\infty}-\mu)\right) = \left\{\begin{pmatrix} v\\ \vdots \\ v\end{pmatrix} \mid A_{0\infty}v = \mu v\right\} \subset \mathbb{C}^{(n-1)N}$$

$$\tilde{\mathcal{K}} := \tilde{\mathcal{K}}_\infty + \bigoplus_{j=1}^{n} \tilde{\mathcal{K}}_j \quad (\text{direct sum} \ \Leftarrow\ \mu\neq 0) \ \Rightarrow\ \tilde{A}_I\text{-invariant} \quad (I\subset L_n)$$

$$\bar{A}_I := \tilde{A}_I|_{\mathbb{C}^{(n-1)N}/\tilde{\mathcal{K}}} \in M\big((n-1)N-\dim\tilde{\mathcal{K}},\mathbb{C}\big)$$

によって $\mathcal{M}$ の middle convolution は $\overline{\mathcal{M}} = \mathrm{mc}_{x_0,\mu}\mathcal{M}$: $\dfrac{\partial\bar{u}}{\partial x_i} = \displaystyle\sum_{0\le\nu<n}\frac{\bar{A}_{i\nu}}{x_i-x_\nu}\bar{u}$ と定義される

$$\mathrm{mc}_{x_0,-\mu}\circ\mathrm{mc}_{x_0,\mu} = \mathrm{id} \quad (\Leftarrow\ \partial^\mu\circ\partial^{-\mu})$$

Fact 2. $A_{L_n}=\kappa,\ I\subset L_n \Rightarrow\ \bar{A}_I = \bar{A}_{\widetilde{L_n\setminus I}}+\kappa+\mu \quad (\kappa=0)$

定義 7. $i\in L_n$ と $I,\,J\subset L_n$ に対し

$$\boxed{\mathrm{md}_{i,J}(I) := \begin{cases} I\cup\{i\} & (I\supset J)\\ I\setminus\{i\} & (I\not\supset J)\end{cases} \qquad \mathrm{me}_{i,J}(I) := \begin{cases} 1 & (i\in I\supset J)\\ 0 & (i\notin I \text{ or } I\not\subset J)\end{cases}}$$

Fact 3. $i\in I\supset J$ または $i\notin I\not\supset J\ \Rightarrow \mathrm{md}_{i,J}(I)=I$
$\mathrm{md}_{i,J}:\{\text{ 可換族 }\}\to\{\text{ 可換族 }\}$

定理 3 [Ok $(n=4)$, Om]. (i) $\mathcal{I}=\{I^{(1)},\dots,I^{(n-1)}\}\in\mathcal{L}_n$, $I\in\mathcal{I}$ とするとき

$$[\tilde{A}_I] = [A_{I\cup\{0\}}+\mu]_{|I|-1}\sqcup[A_{I\setminus\{0\}}]_{n-|I|}$$

$$[\tilde{A}_{I^{(1)}}:\dots:\tilde{A}_{I^{(n-1)}}] = \bigsqcup_{J\in\mathcal{I}}[A^J_{I^{(1)}}:\dots:A^J_{I^{(n-1)}}]$$

$$[\tilde{A}_{I^{(1)}}:\dots:\tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_j} = [A^{\{j\}}_{I^{(1)}}:\dots:A^{\{j\}}_{I^{(n-1)}}]|_{\ker A_{0j}} \quad (j\in L_n^0),$$

$$[\tilde{A}_{I^{(1)}}:\dots:\tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_\infty} = [A^{L_n}_{I^{(1)}}:\dots:A^{L_n}_{I^{(n-1)}}]|_{\ker(A_{0\infty}-\mu)} \quad (A_{0\infty}=A_{L_n^0}-A_{L_n})$$

$$A_I^K := A_{\mathrm{md}_{0,K}(I)}+\mathrm{me}_{0,K}(I)\cdot\mu$$

(ii) $U^{-1}_{b^0(\mathcal{I})}\tilde{A}_{I^{(i)}}U_{b^0(\mathcal{I})}$: 同時ブロック上三角行列 $(i=1,\dots,n-1)$

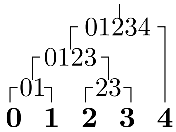
$\mathcal{I}=\{I^{(1)},\dots,I^{(n-1)}\}$ において，以下を仮定してよい（$\mathcal{I}$ の元の順序づけ）.
$\mathcal{I}_0 := \{I\mid 0\in I\in\mathcal{I}\} = \{I^{(k_1)},\dots,I^{(k_m)}\}$

1. $k_p<k_q \quad\Leftrightarrow\quad I^{(k_p)}\subsetneq I^{(k_q)}$
2. $k_p<i\le j<k_{p+1} \quad\Leftarrow\quad I^{(k_p)}\setminus I^{(k_{p-1})}\supset I^{(i)}\supset I^{(j)} \quad (I^{(k_0)}:=\emptyset)$

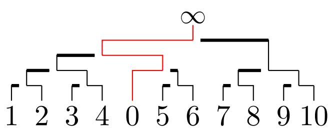
定義 8. For $i\in L_n$ and $I\in\mathcal{I}$
$b^i(I)\in\widetilde{\mathcal{I}}:=\mathcal{I}\cup\bigcup_{\nu\in L_n}\{\{\nu\}\}$ so that $i\notin b^i(I)\subsetneq I$ and $I\setminus b^i(I)\in\widetilde{\mathcal{I}}$

$$GL\big((n-1)N,\mathbb{C}\big)\ni\big(U_{b^0(\mathcal{I})}\big)_{\substack{1\le i\le n-1\\1\le j\le n-1}} := \begin{cases}\mathbf{1}_N & (i\in b^0(I^{(j)}))\\ 0 & (i\notin b^0(I^{(j)}))\end{cases}$$

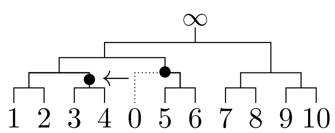
例 3. $L=\{0,1,2,3,4\}$
$\mathcal{I}=\{\{\underline{0,1}\},\{\underline{0,1,2,3}\},\{2,3\},\{\underline{0,1,2,3,4}\}\}$
$b^0(\{0,1\})=\{1\},\ b^0(\{0,1,2,3\})=\{2,3\},\ b^0(\{2,3\})=\{2\},\ b^0(L_5)=\{4\}$



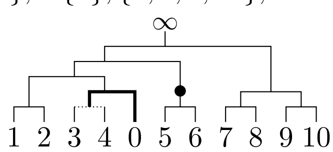
例 4. $L=\{0,1,\dots,10\}$:



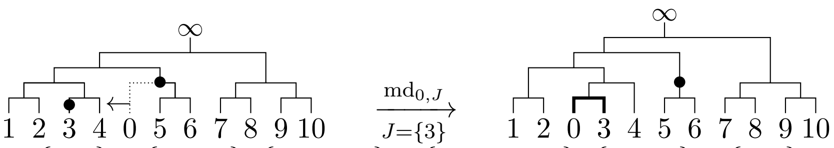
$\mathcal{I}$: $\{\underline{0,5,6}\},\{5,6\},\{\underline{0,\dots,6}\},\{1,2,3,4\},\{1,2\},\{3,4\},\{\underline{0,\dots,10}\},\{7,8,9,10\},\{7,8\},\{9,10\}$
b^0 : $\{5,6\},\ \{5\},\{1,2,3,4\},\ \{1,2\},\ \{1\},\ \{3\},\{7,8,9,10\},\ \{7,8\},\ \{7\},\ \{9\}$



$$\xrightarrow[J=\{3,4\}]{\mathrm{md}_{0,J}}$$

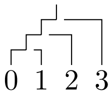


$J=\{3,4\}:\{3,4\}\to\{0,3,4\},\ \{1,2,3,4\}\to\{0,1,2,3,4\},\ \{0,5,6\}\to\{5,6\}$



$j=3:\{3,4\}\to\{0,3,4\},\ \{1,2,3,4\}\to\{0,1,2,3,4\},\ \{0,5,6\}\to\{5,6\},\ +\{0,3\}$

$n=4$ の場合（cf. [Ok, Oi, Or `midkz( )`]）

1. 

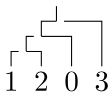
$\mathcal{I}=\big\{\{0,1\},\{0,1,2\},\{0,1,2,3\}\big\}\ \xrightarrow{b^0}\ \big\{\{1\},\{2\},\{3\}\big\}$

$$U=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad V=U^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \tilde A_*\to V\tilde A_*U$$

$$\tilde A_{01}=\begin{pmatrix}A_{01}+\mu&A_{02}&A_{03}\\0&0&0\\0&0&0\end{pmatrix}\to\begin{pmatrix}A_{01}+\mu&A_{02}&A_{03}\\0&0&0\\0&0&0\end{pmatrix}$$

$$\tilde A_{012}=\begin{pmatrix}A_{012}+\mu&0&A_{03}\\0&A_{012}+\mu&A_{03}\\0&0&A_{12}\end{pmatrix}\to\begin{pmatrix}A_{012}+\mu&0&A_{03}\\0&A_{012}+\mu&A_{03}\\0&0&A_{12}\end{pmatrix}$$

$$\begin{aligned}
[\tilde A_{01}:\tilde A_{012}]&=\{[A_{01}+\mu:A_{012}+\mu],[0:A_{012}+\mu],[0:A_{12}]\}\\
[\tilde A_{01}:\tilde A_{012}]|_{\mathcal{K}_1}&=[A_{01}+\mu:A_{012}+\mu]|_{\ker A_{01}}\\
[\tilde A_{01}:\tilde A_{012}]|_{\mathcal{K}_2}&=[0:A_{012}+\mu]|_{\ker A_{02}}\\
[\tilde A_{01}:\tilde A_{012}]|_{\mathcal{K}_3}&=[0:A_{12}]|_{\ker A_{03}}\\
[\tilde A_{01}:\tilde A_{012}]|_{\mathcal{K}_\infty}&=[0:A_{12}]|_{\ker(A_{0\infty}-\mu)}
\end{aligned}$$

2. 

$\mathcal{I}=\big\{\{0,1,2\},\{1,2\},\{0,1,2,3\}\big\}\ \xrightarrow{b^0}\ \big\{\{1,2\},\{1\},\{3\}\big\}$

$$U=\begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad V=U^{-1}=\begin{pmatrix}0&1&0\\1&-1&0\\0&0&1\end{pmatrix},\quad \tilde A_*\to V\tilde A_*U$$

$$\tilde A_{012}=\begin{pmatrix}A_{012}+\mu&0&A_{03}\\0&A_{012}+\mu&A_{03}\\0&0&A_{12}\end{pmatrix}\to\begin{pmatrix}A_{012}+\mu&0&A_{03}\\0&A_{012}+\mu&0\\0&0&A_{12}\end{pmatrix}$$

$$\tilde A_{12}=\begin{pmatrix}A_{012}-A_{01}&-A_{02}&0\\-A_{01}&A_{012}-A_{02}&0\\0&0&A_{12}\end{pmatrix}\to\begin{pmatrix}A_{12}&-A_{01}&0\\0&A_{012}&0\\0&0&A_{12}\end{pmatrix}$$

$$\begin{aligned}
[\tilde A_{012}:\tilde A_{12}]&=\{[A_{012}+\mu:A_{12}],[A_{012}+\mu:A_{012}],[A_{12}:A_{12}]\}\\
[\tilde A_{012}:\tilde A_{12}]|_{\mathcal{K}_1}&=[A_{012}+\mu:A_{012}]|_{\ker A_{01}}\\
[\tilde A_{012}:\tilde A_{12}]|_{\mathcal{K}_2}&=[A_{012}+\mu:A_{012}]|_{\ker A_{02}}\\
[\tilde A_{012}:\tilde A_{12}]|_{\mathcal{K}_3}&=[A_{12}:A_{12}]|_{\ker A_{03}}\\
[\tilde A_{012}:\tilde A_{12}]|_{\mathcal{K}_\infty}&=[A_{12}:A_{12}]|_{\ker(A_{0\infty}-\mu)}
\end{aligned}$$

3. $1\ 2\ 3\ 0$ $\qquad \mathcal{I} = \{\{0,1,2,3\},\{1,2,3\},\{1,2\}\} \xrightarrow{b^0} \{\{1,2,3\},\{1,2\},\{1\}\}$

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{123} = \begin{pmatrix} -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \end{pmatrix} \to \begin{pmatrix} A_{123} & -A_{01}-A_{02} & -A_{01} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} A_{012}-A_{01} & -A_{02} & 0 \\ -A_{01} & A_{012}-A_{02} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{12} & 0 & 0 \\ 0 & A_{12} & -A_{01} \\ 0 & 0 & A_{012} \end{pmatrix}$$

$$\begin{aligned}
[\tilde{A}_{123} : \tilde{A}_{12}] &= \{[A_{123} : A_{12}], [0 : A_{12}], [0 : A_{012}]\} \\
[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_1} &= [0 : A_{012}]|_{\ker A_{01}} \\
[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_2} &= [0 : A_{012}]|_{\ker A_{02}} \\
[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_3} &= [0 : A_{12}]|_{\ker A_{03}} \\
[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_\infty} &= [A_{123} : A_{12}]|_{\ker (A_{0\infty}-\mu)}
\end{aligned}$$

4. $0\ 1\ 2\ 3$ $\qquad \mathcal{I} = \{\{0,1\},\{0,1,2,3\},\{2,3\}\} \xrightarrow{b^0} \{\{1\},\{2,3\},\{2\}\}$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{01} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} A_{012}+\mu & A_{02}+A_{03} & A_{02} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{023}-A_{02} & -A_{03} \\ 0 & -A_{02} & A_{023}-A_{03} \end{pmatrix} \to \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23} & -A_{02} \\ 0 & 0 & A_{023} \end{pmatrix}$$

$$\begin{aligned}
[\tilde{A}_{01} : \tilde{A}_{23}] &= \{[A_{01}+\mu : A_{23}], [0 : A_{23}], [0 : A_{023}]\} \\
[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_1} &= [A_{01}+\mu : A_{23}]|_{\ker A_{01}} \\
[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_2} &= [0 : A_{023}]|_{\ker A_{02}} \\
[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_3} &= [0 : A_{023}]|_{\ker A_{03}} \\
[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_\infty} &= [0 : A_{23}]|_{\ker (A_{0\infty}-\mu)}
\end{aligned}$$

1 の場合

$J\backslash\widetilde{A}$	$\widetilde{01}$	$\widetilde{012}$
01	01+μ	012+μ
012	0	012+μ
0123	0	12
1	01+μ	012+μ
2	0	012+μ
3	0	12
∞	0	12

2 の場合

$J\backslash\widetilde{A}$	$\widetilde{012}$	$\widetilde{12}$
012	012+μ	12
12	012+μ	012
0123	12	12
1	012+μ	012
2	012+μ	012
3	12	12
∞	12	12

3 の場合

$J\backslash\widetilde{A}$	$\widetilde{123}$	$\widetilde{12}$
0123	123	12
123	0	12
12	0	012
1	0	012
2	0	012
3	0	12
∞	123	12

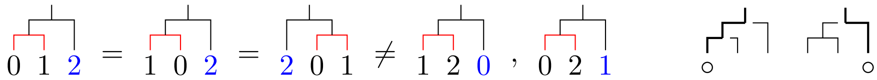
4 の場合

$J\backslash\widetilde{A}$	$\widetilde{01}$	$\widetilde{23}$
01	01+μ	23
0123	0	23
23	0	023
1	01+μ	23
2	0	023
3	0	023
∞	0	23

4　トーナメント戦

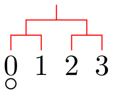
3 teams

2 patterns : 



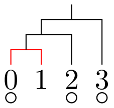
1 type, 2 patterns, 3 (cases of) tournaments, 2 win types

4 teams



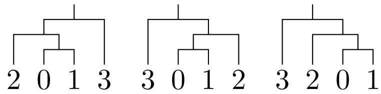
: 1 pattern

: $\frac{4!}{2^3} = 3$ cases



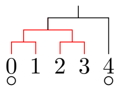
: 4 patterns

: $\frac{4!}{2} = 12$ cases

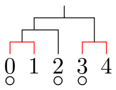


2 types, 5 patterns, 15 tournaments, 4 win types

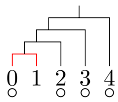
5 teams



: 2 patterns : $\frac{5!}{2^3} = 15$ cases,



: 4 patterns : $\frac{5!}{2^2} = 30$ cases,



: 8 patterns : $\frac{5!}{2} = 60$ cases

3 types, 14 patterns, 105 tournaments, 9 win types

トーナメント戦に関わる場合の数

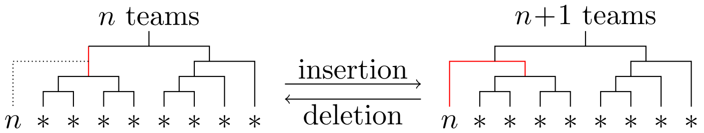
teams	2	3	4	5	6	7	8	9	10	n
patterns	1	2	5	14	42	132	429	1430	4862	$T_n = \frac{(2n-2)!}{n!(n-1)!}$
win types	1	2	4	9	20	46	106	248	582	W_n
types	1	1	2	3	6	11	23	46	98	U_n
tournaments	1	3	15	105	945	10395	135135	2027025	34459425	$K_n = (2n-3)!!$

$$T_n = \sum_{k=1}^{n-1} T_k \cdot T_{n-k}, \quad T_1 = 1, \qquad \text{(patterns)}$$

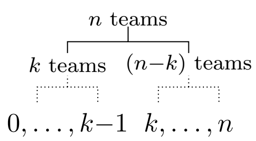
$$W_n = \sum_{1\le k\le n-1} W_k \cdot U_{n-k}, \quad W_1 = 1, \qquad \text{(win types)}$$

$$U_n = \frac{1}{2}\Big(\sum_{1\le k\le n-1} U_k \cdot U_{n-k} \ \big(+U_{\frac{n}{2}} \text{ if } n \text{ is even}\big) \Big), \quad U_1 = 1, \qquad \text{(types)}$$

$$K_n = \frac{1}{2}\sum_{k=1}^{n-1} {}_nC_k \cdot K_k \cdot K_{n-k}, \quad K_1 = 1. \qquad \text{(tournaments)}$$



$2n-1$ vertical line segments $\Rightarrow K_{n+1} = (2n-1)K_n \Rightarrow K_n = (2n-3)!!$.



$$1 \le k < n \ \Rightarrow \begin{cases} T_n \leftarrow T_k \cdot T_{n-k} \\ W_n \leftarrow W_k \cdot U_{n-k} \end{cases}$$

$$k < n-k \ \Rightarrow \begin{cases} U_n \leftarrow U_k \cdot U_{n-k} \\ K_n \leftarrow {}_nC_k \cdot K_k \cdot K_{n-k} \end{cases}$$

$$k = n-k \ \Rightarrow \begin{cases} U_n \leftarrow \frac{1}{2}U_k(U_k-1) + U_k, \\ K_n \leftarrow {}_nC_k\big(\frac{1}{2}K_k(K_k-1)\big) + \big(\frac{1}{2}{}_nC_k\big)K_k \end{cases}$$

Identities

$$K_n = (2n-3)!! = \frac{1}{2}\sum_{k=1}^{n-1} {}_nC_k \cdot (2k-1)!! \cdot (2n-2k-1)!!$$

$$T_n = C_{n-1} = \frac{(2n-2)!}{(n-1)!n!} = \sum_{k=1}^{n-k} \frac{(2k-2)!}{(k-1)!k!}\frac{(2(n-k)-2)!}{(n-k-1)!(n-k)!}$$

$$1 = \Big(1-\sum_{k=1}^{\infty} U_n x^k\Big)\Big(\sum_{\ell=0}^{\infty} W_{\ell+1}x^{\ell}\Big),$$

C_n : Catalan number

U_n : Wedderburn-Etherington number

5 定理 3 の証明のキーポイント

$$\begin{pmatrix} A_{12}{+}A_{02} & -A_{02} & 0 \\ -A_{01} & A_{12}{+}A_{01} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} + \begin{pmatrix} A_{13}{+}A_{03} & 0 & -A_{03} \\ 0 & A_{13} & 0 \\ -A_{01} & 0 & A_{13}{+}A_{01} \end{pmatrix} + \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23}{+}A_{03} & -A_{03} \\ 0 & -A_{02} & A_{23}{+}A_{02} \end{pmatrix}$$

$$= \begin{pmatrix} A_{0123}{-}A_{01} & -A_{02} & -A_{03} \\ -A_{01} & A_{0123}{-}A_{02} & -A_{03} \\ -A_{01} & -A_{02} & A_{0123}{-}A_{03} \end{pmatrix} \qquad (\Leftarrow A_{12}+A_{13}+A_{23}=A_{123})$$

$$\tilde{A}_{1\cdots k} = \begin{pmatrix} A_{0\cdots k}{-}A_{01} & \cdots & -A_{0k} & & & \\ \vdots & \cdots & \vdots & & & \\ -A_{01} & \cdots & A_{0\cdots k}{-}A_{0k} & & & \\ & & & A_{1\cdots k} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots k} \end{pmatrix}$$

$I \subset L_n \,(=\{1,\ldots,n{-}1\}),\quad i\in I,\quad j\in L_n\setminus I$

$$\tilde{A}_I(v)_J = (A_I v)_J \ \text{ in } \ V_J := \iota_J(\mathbb{C}^N) \qquad (I\subset \forall J\subset L_n)$$

$$\tilde{A}_I(v)_i = (A_{0I}v)_i - (A_{0i}v)_I \equiv (A_{0I}v)_i \mod V_I \qquad (i\in I)$$

$$\tilde{A}_I(v)_j = (A_I v)_j \qquad (j\notin I)$$

$$A_I(v)_{L_n} = (A_I v)_{L_n}$$

$$[\tilde{A}_I] = [A_I]_{n-|I|} \cup [A_{0I}]_{|I|-1}$$

$$\tilde{A}_I(v)_i = (A_{0I}v)_i \qquad (v\in\ker A_{0i})$$

$$\tilde{A}_{0\cdots k-1} = \begin{pmatrix} A_{01\cdots k-1}+\mu & & & A_{0k} & \cdots & A_{0n-1} \\ & \ddots & & \vdots & \cdots & \vdots \\ & & A_{01\cdots k-1}+\mu & A_{0k} & \cdots & A_{0n-1} \\ & & & A_{1\cdots k} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots k} \end{pmatrix}$$

$I \subset L_n \quad (I=\{1,\ldots k{-}1\}\uparrow),\quad i\in I,\quad j\in L_n\setminus I$

$$\tilde{A}_{0I}(v)_i = ((A_{0I}+\mu)v)_i$$

$$\tilde{A}_{0I}(v)_j = (A_{0j}v)_I + (A_I v)_j \equiv (A_I v)_j \mod V_I$$

$$(A_{0I}+\mu)v + \sum_{\nu\in L_n\setminus I} A_{0\nu}v = A_I v + \Big(\sum_{\nu=1}^{n-1} A_{0\nu}+\mu\Big)v$$

$$\equiv A_I v \mod \ker(A_{0\infty} - \mu)$$

$$[\tilde{A}_{0I}] = [A_I]_{n-1-|I|} + [A_{0I} + \mu]_{|I|}$$

補題. $W^{(\ell)} := W_I := \sum_{\nu=1}^{\ell} V_{b^0(I^{(\nu)})} \quad (I = I^{(\ell)} \in \mathcal{I})$ とおくと

(i) $0 \notin I \in \mathcal{I}$ に対し $V_I, V_{b^0(I)}, V_{I \setminus b^0(I)} \subset W_I$.

(ii) $\dim W^{(\ell)} = \ell N \quad (\ell = 1, \ldots, n-1)$.

(iii) $I, K \in \mathcal{I}$ とし，$J = b^0(K)$ とおく．$v \in \mathbb{C}^N$ に対し

$0 \in I$ ならば

$$\tilde{A}_I(v)_J = \begin{cases} (A_{I \setminus \{0\}} v)_J & (I \supset K), \\ \big((A_I + \mu)v\big)_J + \big(\sum_{\nu \in J} A_{0\nu} v\big)_{I \setminus \{0\}} & (I \not\supset K). \end{cases}$$

$0 \notin I$ ならば

$$\tilde{A}_I(v)_J = \begin{cases} (A_I v)_J & (I \not\supset K), \\ (A_{0I} v)_J - \big(\sum_{\nu \in J} A_{0\nu} v\big)_I & (I \supset K). \end{cases}$$

(iv) $\tilde{A}_I W^{(\ell)} \subset W^{(\ell)} \quad (I \in \mathcal{I},\ 1 \le \ell < n)$.

6 補足

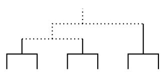
KZ 型方程式とトーナメント戦の対応

n **変数の KZ 型方程式**	n **チームのトーナメント戦**
極大可換留数行列族	トーナメント戦
KZ 型方程式のスペクトル	トーナメント戦の全体
特異点	準決勝
特異点解消	準決勝に至るまでのトーナメント戦の結果
middle convolution を行う変数	優勝者を決める
極大可換留数行列族の上三角化の基底	トーナメント戦の各ゲームの勝者を決める
middle convolution	優勝チームの deletion と insertion
middle convolution の定義の kernel	上で basic/top insertion
さらに m 個の固定特異点がある場合	n チームを m 個にグループ分け

例 5. 変数 x_0, x_1, x_2, 固定特異点 y_3, y_4, y_5 $\Rightarrow$ 105 個の極大可換留数行列族

$$\frac{\partial u}{\partial x_i} = \sum_{\substack{0 \le \nu \le 2 \\ \nu \ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u + \sum_{\nu=3}^{5} \frac{A_{i\nu}}{x_i - y_\nu} u \quad (i = 0, 1, 2)$$

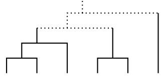
$\overset{3}{\{0\}} \sqcup \overset{4}{\{1\}} \sqcup \overset{5}{\{2\}}$



0 **3** 1 **4** 2 **5**

$\{\{0,3\},\{1,4\},\{2,5\}\}$

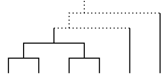
$\overset{3}{\{0,2\}} \sqcup \overset{4}{\{1\}} \sqcup \overset{5}{\emptyset}$



0 **3** 2 1 **4** **5**

$\{\{0,3\},\{1,4\},\{0,2,3\}\}$

$\overset{3}{\{0,1,2\}} \sqcup \overset{4}{\emptyset} \sqcup \overset{5}{\emptyset}$



0 1 2 **3** **4** **5**

$\{\{0,1\},\{2,3\},\{0,1,2,3\}\}$

例 6. Dettweiler–Reiter [DR] は，変数が 1 個でいくつかの固定特異点がある場合に対応する．

例 7. [MO, Oi] で考察されたレベル 0 の多変数の一般超幾何級数

$$\sum_{(m_1,\ldots,m_n) \in \mathbb{Z}_{\ge 0}^n} \frac{(\boldsymbol{\alpha}_1)_{m_1} \cdots (\boldsymbol{\alpha}_n)_{m_n} (\boldsymbol{\gamma})_{m_1 + \cdots + m_n}}{(1 - \boldsymbol{\alpha}'_1)_{m_1} \cdots (1 - \boldsymbol{\alpha}'_n)_{m_n} (1 - \boldsymbol{\gamma}')_{m_1 + \cdots + m_n}} x_1^{m_1} \cdots x_n^{m_n},$$

$$(a)_m := a(a+1)\cdots(a+m-1), \quad (\boldsymbol{\alpha}_1)_{m_1} := (\alpha_{i,0})_{m_1} \cdots (\alpha_{i,p_i-1})_{m_i},$$

$$\boldsymbol{\alpha}_i,\ \boldsymbol{\alpha}'_i \in \mathbb{C}^{p_i},\ \boldsymbol{\beta}_j,\ \boldsymbol{\beta}'_i \in \mathbb{C}^{q_j},\ \boldsymbol{\gamma}_k,\ \boldsymbol{\gamma}'_k \in \mathbb{C}^q,\ \alpha'_{i,0} = \beta'_{j,0} = 0$$

は rigid な KZ 型方程式の解となる．一方，レベル 1 の多変数一般超幾何級数の満たす方程式の特異集合も超平面の合併であるが，braid arrangement とは限らない．一般の超平面配置に対数型特異点を持つ Pfaff 系として [Oh] で考察される．なお，[MO] では上の超幾何級数の定義式において $\alpha'_i \in \mathbb{C}^{p_i+L}$，$\beta'_j \in \mathbb{C}^{q_j+L}$，$\gamma'_k \in \mathbb{C}^{r_k-L}$ としたものを考察し，整数 L をレベルとよんだ．

問題. 自明な方程式 $u' = 0$ から addition と middle convolution を繰り返してできる既約な KZ 型方程式 $\mathcal{M}$ は rigid となる（すなわち Sp $\mathcal{M}$ から $\mathcal{M}$ が一意に定まる）が，それ以外に既約で rigid な KZ 型方程式は存在するか？ より一般に，既約な KZ 型方程式の空間において，addition, middle convolution および KZ 型を保つ座標変換から生成される群の軌道を決定せよ．

不確定特異点をもつ KZ 型方程式に対して Sp $\mathcal{M}$ を拡張せよ (cf. [Ov]).

一般の超平面配置に対数的特異点をもつ Pfaff 系に対しても，同様な問題を考察せよ (cf. [Oh]).

パラメータを変数とみなしたシフト作用素も含めて方程式系を統一的に考察せよ（cf. [Os]）.

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