PALEY-WIENER THEOREMS ON A SYMMETRIC SPACE AND ITS APPLICATION

Toshio OSHIMA (University of Tokyo)
Yutaka SABURI (Chiba Jr. College)
Masato WAKAYAMA (Tottori University)

Table of Contents

0. Introduction

The principal aim of this paper is to study the Fourier analysis on Riemannian symmetric spaces. In particular, the Paley-Wiener theorems of some types of function spaces on the symmetric space will be discussed throughout the paper. As the application of these results, the solvability of the single differential equation defined by the invariant differential operator on the symmetric space will be shown in the last section. Furthermore, in the forthcoming paper we will discuss Eherenpreis' fundamental principle on Riemmanian symmetric space. The statement of this fundamental principle is appeared in [OSW1]. In this point of view, the results of this paper is the preliminary investigation for the proof of the fundamental principle.

We introduce distributions and hyperfunctions of exponential type on a Riemannian symmetric space and discuss their Fourier transforms. Our study is originated from the preceding works on two subjects. One is the theory of the Fourier transformation on a Riemannian symmetric space developed by Harish-Chandra, Helgason, Trombi-Varadarajan, Kostant, Eguchi and others. The another one is the theory of the Fourier transforms of distributions and hyperfunctions of some types on a Euclidean space. As to the theory of the Fourier transformation of distributions of exponential type on a Euclidean space, there are works by Hasumi, Sebastiaõe Silva and others. As to the theory of Fourier transforms of hyperfunctions of some types on a Euclidean space, there are works by Sato, Kawai, Morimoto, Zharinov, Nagamachi, Saburi, Kaneko and others. The basic means and facts on introducing these generalized functions are of the establishment of Eherenpreis' fundamental principle.

Now we explain the series of contents of this paper. In §1 we set up the notations and basic facts about the real reductive linear Lie group and define a Riemmanian symmetric space which we deal with throughout the paper briefly. In §2 we introduce the notion of the invariant differential operator of infra exponential type which is, in fact, the infinite order differential operator on our symmetric space. Using these differential operator we define the function space $\mathcal{A}_*(G/K)$. Also we define the function space $\mathcal{C}_*(G/K)$. This space is the inductive limit of the space of L^p Shwartz functions discussed in [Eg1] and has the structure of the FS space. The space $\mathcal{A}_*(G/K)$ is the subspace of $\mathcal{C}_*(G/K)$ and it consists to the real analytic functions those are belong to $\mathcal{C}_*(G/K)$. In §3 we review the Fourier-Laplace transformations on symmetric spaces and state the main results. We devote the proof of the theorem in the case of $\mathcal{C}_*(G/K)$ in §4. Since the proof of it is owed to the results of [Eg1], we review it in this section. In order to prove the theorem in the case of $\mathcal{A}_*(G/K)$, we must get the knowledge of the Fourier series of the analytic functions on the isotropy group at the origin of the symmetric space. We discuss them in detail in §5. By means of the theorem in the case of $\mathcal{C}_*(G/K)$ and the results of §5, we can get the proof of the theorem in the case of $\mathcal{A}_*(G/K)$ in §6. For the sake of studies in the forthcoming paper, we describe the Fourier coefficients of the Fourier-Laplace images of our function spaces on the boundary of the symmetric space. In the last section we derive some results on the solvability of a single differential equation on the symmetric space.

In Appendix 1, we devote to study the fundamental lemma for our discussion on the differential operator of infra exponential type. The elementary lemmas on the infra exponential functions and the important properties of the differential operators of infra exponential type deduced from the results in Appendix 1 is collected in Appendix 2.

1. Notation and preliminaries

We use the standard notation $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the ring of integers, the field of real numbers and the field of complex numbers, respectively. We denote by $\mathbb{N}$ the set of nonnegative integers. For any finite set F, we denote the number of elements of F by $\#F$. For a C^∞ manifold V, we denote by $C^\infty(V)$ the space of C^∞ functions on V. Let G be a connected real reductive linear Lie group and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Then $\mathfrak{g} = \mathfrak{c} + \mathfrak{g}_1$ where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}_1$ is the derived algebra of $\mathfrak{g}$. Let θ be the Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Let $\mathfrak{p}$ be the subspace of $\mathfrak{g}$ corresponding to the eigenvalue -1 of θ. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. If we put $\mathfrak{c}_\mathfrak{k} = \mathfrak{c} \cap \mathfrak{k}$ and $\mathfrak{c}_\mathfrak{p} = \mathfrak{c} \cap \mathfrak{p}$, we have $\mathfrak{c} = \mathfrak{c}_\mathfrak{k} + \mathfrak{c}_\mathfrak{p}$. Let $\langle\ ,\ \rangle_\mathfrak{k}$ (resp. $\langle\ ,\ \rangle_\mathfrak{p}$) be a positive definite symmetric bilinear form on $\mathfrak{c}_\mathfrak{k}$ (resp. $\mathfrak{c}_\mathfrak{p}$). Let $B(\ ,\ )$ denote the Killing form on $\mathfrak{g}_1$. We define and fix an $Ad(G)$ invariant form $Q(\ ,\ )$ on $\mathfrak{g}$ by the following equation:

$$Q(X_1 + Y_1 + Z_1, X_2 + Y_2 + Z_2) = B(X_1,\, X_2) + \langle Y_1, Y_2\rangle_\mathfrak{k} + \langle Z_1, Z_2\rangle_\mathfrak{p}$$
$$\text{for } X_i \in \mathfrak{g}_1, Y_i \in \mathfrak{c}_\mathfrak{k} \text{ and } Z_i \in \mathfrak{c}_\mathfrak{p} \qquad (i = 1, 2).$$

It is clear that $Q(\ ,\ )$ defines a scalar product on $\mathfrak{p}$. For any subspace $\mathfrak{l}$ of $\mathfrak{g}$, we denote by $\mathfrak{l}_c$ and $\mathfrak{l}^*$ the complexification and the real dual space of $\mathfrak{l}$ respectively. Furthermore denote by $\mathfrak{l}_c^*$, the complexification of $\mathfrak{l}^*$. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and introduce a linear order in $\mathfrak{a}^*$. Let Σ and Σ^+ be the set of all restricted roots and restricted positive roots, respectively. For any root α in Σ, we denote by $\mathfrak{g}_\alpha$ the root space in $\mathfrak{g}$ corresponding to α and by m_α the dimension of $\mathfrak{g}_\alpha$. We put $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ and $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. Let A and N be the analytic subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then we have the Iwasawa decomposition $G = KAN$ (resp. $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$) of G (resp. $\mathfrak{g}$). For any g in G, we denote $\kappa(g)$ and $H(g)$ for the unique elements k in K and H in $\mathfrak{a}$ such that $g = k \exp Hn$ where n in N. For any a in A, we write $H(a)$ by $\log a$. Let M' and M be the normalizer and centralizer of A in K, respectively. The quotient group $W = M'/M$ is the Weyl group and acts on $\mathfrak{a}_c, \mathfrak{a}^*$ and $\mathfrak{a}_c^*$ in the obvious way. Let l denote a real rank of G, that is $l = \dim A$. The form $Q(\ ,\ )$ induces Euclidean measures on A and $\mathfrak{a}^*$; multiplying these by the factor $(2\pi)^{-l/2}$ we have measures da and $d\lambda$ on A and $\mathfrak{a}^*$, respectively, and the Fourier transform

$$f^*(\lambda) = \int_A f(a) \exp -i\lambda(\log a) da \qquad (\lambda \in \mathfrak{a}_c^*)$$

is inverted without any multiplicative constant, where i denotes the square root of -1. The Haar measures dk on K and dm on M are so normalized that the total measures are one, respectively. Moreover, we denote the induced invariant measure on K/M by dkM. The Haar measures of the nilpotent groups N and $\overline{N} = \theta(N)$ are normalized so that $\theta(dn) = d\overline{n}$ and the integral of $\exp -2\rho(H(\overline{n}))$ with respect to $d\overline{n}$ over $\overline{N}$ is one. Then the Haar measure dg on G can be normalized with $dg = \exp 2\rho(\log a) dk da dn$ $(g = kan)$. The homogeneous space G/K is a Riemannian symmetric space. By means of the polar decomposition of G, there is a real analytic diffeomorphism:

$$\exp : \mathfrak{p} \ni X \stackrel{\sim}{\longmapsto} (\exp X)K \in G/K.$$

Also, we can define a norm on G/K by

$$|x| = Q(\exp^{-1}(x), \exp^{-1}(x))^{\frac{1}{2}} \qquad (x \in G/K).$$

2. Function space $\mathcal{C}_*(G/K)$ and $\mathcal{A}_*(G/K)$

Let $X_1, X_2, \cdots, X_m$ be an ordered basis of $\mathfrak{g}$. X_j determine right G-invariant vector fields on G:

$$(X_j f)(g) = \frac{d}{dt} f(\exp(-tX_j)g)|_{t=0} \qquad (g \in G),$$

where f is a C^∞ function on G. They act also on functions on G/K which are identified with right K-invariant functions on G. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}_c$. Naturally, each element of $U(\mathfrak{g})$ is regarded as a right invariant differential operator on G. We put $X = (X_1, \dots, X_m)$. For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, we define a differential operator X^α on G/K by $X^\alpha = X_1^{\alpha_1} \cdots X_m^{\alpha_m}$.

For a finite dimensional $\mathbb{C}$-vector space V, we denote by $\mathcal{O}(V)$ the space of entire holomorphic functions on V. A function $f(z)$ in $\mathcal{O}(V)$ is called to be infra-exponential type if it satisfies the growth condition:

$$\lim_{|z|\to\infty} f(z)e^{-\varepsilon|z|} = 0 \qquad \text{for any } \varepsilon > 0,$$

where $|z|$ denotes a norm of z in V. Let $\widetilde{\mathcal{O}}(V)$ be the set of all functions of infra-exponential type on V.

Definition 1. *Let $\{X_1, \dots, X_m\}$ be an ordered basis of $\mathfrak{g}$. We call the right G-invariant differential operator $J(X) = \sum_{\alpha\in\mathbb{N}^m} C_\alpha X^\alpha$ of infinite order is of infra exponential type if and only if the function $J(\xi) = \sum_{\alpha\in\mathbb{N}^m} C_\alpha \xi^\alpha$ with $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{C}^m$ belongs to $\widetilde{\mathcal{O}}(\mathbb{C}^m)$. Further, we denote by $\widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$ the set of all right G-invariant differential operators of infra exponential type on G.*

The following lemma can be proved directly from Lemma 1 in Appendix 1 and guarantees the well-definedness of $\widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$.

Lemma 1. *Let $\{X_1, \dots, X_m\}$ and $\{Y_1, \dots, Y_m\}$ be two ordered basis of $\mathfrak{g}$ and we put*

$$Y_i = \sum_{j=1}^{m} a_{i,j} X_j \qquad \text{for } i = 1, \dots, m.$$

Suppose that $J(Y) = \sum_{\alpha\in\mathbb{N}^m} C_\alpha Y^\alpha$ is a right G-invariant differential operator of infra exponential type. If we put

$$\begin{aligned}\tilde{J}(X) &= \sum_{\alpha\in\mathbb{N}^m} \tilde{C}_\alpha X^\alpha \\ &= \sum_{\alpha\in\mathbb{N}^m} C_\alpha \big(\sum a_{1,j} X_j\big)^{\alpha_1} \cdots \big(\sum a_{m,j} X_j\big)^{\alpha_m},\end{aligned}$$

then $\tilde{J}(X)$ is also the right G-invariant differential operator of infra exponential type.

The elementary and necessary facts on the functions and differential operators of infra exponential type are collected in Appendix 2.

If $J(X) = \sum_{\alpha\in\mathbb{N}^m} C_\alpha X^\alpha$ belongs to $\widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$, then we put $J_\alpha(X) = C_\alpha X^\alpha$ for all $\alpha \in \mathbb{N}^m$. We now introduce the following two function spaces on G/K. These are main objects of our study throughout this paper.

$$\mathcal{C}_*(G/K) = \{\phi \in C^\infty(G/K); \|\phi\|_{\alpha,r} = \sup_{x\in G/K} |(X^\alpha\phi)(x)|e^{r|x|} < \infty \quad \text{for } \forall\alpha \in \mathbb{N}^m,\ \forall r \geq 0\}$$

$$\mathcal{A}_*(G/K) = \{\phi \in C^\infty(G/K); \|\phi\|_{J,r} = \sup_{x\in G/K}\sup_{\alpha\in\mathbb{N}^m} |(J_\alpha(X)\phi)(x)|e^{r|x|} < \infty \quad \text{for } \forall J \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*),\ \forall r \geq 0\}$$

It is easy to see that $\mathcal{A}_*(G/K)$ is a subspace of $\mathcal{C}_*(G/K)$ and the elements of ϕ of $\mathcal{A}_*(G/K)$ are real analytic on G/K. The space $\mathcal{C}_*(G/K)$ is a so-called FS space. We denote the dual space of $\mathcal{C}_*(G/K)$ (resp. $\mathcal{A}_*(G/K)$) by $\mathcal{C}'_*(G/K)$ (resp. $\mathcal{A}'_*(G/K)$) and call its element a distribution (resp. hyperfunction) of exponential type on G/K.

Remark 1. *Let $\widetilde{G/K}$ be the compactification of G/K discussed in* [O1] *(see* [Sc]*).* Then the element in $\mathcal{C}'_*(G/K)$ is characterized by the fact that it is in fact the restriction of a distribution on $\widetilde{G/K}$.

Remark 2. *Suppose G is abelian. Then as a linear space $\mathcal{A}_*(G/K)$ is nothing but the inductive limit of the following spaces:*

$$\mathcal{A}_*(G/K) = \operatorname*{inj\,lim}_{\varepsilon>0} \mathcal{O}_E(\mathbb{R}^m \times i(-\varepsilon,\varepsilon)^m),$$

where

$$\mathcal{O}_E(\mathbb{R}^m \times i(-\varepsilon,\varepsilon)^m) = \{f \in \mathcal{O}(\mathbb{R}^m \times i(\varepsilon,\epsilon)^m); \text{ for } \forall r > 0, \exists\delta > 0 \text{ such that } \sup_{z\in\mathbb{R}^m\times i(-\delta,\delta)^m} |f(z)|e^{r|z|} < \infty\}.$$

Here $\mathcal{O}(\mathbb{R}^m \times i(-\varepsilon,\varepsilon)^m)$ stands for the space of holomorphic functions of the tube domain $\mathbb{R}^m \times i(-\varepsilon,\varepsilon)^m$.

Remark 3. *A C^∞-function ϕ on G/K belongs to $\mathcal{A}_*(G/K)$ if and only if*

$$\sup_{x\in G/K} |\sum_{\alpha\in\mathbb{N}^m} C_\alpha(X^\alpha f)(x)|e^{r|x|} < \infty$$

for any $J(X) = \sum_{\alpha\in\mathbb{N}^m} C_\alpha X^\alpha \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$ and $r \geq 0$ (See Appendix 2, Lemma 2). Moreover, the topology of $\mathcal{A}_*(G/K)$ is independent to the choice of our ordered base $\{X_1,\dots,X_m\}$. This is non-trivial fact, but in the proof of the Paley-Wiener theorem (see §6), we will see that it is the legitimate one.

3. Fourier-Laplace transformations and main results.

The Fourier-Laplace transform $F\phi$ of $\phi \in \mathcal{C}_*(G/K)$ is a function on $\mathfrak{a}_c^* \times K/M$ and defined by

$$F\phi(\lambda : kM) = \int_G \phi(x) e^{(i\lambda-\rho)H(x^{-1}k)} dx \qquad \text{for } (\lambda, kM) \in \mathfrak{a}_c^* \times K/M.$$

We now introduce the following two function spaces on $\mathfrak{a}_c^* \times K/M$:

$$\mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M) = \{\Phi \in C^\infty(\mathfrak{a}_c^* \times K/M); \Phi \text{ is holomorphic in } \lambda \in \mathfrak{a}_c^* \text{ and } \|\Phi\|^{j,r} < \infty \text{ for } \forall j \in \mathbb{N},\ \forall r \geqq 0\}$$

and

$$\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M) = \{\Phi \in \mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M); \|\Phi\|^{J,r} < \infty \text{ for } \forall J = (J_1, J_2) \in \widetilde{\mathcal{O}}(\mathfrak{a}_c^*) \times \mathcal{O}_2(\mathbb{C}),\ \forall r \geqq 0\}$$

where, denoted by $\Delta_{K/M}$ the Laplacian on K/M,

$$\|\Phi\|^{j,r} = \sup_{\substack{|\operatorname{Im}\lambda| \leqq r \\ k \in K}} |\Delta_{K/M}^j \Phi(\lambda : kM)|(1 + |\lambda|)^j,$$

$$\|\Phi\|^{J,r} = \sup_{\substack{|\operatorname{Im}\lambda| \leqq r \\ k \in K}} \sup_n |J_1(\lambda) J_{2,n}(\Delta_{K/M}) \Phi(\lambda : kM)|$$

and

$$\mathcal{O}_2(\mathbb{C}) = \{J \in \mathcal{O}(\mathbb{C}); J(z^2) \text{ is of infra-exponential type}\}.$$

Of course, the meaning of the differential operator $J_2(\Delta_{K/M})$ of infinite order is similar to that of $J(X)$ is the definition of $A_*(G/K)$, and if $J_2(\Delta_{K/M})$ is of the form $J_2(\Delta_{K/M}) = \sum_{n=0}^\infty c_n \Delta_{K/M}^n$, then we put $J_{2.n}(\Delta_{K/M}) = c_n \Delta_{K/M}^n$.

Let U be the subset of $\mathfrak{a}_c^*$. For a continuous function Φ on $U \times K/M$ we define its Poisson integral $\check{\Phi}$ by

$$\check{\Phi}(\lambda : x) = \int_K \Phi(\lambda : kM) e^{-(i\lambda+\rho)H(x^{-1}k)} dk \qquad \text{for } (\lambda, x) \in U \times G.$$

We put

$$\mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W = \{\Phi \in \mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M); \check{\Phi}(w\lambda : x) = \check{\Phi}(\lambda : x) \text{ for } \forall w \in W,\ \forall (\lambda, x) \in \mathfrak{a}_c^* \times G\}$$

and

$$\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W = \mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M) \cap \mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W.$$

Then it is clear that $\mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W$ (resp. $\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W$) is the closed subspace of $\mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)$ (resp. $\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)$).

Now we state the following Paley-Wiener theorem.

Theorem 1. *Let G be a connected real reductive linear Lie group and K its maximal compact subgroup. Then we have the following linear topological isomorphisms:*

$$F: \mathcal{C}_*(G/K) \tilde{\to} \mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W.$$
$$F: \mathcal{A}_*(G/K) \tilde{\to} \mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W.$$

The inverse F^{-1} of F is given by

$$F^{-1}\Phi(xK) = \frac{1}{\#W} \int_{\mathfrak{a}_c^*} \check{\Phi}(\lambda, x)|c(\lambda)|^{-2} d\lambda \qquad \text{for } x \in G,$$

where $c(\lambda)$ is Harish-Chandra's c-function for the principal series of class 1 with respect to K.

Remark 4. *As to the theory of Fourier transformation of distributions of exponential type on the Euclidean space, there are works by* [Ha], [M] [SS] *and others. Also the Fourier hyperfunctions on the Euclidean space was studied in* [Sat] [Kaw] [Kan] *and others.*

4. Proof of the theorem in the case of $\mathcal{C}_*(G/K)$.

We can get the proof of the theorem with respect to $\mathcal{C}_*(G/K)$ thanks to Theorem 4.1.1 in [Eg 1]. Also, our proof of the theorem in the case of $\mathcal{A}_*(G/K)$ is owed to the works of [Kan] in addition to that of [Eg 1]. So we review the result of it in the first place.

In this section, we will identify $U(\mathfrak{g})$ with the algebra of left-invariant differential operators on G, as usual. It is known that there exists an anti isomorphism ι from $U(\mathfrak{g})$ with the algebra of right-invariant differential operators on G. If f is a function on G and $u, u' \in U(\mathfrak{g})$, then we write

$$((^\iota u)u'f)(x) = f(u : x; u') \qquad (x \in G).$$

For any element v of the symmetric algebra $S(\mathfrak{a}_c^*)$ over $\mathfrak{a}_c^*$, let $\partial(v)$ denote the corresponding differential operator on $\mathfrak{a}^*$. Then $S(\mathfrak{a}_c^*)$ can be regarded as the algebra of all differential operators with constant coefficients on $\mathfrak{a}_c^*$.

Let $0 < p \leq 2$ and let $\mathcal{C}^p(G/K)$ denote the set of C^∞ function ϕ on G which satisfy the following conditions: (i) $\phi(xk) = \phi(x)$ for any $x \in G$ and $k \in K$; (ii) for any $j \in \mathbb{N}$ and $u, u' \in U(\mathfrak{g})$,

$$\tau^p_{u,u',j}(\phi) = \sup_{x \in G} |\phi(u : x; u')| \Xi^{-2/p}(x)(1 + |x|)^j < \infty,$$

where Ξ is the zonal spherical function defined by

$$\Xi(x) = \int_K e^{-\rho(H(xk))} dk \qquad (x \in G).$$

The seminorms $\tau^p_{u,u',j}$ convert $\mathcal{C}^p(G/K)$ into an FS space.

Put $\epsilon = 2/p - 1$. We define the tube domain F^p by setting

$$F^p = \{\lambda \in \mathfrak{a}_c^*; |\operatorname{Im}(w\lambda)(H)| \leq \epsilon\rho(H) \text{for any} H \in \mathfrak{a}^+ \text{and any} w \in W\},$$

where $\mathfrak{a}^+$ denotes the positive Weyl chamber of $\mathfrak{a}$. We denote the interior of the domain F^p by $\operatorname{Int} F^p$. For brevity of notation, we put $F^2 = \operatorname{Int} F^2 = \mathfrak{a}^*$. We define $\mathcal{Z}^p(\mathfrak{a}^* \times K/M)$ to the space of all C^∞ complex valued functions Φ on $\mathfrak{a}^* \times K/M$ which satisfy the following conditions : (i) For any $k \in K$ the function $\lambda \to \Phi(\lambda : kM)$ extends holomorphically to $\operatorname{Int} F^p$; (ii) for any $(n, m) \in \mathbb{N}^2$ and $v \in S(\mathfrak{a}_c^*)$,

$$\zeta^p_{v,n,m}(\Phi) = \sup_{\lambda \in \operatorname{Int} F^p, k \in K} |\Phi(\lambda; \partial(v) : kM; \Delta^m_{K/M})|(1 + |\lambda|)^n < \infty,$$

where the condition (i) is omitted when $p = 2$. The seminorms $\zeta^p_{v,n,m}$ convert $\mathcal{Z}^p(\mathfrak{a}^* \times K/M)$ into an FS space.

For any element Φ in $\mathcal{Z}^p(\mathfrak{a}^* \times K/M)$, the Poisson integral $\check{\Phi}$ is a function on $\operatorname{Int} F^p \times G$. We denote by $\mathcal{Z}^p(\mathfrak{a}^* \times K/M)_W$ the closed subspace of all elements Φ of $\mathcal{Z}^p(\mathfrak{a}^* \times K/M)$ which satisfy the condition $\check{\Phi}(w\lambda : x) = \check{\Phi}(\lambda : x)$ for all $x \in G, \lambda \in \operatorname{Int} F^p$ and $w \in W$. The following theorem, which is fundamental for our proof of the theorem, is the main result of [Eg 1].

Lemma 2. *Let $0 < p \leq 2$. Then the Fourier-Laplace transform F is a linear topological isomorphism of $\mathcal{C}^p(G/K)$ onto $\mathcal{Z}^p(\mathfrak{a}^* \times K/M)_W$.*

By the definition of $\mathcal{Z}^p(\mathfrak{a}^* \times K/M)_W$, it is clear that

$$\mathcal{Z}^p(\mathfrak{a}^* \times K/M)_W \subset \mathcal{Z}^q(\mathfrak{a}^* \times K/M)_W$$

if $0 < p \leq q \leq 2$. So we can consider the projective limit of them. We define

$$\widetilde{\mathcal{Z}}(\mathfrak{a}_c^* \times K/M)_W = \operatorname*{proj\,lim}_{0<p\leq 2} \mathcal{Z}^p(\mathfrak{a}^* \times K/M)_W.$$

Then $\widetilde{\mathcal{Z}}(\mathfrak{a}_c^* \times K/M)_W$ is also an FS space. Since each element Φ of $\widetilde{\mathcal{Z}}(\mathfrak{a}_c^* \times K/M)_W$ is an entire holomorphic function with respect to the variable λ, it is easy to see that the topology of this space is given by the seminorms

$$\widetilde{\zeta}_{j,r}(\Phi) = \sup_{|\operatorname{Im}\lambda|\le r, k\in K} |\Phi(\lambda : kM; \Delta^j_{K/M})|(1+|\lambda|)^j \qquad (j\in\mathbb{N}, r\ge 0).$$

This means that $\widetilde{\mathcal{Z}}(\mathfrak{a}_c^* \times K/M)_W$ coincides with our space $\mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W$ defined in §3.

Moreover we define

$$\widetilde{\mathcal{C}}(G/K) = \operatorname*{proj\,lim}_{0<p\le 2} \mathcal{C}^p(G/K).$$

It is well known that there exist constants $c>0$ and $d>0$ such that for all $h\in\mathcal{C}\ell(A^+)$,

$$e^{-\rho(\log h)} \le \Xi(h) \le c\, e^{-\rho(\log h)}(1+|h|)^d, \tag{4.1}$$

where $A^+ = \exp\mathfrak{a}^+$ and $\mathcal{C}\ell(A^+)$ stands for the closure of A^+. Using this estimate, we can easily find that $\widetilde{\mathcal{C}}(G/K)$ consists of the functions $\phi\in C^\infty(G)$ which satisfy the conditions: (i) $\phi(xk)=\phi(x)$ for any $x\in G$ and $k\in K$; (ii) for any $r\ge 0$ and $u,u'\in U(\mathfrak{g})$,

$$\tau_{r,u,u'}(\phi) = \sup_{x\in G} |\phi(u:x;u')|e^{r|x|} < \infty.$$

Of course, $\widetilde{\mathcal{C}}(G/K)$ is an FS space under the system of seminorms $\tau_{r,u,u'}$. Furthermore, by means of Lemma 2, we see that the Fourier-Laplace transform F gives a linear topological isomorphism of $\widetilde{\mathcal{C}}(G/K)$ onto $\mathcal{Z}_*^C(\mathfrak{a}_c^*\times K/M)_W$.

Therefore, in order to prove the Paley-Wiener theorem in the case of $\mathcal{C}_*(G/K)$, it is sufficient to show that $\widetilde{\mathcal{C}}(G/K)$ coincides with $\mathcal{C}_*(G/K)$. As to the set theoretically, it is clear that

$$\mathcal{C}_*(G/K) \supset \widetilde{\mathcal{C}}(G/K).$$

Hence we nee to show that $\mathcal{C}_*(G/K)\subset\widetilde{\mathcal{C}}(G/K)$. In other words, it suffices to prove that the Fourier-Laplace image of $\mathcal{C}_*(G/K)$ is contained in $\mathcal{Z}_*^C(\mathfrak{a}_c^*\times K/M)_W$.

The following lemma can be proved easily in the similar way of proving Lemma 4.1 and Lemma 4.2 of [EK]. So we omit the proof.

Lemma 3. *Let $\phi \in \mathcal{C}_*(G/K)$. The integral*

$$(F\phi)(\lambda : kM) = \int_G \phi(x)e^{(i\lambda-\rho)H(x^{-1}k)}dx$$
$$= \int_{AN} \phi(kan)e^{(-i\lambda-\rho)(\log a)}dadn$$

is uniformly convergent on $F^p \times K/M$ for any $p(0 < p \leq 2)$, and for any fixed k in K the function $\lambda \to (F\phi)(\lambda : kM)$ is an entire holomorphic function on $\mathfrak{a}_c^*$. Moreover the Poisson integral of $F\phi$ satisfies the following functional equation with respect to the Weyl group:

$$(F\phi)(w\lambda : x) = (F\phi)(\lambda : x) \qquad (w \in W, \lambda \in \mathfrak{a}_c^* and x \in G).$$

For the purpose of proving the theorem, we now prove the following estimate.

Lemma 4. *For any $r \geq 0$ and $(m, n) \in \mathbb{N}^2$ we can select $r' \geq 0$ and a positive constant C such that*

$$\sup_{|\operatorname{Im}\lambda| \leq r, k \in K} |(F\phi)(\lambda : kM; \Delta_{K/M}^m)|(1 + |\lambda|)^{2n}$$
$$\leq C \sup_{x \in G} |\phi(\Delta^n \Delta_{K/M}^m : x)e^{r'|x|},$$

where Δ is the Laplacian on G/K.

Proof. Since Δ is G-invariant differential operator on G/K,

$$\Delta e^{(i\lambda-\rho)(H(x^{-1}k))} = -(|\lambda|^2 + |\rho|^2)e^{(i\lambda-\rho)(H(x^{-1}k))},$$

it is easy to see that

$$(F\Delta^n\phi)(\lambda : kM) = (-1)^n(|\lambda|^2 + |\rho|^2)^n(F\phi)(\lambda : kM).$$

Also, it is clear that $\Delta_{K/M}$ and F commute with each other we obtain

$$(|\lambda|^2 + |\rho|^2)^n(F\phi)(\lambda : kM; \Delta_{K/M}^m)$$
$$= (-1)^n \int_G \phi(\Delta^n \Delta_{K/M}^m : x)e^{(i\lambda-\rho)(H(x^{-1}k))}dx.$$

Hence we have

$$(|\lambda|^2 + |\rho|^2)^n|(F\phi)(\lambda : kM; \Delta_{K/M}^m)|$$
$$\leq \sup_{x \in G}(|\phi(\Delta^n \Delta_{K/M}^m : x)|e^{r'|x|}) \int_G e^{-r'|x|}e^{-(\operatorname{Im}\lambda+\rho)(H(x^{-1}k))}dx.$$

Since the function $x \to |x|$ $(x \in G)$ is the K-invariant function, we have

$$
\begin{aligned}
&\int_G e^{-r'|x|} e^{-(\operatorname{Im}\lambda+\rho)(H(x^{-1}k))} dx \\
&\qquad = \int_{AN} e^{-r'|an|} e^{(\operatorname{Im}\lambda+\rho)(\log a)} da dn \\
(4.2) \qquad &\qquad = \int_G e^{-r'|x|} e^{(\operatorname{Im}\lambda-\rho)(H(x))} dx \\
&\qquad = \int_{A^+K} e^{-r'|a|} e^{(\operatorname{Im}\lambda-\rho)(H(ak))} \delta(a) da dk.
\end{aligned}
$$

Here we used the well known formula $dx = \delta(a)dk'dadk$ $(x = k'ak \in KA^+K)$,where the function $\delta(a)$ on A^+ is defined by

$$\delta(a) = c \prod_{\alpha \in \Sigma^+} \{\sinh \alpha(\log a)\}^{m_\alpha}$$

for a suitable constant c. Since it is known that

$$\int_K e^{(\operatorname{Im}\lambda-\rho)(H(ak))} dk \leq e^{(\operatorname{Im}\lambda)(\log a)} \Xi(a) \qquad (a \in A^+),$$

by means of (4.1) it follows that (4.2) is bounded by

$$c \int_{A^+} e{-}r'|a|(1+|a|)^d e^{(\operatorname{Im}\lambda-\rho)(\log a)} \delta(a) da.$$

By the way, since $|\operatorname{Im}\lambda| \leq r'$, for a sufficiently large positive number r' the last integral is finite. This completes the proof of the lemma.

Combining these lemmas, it is easy to see that

$$F(\mathcal{C}_*(G/K)) \subset \mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W.$$

Hence we have the desired result.

5. Fourier expansions of the function on K/M.

In order to prove the theorem in the case of $\mathcal{A}_*(G/K)$, we review some results on the Fourier series of analytic functions on K.

Let T be a maximal toral subgroup of K. We denote by $\mathfrak{t}$ the Lie algebra of T. Let $\widehat{K}$ denotes the set of equivalence classes of irreducible unitary representations of K. We fix a norm $|\ |$ on $\mathfrak{t}_c^*$, induced by the positive definite bilinear form $Q(\ ,\theta\ )$ on $\mathfrak{k}$. We choose once for all a lexicographic order in $\mathfrak{t}^*$. Since an irreducible unitary representation δ of K is uniquely determined, up to equivalence, by its highest weight $\bar{\delta}\,(\in \mathfrak{t}_c^*)$, we will identify δ with its highest weight $\bar{\delta}$, throughout this paper. For $\delta \in \widehat{K}$ we take a Hilbert space V_δ which realizes a representation δ of K. Let $d(\delta)$ and χ_δ denote the dimension of V_δ and the character of δ, respectively. For each $\delta \in \widehat{K}$, we take an orthonormal basis $\{v_1, \cdots, v_{d(\delta)}\}$ of V_δ, and put

$$f_{ij}^\delta(k) = (v_j, \delta(k)v_i),$$

where $(\ ,\ )$ denotes the inner product of V_δ. By Peter-Weyl's theorem, the set

$$\Upsilon = \{d(\delta)^{\frac{1}{2}} f_{ij}^\delta; \delta \in \widehat{K},\ 1 \leq i, j \leq d(\delta)\}$$

is a complete orthonormal basis of $L^2(K)$. Therefore any function Φ on $L^2(K)$ can be expanded by a mean convergent Fourier series of the functions in Υ :

$$\Phi(k) = \sum_{\delta \in \widehat{K}} d(\delta) \sum_{i,j=1}^{d(\delta)} \Phi_{ij}^\delta\, f_{ij}^\delta(K) \qquad (k \in K), \tag{5.1}$$

where we put

$$\Phi_{ij}^\delta = \int_K \Phi(k) \overline{f_{ij}^\delta(k)}\, dk.$$

Let $F\Phi(\delta)$ be a matrix of degree $d(\delta)$ where its (i,j)-entry is given by Φ_{ij}^δ. Then it is clear that

$$\sum_{i,j=1}^{d(\delta)} \Phi_{ij}^\delta f_{ij}^\delta(k) = \mathrm{Tr}(F\Phi(\delta)\delta(k)). \tag{5.2}$$

Also we have the Parseval's equality:

$$\|\Phi\|_2^2 = \sum_{\delta \in \widehat{K}} d(\delta) \sum_{i,j=1}^{d} (\delta)|\Phi_{ij}^\delta|^2,$$

where $\|\Phi\|_2$ denotes the L^2-norm of Φ on K.

Let Δ_K be the Casimir operator of K. Let $d\delta$ be the differential of the representation δ. Then it is well known that $d\delta(\Delta_K) = (\delta,\, \delta + 2\rho_T)I$, where ρ_T denotes the half sum of positive roots with respect to the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$.For any matrix A we denote by $\|A\|_{HS}$ the

Hilbert-Schmidt norm of A. If Φ is a C^∞ function on K, then it is known that the Fourier series of Φ given by (5.1) converges to Φ absolutely and uniformly on K. Therefore, since $\|\delta(k)\|_{HS} = d(\delta)^{\frac{1}{2}}$, by means of (5.1) and(5.2) we have

$$\begin{aligned} \sup_{k\in K} |\Phi(k)| &\leq \sum_{\delta\in\widehat{K}} d(\delta)^{\frac{3}{2}} \|F\Phi(\delta)\|_{HS} \\ &= \sum_{\delta\in\widehat{K}} d(\delta)^{\frac{3}{2}} \omega(\delta)^{-N} \|F\Delta_K^N\Phi(\delta)\|_{HS} \end{aligned} \tag{5.3}$$

for any $N \in \mathbb{N}$, where we put $\omega(\delta) = (\delta, \delta + 2\rho_T)$. By the Schwartz inequality, the right hand side of (5.3) is dominated by

$$\{\sum_{\delta\in\widehat{K}} d(\delta)\|F\Delta_K^N\Phi(\delta)\|_{HS}^2\}^{\frac{1}{2}} \cdot \{\sum_{\delta\in\widehat{K}} d(\delta)^2\omega(\delta)^{-2N}\}^{\frac{1}{2}}.$$

Moreover by Weyl's dimension formula, for any non-trivial representation δ we have

$$d(\delta) \leq D|\delta|^m,$$

where m is the number of positive roots and D is some positive constant. Hence, for sufficiently large N, the series $\sum_{\delta\in\widehat{K}} d(\delta)^2\omega(\delta)^{-2N}$ converges. Therefore by the Parseval's equality, there exists a constant C_N such that

$$\sup_{k\in K} |\Phi(k)| \leq C_N \|\Delta_K^N \Phi\|_2.$$

Let $A(K)$ denotes the set of all real analytic functions on K. The above argument and Lemma 3 in Appendix 2 imply the following lemma.

Lemma 5. *Let $\Phi \in A(K)$. Then the following conditions (i) and (ii) are equivalent:*

(i) $$\sup_{k\in K} |J(Y)\Phi(k)| < \infty \quad \textit{for any} \quad J \in \widetilde{\mathcal{O}}(\mathfrak{k}_c^*),$$

(ii) $$\|J(Y)\Phi\|_2 < \infty \quad \textit{for any} \quad J \in \widetilde{\mathcal{O}}(\mathfrak{k}_c^*).$$

For later use, we need the following estimate.

Lemma 6. *Let $\Phi \in A(K)$. Suppose that $J \in \widetilde{\mathcal{O}}(\mathfrak{k}_c^*)$. Then there exist a positive integer N, and a positive constant C and $\widetilde{J} \in \mathcal{O}_2(\mathbb{C})$ such that*

$$\|J(Y)\Phi\|_2 \leq C\,\|\Delta_K^N \widetilde{J}(\Delta_K)\Phi\|_2.$$

Proof. For $\Phi \in A(K)$, we put $(\pi(k)\Phi)(x) = \Phi(k^{-1}x)$ $(k, x \in K)$. For each $\delta \in \widehat{K}$, we define

$$P_\delta = d(\delta)^{-1} \int_K \overline{\chi_\delta(k)} \pi(k) dk.$$

Moreover we put

$$A(K)_\delta = \{ \sum_{i,j=1}^{d(\delta)} c_{ij} f_{ij}^\delta \ ; \ c_{ij} \in \mathbb{C} \ \}.$$

Then the operator P_δ is an orthogonal projection of $A(K)$ onto $A(K)_\delta$. Let $\{Y_1, \ldots, Y_s\}$ be an ordered orthonormal basis of $\mathfrak{k}$. For any $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{N}^s$ we put $Y^\beta = Y_1^{\beta_1} \cdots Y_s^{\beta_s}$. As usual, we regard Y^β as a right invariant differential operator on K. We define an operator norm of Y^β with respect to δ by

$$\|Y^\beta\|^\delta = \sup_{\|f\|_2 = 1, f \in A(K)_\delta} \|Y^\beta f\|_2.$$

Then it is easy to see that

$$\|Y^\beta\|^\delta \leq (\|Y_1\|^\delta)^{\beta_1} \cdots (\|Y_s\|^\delta)^{\beta_s}, \tag{5.4}$$

and each $\|Y_t\|^\delta$ $(t = 1, \ldots, s)$ is, in fact, given by

$$\|Y_t\|^\delta = d(\delta)^{\frac{1}{2}} \max_{1 \leq i,j \leq d(\delta)} \|Y_t f_{ij}^\delta\|_2$$

for any choice of orthonormal basis $\{v_j\}_{1 \leq j \leq d(\delta)}$ of V_δ. For any Y in $\mathfrak{k}$, there exists a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ such that Y is contained in $\mathfrak{t}$. We now take a weight basis $\{v_j\}_{1 \leq j \leq d(\delta)}$ as an orthonormal basis of V_δ. Then it is easy to see that $Y f_{ij}^\delta = \mu(Y) f_{ij}^\delta$ for some weight μ of the representation δ. This means that $\|Y\|^\delta \leq |\delta||Y|$ for each Y in $\mathfrak{k}$. Therefore, by virtue of (5.4) we have

$$\|Y^\beta\|^\delta \leq |\delta|^{|\beta|}, \tag{5.5}$$

where $|\beta| = \beta_1 + \cdots + \beta_s$.

Since

$$(J(Y)\Phi)(k) = \sum_{\delta \in \widehat{K}} d(\delta) \sum_{i,j=1}^{d(\delta)} \Phi_{ij}^\delta (J(Y) f_{ij}^\delta)(k),$$

if we write $J(Y) = \sum_{\beta \in \mathbb{N}^s} a_\beta Y^\beta$, then we obtain

$$\|J(Y)\Phi\|_2 \leq \sum_{\delta \in \widehat{K}} d(\delta) \sum_{i,j=1}^{d(\delta)} \|\Phi_{ij}^\delta\| \sum_{\beta \in \mathbb{N}^s} |a_\beta| \|Y^\beta f_{ij}^\delta\|_2.$$

Furthermore, thanks to the estimate (5.5), we have

$$\begin{aligned} \|J(Y)\Phi\|_2 &\leq \sum_{\delta \in \widehat{K}} d(\delta) \sum_{i,j=1}^{d(\delta)} \|\Phi_{ij}^\delta\| \sum_{\beta \in \mathbb{N}^s} |a_\beta| |\delta|^{|\beta|} d(\delta)^{-\frac{1}{2}} \\ &\leq \sum_{\delta \in \widehat{K}} d(\delta) \|F\Phi(\delta)\|_{HS} d(\delta)^{2-\frac{1}{2}} \widetilde{J}(|\delta|), \end{aligned}$$

where we put

$$\widetilde{J}(t) = \sum_{\beta \in \mathbb{N}^s} |a_\beta| t^{|\beta|}.$$

Of course, $\widetilde{J}(t)$ is a function of infra exponential type of one variable. By Lemma 5 in Appendix 2, we see that there exists $J' \in \mathcal{O}_2(\mathbb{C})$ such that

$$\widetilde{J}(|\delta|) \leq |J'(\omega(\delta))|.$$

Hence the similar argumentation as in the proof of Lemma 5 shows that for sufficiently large N there is a constant C such that $\|J(Y)\Phi\|_2$ is dominated by $C\|\Delta_K^N J'(\Delta_K)\Phi\|_2$. This completes the proof of the assertion

Combining the results described in the preceding two lemmas, one can easily find that the following fact.

Lemma 7. *For $\Phi \in C^\infty(K)$, the following conditions (i) and (ii) are equivalent with each other:*

(i) $\quad \sup_{k \in K} |J(Y)\Phi(k)| < \infty \quad$ *for any* $\quad J \in \widetilde{\mathcal{O}}(\mathfrak{k}_c^*)$,

(ii) $\quad \sup_{k \in K} |\widetilde{J}(\Delta_K)\Phi(k)| < \infty \quad$ *for any* $\quad \widetilde{J} \in \mathcal{O}_2(\mathbb{C})$.

Proof. Since the L^2-norm of a function is always dominated by its sup-norm on the compact manifold K, by virtue of Lemma 6 and Lemma 7, it is clear that (i) is deduced from (ii). On the other hand, for any $\widetilde{J} \in \mathcal{O}_2(\mathbb{C})$, if we put $J(Y_1, \ldots, Y_s) = \widetilde{J}(\Delta_K)$ then $J \in \widetilde{\mathcal{O}}(\mathfrak{k}_c^*)$ by Lemma 4 in Appendix 2. This proves the lemma.

As the corollary, we get also the following result.

Lemma 8. *For $\Phi \in C^\infty(K/M)$, the following two conditions (i) and (ii) are equivalent with each other:*

$$\text{(i)} \qquad \sup_{k\in K} |J(Y)\Phi(kM)| < \infty \qquad \textit{for any} \qquad J \in \widetilde{\mathcal{O}}(\mathfrak{k}_c^*),$$

$$\text{(ii)} \qquad \sup_{k\in K} |\widetilde{J}(\Delta_{K/M})\Phi(kM) < \infty \qquad \textit{for any} \qquad \widetilde{J} \in \mathcal{O}_2(\mathbb{C}).$$

6. Proof of the theorem in the case of $\mathcal{A}_*(G/K)$.

By the theorem in the case of $\mathcal{C}_*(G/K)$, it is sufficiently to show that $F(\mathcal{A}_*(G/K)) \subset \mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W$ and $F^{-1}(\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W) \subset \mathcal{A}_*(G/K)$.

Let $\Phi \in \mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W$. Put $\phi(x) = (F^{-1}\Phi)(x)$. It is clear that $\phi \in \mathcal{C}_*(G/K)$. If we can show that

$$\text{(6.1)} \qquad (FJ(X)\phi)(\lambda : kM) \in \mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W,$$
$$(\lambda,\, kM) \in \mathfrak{a}_c^* \times K/M$$
$$\text{for arbitrary} \qquad J \in \widetilde{\mathcal{Z}}(\mathfrak{g}_c^*),$$

then, by means of the theorem in the case of $\mathcal{C}_*(G/K)$, we see that $(J(X)\phi)(x) \in \mathcal{C}_*(G/K)$. This means that $\phi \in \mathcal{A}_*(G/K)$. Therefore it is enough to verify that (6.1) for proving $F^{-1}(\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W) \subset \mathcal{A}_*(G/K)$. Since

$$\Delta_{K/M}^j (FJ(X)\phi)(\lambda : kM) = F(\Delta_{K/M}^j J(X)\phi)(\lambda : kM)$$

for any $j \in \mathbb{N}$, by Lemma 3 in Appendix 2, the condition (6.1) is equivalent to the following condition:

$$\text{(6.2)} \qquad \sup_{|\operatorname{Im}\lambda|\le r, k\in K} |(FJ(X)\phi)(\lambda : kM)|(1+|\lambda|)^{r'} < \infty$$
$$\text{for} \qquad \forall r > 0,\ \forall r' > 0 \qquad \text{and} \qquad \forall J \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*).$$

Now we put $J(\zeta) = \sum_{\alpha\in\mathbb{N}^m} \frac{a_\alpha}{\alpha!}\zeta^\alpha$, where $m = \dim \mathfrak{g}$. By Lemma 2 in Appendix 2, it is also equivalent to the following condition:

$$\text{(6.3)} \qquad \sup_{|\operatorname{Im}\lambda|\le r,\, k\in K}\ \sup_{\alpha\in\mathbb{N}^m} |\frac{a_\alpha}{\alpha!}(FX^\alpha\phi)(\lambda : kM)|(1+|\lambda|)^{r'} < \infty$$
$$\text{for} \qquad \forall r > 0 \qquad \text{and} \qquad \forall r' > 0.$$

The following lemma is essential for our proof.

Lemma 9. *Let $\{X_1, \ldots, X_m\}$ (resp. $\{Y_1, \ldots, Y_s\}$) be an ordered basis of $\mathfrak{g}$ (resp. $\mathfrak{k}$). Let $\phi \in \mathcal{C}_*(G/K)$. Then for any $\alpha \in \mathbb{N}^m$, the Fourier-Laplace transform of the function $X^\alpha \phi$ can be written uniquely as*

$$F(X^\alpha \phi)(\lambda : kM) = \sum_{\substack{|\beta|+|\gamma|\leq|\alpha| \\ \beta\in\mathbb{N}^s, \gamma\in\mathbb{N}^\ell}} p_{\beta,\gamma}(\lambda : kM : \alpha) Y^\beta (F\phi)(\lambda : kM),$$

where $p_{\beta,\gamma}(\lambda : kM : \alpha)$ are real analytic functions with respect to the variable kM in K/M and holomorphic in $\lambda \in \mathfrak{a}_c^*$. Moreover we can find a constant C such that

$$|p_{\beta,\gamma}(\lambda : kM : \alpha)| \leq C^{|\alpha|}(1+|\lambda|)^{|\gamma|} \frac{|\alpha|!}{(|\beta|+|\gamma|)!}$$

for any $\alpha \in |\mathbb{N}^m$, $\beta \in \mathbb{N}^s$ and $\gamma \in \mathbb{N}^\ell$.

Proof. For any $X \in \mathfrak{g}$ and $\phi \in \mathcal{C}_*(G/K)$, we observe that

$$\begin{aligned}(FX\phi)(\lambda : kM) &= \int_G \frac{d}{dt}\phi(\exp(-tX)x)|_{t=0} e^{(i\lambda-\rho)H(x^{-1}k)} dx \\ &= \int_G \phi \frac{d}{dt} e^{(i\lambda-\rho)H(x^{-1}k\exp(-tX^{k-1}))}|_{t=0} dx\end{aligned}$$

where X^{k-1} denoting the adjoint action of k^{-1} on $\mathfrak{g}$. According to the Iwasawa decomposition of $\mathfrak{g}$, one can write X^{k-1} as the following form:

$$\begin{aligned}X^{k-1} &= X_{\mathfrak{k}}(k) + X_{\mathfrak{a}}(k) + X_{\mathfrak{n}}(k) \\ &\quad (X_{\mathfrak{k}}(k) \in \mathfrak{k},\, X_{\mathfrak{a}}(k) \in \mathfrak{a},\, X_{\mathfrak{n}}(k) \in \mathfrak{n}).\end{aligned}$$

Now, let $\{H_1, \ldots, H_\ell\}$ be an orthonormal basis of $\mathfrak{a}$ with respect to $Q(\ ,\ )$. Suppose that $X = X_j$. Then we can express $X_{\mathfrak{a}}(k)$ as

$$X_{\mathfrak{a}}(k) = \sum_{j=1}^{\ell} a_j^i(k) H_j.$$

Since $X_{\mathfrak{a}}(km) = X_{\mathfrak{a}}(k)$ for all $m \in M$, it is clear that $a_j^i(k)$ are the real analytic functions on K/M. We define functions $b_j^i(k)$ $(j = 1, \ldots, s)$ by the formula

$$X_{\mathfrak{k}}(k)^k = \sum_{j=1}^{s} b_j^i(k) Y_j,$$

By the similar way as above, we see that $b^i_j(k)$ are also the real analytic functions on K/M. Let K_ϵ be a relatively compact complex neighborhood of K in K_c, the complexification of K, such that the functions $a^i_j(k)$ and $b^i_j(k)$ are holomorphic on there. It is clear that there is a positive constant L such that the absolute values of $a^i_j(k)$ and $b^i_j(k)$ are less than L on K_ϵ.

On the other hand, since $H(xa) = H(x) + \log a\,(a \in A)$ and $H(xn) = H(x)\,(n \in N)$ we have

$$
\begin{aligned}
&(FX_i\phi)(\lambda : kM)\\
&= (\rho - i\lambda)(X_{\mathfrak{a}}(k))(F\phi)(\lambda : kM) + \ (X_{\mathfrak{k}}(k)^k F\phi)(\lambda : kM)\\
&= \sum_{j=1}^{\ell} a^i_j(k)(\rho - i\lambda)(H_j)(F\phi)(\lambda : kM) + \sum_{j=1}^{s} b^i_j(k)(Y_j F\phi)(\lambda : kM).
\end{aligned}
$$

Now, we define differential operators $\frac{\partial}{\partial\lambda_j}$ $(j = 1, \ldots, \ell)$ on $\mathfrak{a}_c$ by

$$\frac{\partial}{\partial\lambda_j} f(H) = \frac{d}{dt} f(H - tH_j)|_{t=0}$$

for $f \in C^\infty(\mathfrak{a}^*_c)$. Furthermore we put

$$\widetilde{X}_i = \sum_{j=1}^{\ell} a^i_j(k)\frac{\partial}{\partial\lambda_j} + \sum_{j=1}^{s} b^i_j(k)Y_j.$$

Since the above equality implies that

$$FX_i\phi = \widetilde{X}_i F\phi$$

we have

$$FX^\alpha\phi = \widetilde{X}^\alpha F\phi.$$

Thanks to Lemma 1 in Appendix 1, we can uniquely write

$$\widetilde{X}^\alpha = \sum_{\substack{|\beta|+|\gamma|\le|\alpha| \\ \beta\in\mathbb{N}^s,\gamma\in\mathbb{N}^\ell}} q_{\beta,\gamma}(kM : \alpha)Y^\beta(\frac{\partial}{\partial\lambda_1})^{\gamma_1}\cdots(\frac{\partial}{\partial\lambda_\ell})^{\gamma_\ell}$$

and we have the estimate

$$|q_{\beta,\gamma}(kM : \alpha)| \le D^{|\beta|+|\gamma|+1}L^{|\alpha|}\frac{|\alpha|!}{(|\beta| + |\gamma|)!} \qquad (k \in K)$$

with some positive number D which depends only on s, ℓ and ϵ.

On the other hand, it should be noted that there is a constant C_1 such that

$$\sum_{j=1}^{\ell} |(\rho - i\lambda)(H_j)| \le C_1(1+|\lambda|).$$

We put

$$p_{\beta,\gamma}(\lambda : kM : \alpha) = q_{\beta,\gamma}(kM : \alpha)((\rho - i\lambda)(H_1))^{\gamma_1} \cdots ((\rho - i\lambda)(H_\ell))^{\gamma_\ell}.$$

Of course, we may assume that $D > 1$. Therefore, if we put

$$C = C_1 D^2 L$$

then the assertion of this lemma follows immediately from the above argument.

This Lemma implies that

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{N}^m} |\frac{a_\alpha}{\alpha!}(FX^\alpha\phi)(\lambda : kM)| \\
&\le \sup_{\alpha\in\mathbb{N}^m} |\frac{a_\alpha}{\alpha!}| \sum_{\substack{|\beta|+|\gamma|\le|\alpha| \\ \beta\in\mathbb{N}^s, \gamma\in\mathbb{N}^\ell}} |p_{\beta,\gamma}(\lambda : kM : \alpha)||Y^\beta F\phi(\lambda : kM)| \\
&\le \sup_{N\in\mathbb{N}} \sup_{\substack{|\alpha|=N \\ \alpha\in\mathbb{N}^m}} |\frac{a_\alpha}{\alpha!} C^{|\alpha|}|\alpha|!| \sum_{\substack{|\beta|+|\gamma|\le N \\ \beta\in\mathbb{N}^s, \gamma\in\mathbb{N}^\ell}} \frac{(1+|\lambda|)^{|\gamma|}|Y^\beta|\Phi(\lambda : kM)|}{(|\beta|+|\gamma|)!}.
\end{aligned}$$

If we put $a_N = \sup_{|\alpha|=N, \alpha\in\mathbb{N}^m} \frac{|\alpha|!}{\alpha!}|a_\alpha|C^{|\alpha|}$, then $a_N \in \mathcal{A}_1$, where we put

$$\mathcal{A}_j = \{ \{d_\alpha\}_{\alpha\in\mathbb{N}^j} \ : \ \overline{\lim}_{|\alpha|\to\infty}|d_\alpha|^{\frac{1}{|\alpha|}} = 0 \}.$$

Therefore we have

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{N}^m} |\frac{a_\alpha}{\alpha!}(FX^\alpha\phi)(\lambda : kM)| \\
&\le \sup_{N\in\mathbb{N}} a_N \sum_{|\beta|+|\gamma|\le N} (1+|\lambda|)^{|\gamma|}|Y^\beta\Phi(\lambda : kM)|\frac{1}{|\beta|!|\gamma|!} \\
&\le \sup_{\beta\in\mathbb{N}^s, \gamma\in\mathbb{N}^\ell} \sup_{N\ge|\beta|+|\gamma|} a_N(1+|\lambda|)^{|\gamma|}|Y^\beta\Phi(\lambda : kM)|\frac{1}{|\beta|!|\gamma|!}.
\end{aligned}$$

Moreover if we define

$$\widetilde{a}_L = \sup_{N\ge L} a_N$$

then it is clear that $\widetilde{a}_L \in \mathcal{A}_1$. By means of Lemma 1 in Appendix 2, there exist an element $\{b_k\}$ of $\mathcal{A}_1$ such that

$$\widetilde{a}_{|\beta|+|\gamma|} \le b_{|\beta|} b_{|\gamma|}$$

for any $\beta \in \mathbb{N}^s$ and $\gamma \in \mathbb{N}^\ell$.

Therefore, in order to show that (6.3), it is sufficiently to show that

$$\sup_{\substack{|\operatorname{Im}\lambda|\le r\\ k\in K}} \sup_{\substack{\beta\in\mathbb{N}^s\\ \gamma\in\mathbb{N}^\ell}} \frac{b_{|\beta|}b_{|\gamma|}}{|\beta|!|\gamma|!}(1+|\lambda|)^{(|\gamma|+r')}|Y^\beta\Phi(\lambda : kM)| < \infty \tag{6.4}$$

for $\forall r > 0$, $\forall r' \ge 0$ and $\{b_k\} \in \mathcal{A}_1$.

Using Lemma 2 in Appendix 2, the condition (6.4) is equivalent to

$$\sup_{\substack{|\operatorname{Im}\lambda|\le r\\ k\in K}} |J_1(\lambda)J_2(Y)\Phi(\lambda : kM)| < \infty \tag{6.5}$$

for $\forall r > 0$, $\forall J_1 \in \widetilde{\mathcal{O}}(\mathfrak{a}_c^*)$ and $\forall J_2 \in \widetilde{\mathcal{O}}(\mathfrak{k}_c^*)$.

By the way, since $\Phi \in \mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W$, (6.5) follows immediately from Lemma 8 in §5. This asserts $F^{-1}(\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W) \subset \mathcal{A}_*(G/K)$.

Inversely, we will prove that the relation : $F(\mathcal{A}_*(G/K)) \subset \mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W$. Suppose that $\phi \in \mathcal{A}_*(G/K)$. Since $J(X)\phi \in \mathcal{C}_*(G/K)$, we see that $FJ(X)\phi \in \mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W$ for any $J \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$. In particular we find that

$$\sup_{\substack{|\operatorname{Im}\lambda|\le r\\ k\in K}} |(FJ(X)\phi)(\lambda : kM)| < \infty \tag{6.6}$$

for any $r > 0$ and $J \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$.

Let Δ be the Laplace operator on G/K. Then it is well known that there is a polynomial χ of degree two on $\mathfrak{a}_c^*$ such that

$$(F\Delta^N\phi)(\lambda : kM) = \chi(\lambda)^N F\phi(\lambda : kM)$$

for any $N \in \mathbb{N}$. For any $J_1 \in \widetilde{\mathcal{O}}(\mathfrak{a}_c^*)$ and any $r > 0$, by Lemma 5 in Appendix 2, there is $\widetilde{J}_1 \in \mathcal{O}_2(\mathbb{C})$ such that

$$|J_1(\lambda)| \le |\widetilde{J}_1(\chi(\lambda))| \qquad \text{for} \qquad |\operatorname{Im}\lambda| \le r.$$

Let $J_2 \in \mathcal{O}_2(\mathbb{C})$. Thanks to Lemma 3 and Lemma 4 in Appendix 2, if we put

$$J(X_1, \ldots, X_m) = \widetilde{J}_1(\Delta)J_2(\Delta_{K/M})$$

then we see that $J \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$. So we have

$$\begin{aligned}
&\sup_{\substack{|\operatorname{Im}\lambda|\le r\\ k\in K}} |J_1(\lambda)J_2(\Delta_{K/M})F\phi(\lambda : kM)| \\
&\le \sup_{\substack{|\operatorname{Im}\lambda|\le r\\ k\in K}} |\widetilde{J}_1(\chi(\lambda))J_2(\Delta_{K/M})F\phi(\lambda : kM)| \\
&= \sup_{\substack{|\operatorname{Im}\lambda|\le r\\ k\in K}} |F(\widetilde{J}_1(\Delta)J_2(\Delta_{K/M})\phi)(\lambda : kM)| \\
&= \sup_{\substack{|\operatorname{Im}\lambda|\le r\\ k\in K}} |F(J(X)\phi)(\lambda : kM)|.
\end{aligned}$$

This last quantity is in fact finite by the estimate (6.6). This completes the proof of theorem.

As the corollary to the above proof of the theorem we get the following result.

Theorem 2. *The topology of $\mathcal{A}_*(G/K)$ defined by the family of seminorms $\|\phi\|_{J,r}$ ($J \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$, $r \ge 0$) coincides with the topology defined by the family of seminorms*

$$\|\phi\|_{J_1,J_2,r} = \sup_{x\in G/K}\sup_{n,k} |(J_{1,n}(\Delta)J_{2,k}(\Delta_{K/M})\phi)(x)|e^{r|x|},$$

where $J_j \in \mathcal{O}_2(\mathbb{C})$ ($j = 1, 2$) and $r \ge 0$.

7. Fourier expansions on K/M revisited.

In this section we will describe the images of $\mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W$ and $\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W$ under the Fourier transform $F_{K/M}$ on K/M. At the first place, we review certain results on the so-called Kostant matrix [Kos1,Kos2].

For each $\delta \in \widehat{K}$, we put

$$\begin{aligned}
&V_\delta^M = \{\, v \in V_\delta \ ; \ \delta(m)v = v \qquad \forall m \in M \,\}, \\
&\ell(\delta) = \dim V_\delta^M \qquad \text{and} \qquad \widehat{K}_M = \{\, \delta \in \widehat{K} \ ; \ \ell(\delta) > 0 \,\}.
\end{aligned}$$

Now the Poincaré-Birchoff-Witt theorem implies that $U(\mathfrak{g}) = U(\mathfrak{a}) + (U(\mathfrak{g})\mathfrak{k} + \mathfrak{n}U(\mathfrak{g}))$, where $U(\mathfrak{a})$ stands for the universal enveloping algebra of $\mathfrak{a}_c^*$. If $u \in U(\mathfrak{g})$, let $q^u \in U(\mathfrak{a})$ be the component of u in $U(\mathfrak{a})$ relative to the direct sum decomposition of $U(\mathfrak{g})$. If $a \in U(\mathfrak{a})$ we look upon a as a polynomial on $\mathfrak{a}_c^*$. Let $S(\mathfrak{g})$ and $S(\mathfrak{p})$ denote the symmetric algebra over $\mathfrak{g}$ and $\mathfrak{p}$, respectively, and $p \to p^*$ the symmetrization map of $S(\mathfrak{g})$ onto $U(\mathfrak{g})$. Let $H = H(\mathfrak{p}) \subset S(\mathfrak{p})$ denotes the subspace of harmonic polynomials relative to $Ad(K)$. Of course, for each $h \in H$, we consider h as a function on $\mathfrak{p}$ via the form $Q(\ ,\ )$. Let X be a regular element in $\mathfrak{a}$. Consider the imbedding $K/M = Ad(K)X \subset \mathfrak{p}$. Since it is known [Hel2] that each h in H is determined by its values on the orbit $Ad(K)X$ of X, and each K-finite function on K/M can be taken by the restriction of a harmonic polynomial, we observe that the restriction mapping $\iota : h \to h|_{Ad(K)X}$ is a bijection of H onto the space $\mathcal{E}^\infty(K/M)$ of K-finite functions on K/M. Let $\mathcal{E}_\delta(K/M) = P_\delta \mathcal{E}^\infty(K/M)$, where P_δ is the projection operator for δ defined in §5. Put $H_\delta = \iota^{-1}(\mathcal{E}_\delta(K/M)) \subset H$. Then there is a bijection H_δ onto $\mathcal{E}_\delta(K/M)$. Let $\{v_1, \dots, v_{d(\delta)}\}$ be an orthonormal basis of V_δ such that $\{v_1, \dots, v_{\ell(\delta)}\}$ spans V_δ^M. Since the functions $f_{ij}^\delta(kM) = (v_j, \delta(k)v_i)$ $(1 \le i \le \ell(\delta),\ 1 \le j \le d(\delta))$ form a basis of $\mathcal{E}_\delta(K/M)$, we see that $\dim H_\delta = \ell(\delta)$. We now take and fix a basis $\varepsilon_1, \dots, \varepsilon_{\ell(\delta)}$ of $\mathrm{Hom}_K(V_\delta^M,\ H^*)$. We define the $\ell(\delta) \times \ell(\delta)$ matrix $Q^\delta(\lambda)$ by

$$Q^\delta(\lambda)_{ij} = q^{\varepsilon_j(v_i)}(\rho - i\lambda) \qquad (1 \le i, j \le \ell(\delta)),$$

(cf. [Kos1,Kos2]). This is the so-called Kostant matrix.

Let $\Phi \in C^\infty(\mathfrak{a}_c^* \times K/M)$. For each $\delta \in \widehat{K}_M$, we define

$$F_{K/M}^\delta \Phi(\lambda) = \Phi^\delta(\lambda) = \int_K \Phi(\lambda : kM) f^\delta(kM) dk,$$

where f^δ is the $\ell(\delta) \times d(\delta)$ matrix with entries f_{ij}^δ. Moreover we put

$$F_{K/M}\Phi(\lambda) = \{\Phi^\delta(\lambda)\}_{\delta \in \widehat{K}_M}.$$

Then the following result is well known (cf. [He2]).

Lemma 10. *Let U be a W-invariant subset in $\mathfrak{a}_c^*$. Suppose that $\Phi(\lambda : kM)$ is a continuous function on $U \times K/M$. Then the Poisson integral $\check{\Phi}(\lambda : x)$ ($(\lambda, x) \in U \times G$) of Φ is invariant under W with respect to the variable λ if and only if the Fourier coefficients $F_{K/M}\Phi(\lambda) = \{\Phi^\delta(\lambda)\}_{\delta \in \widehat{K}_M}$ satisfy the following conditions : for each $\delta \in \widehat{K}_M$, the function $Q^{\check{\delta}}(\lambda)^{-1}\Phi^\delta(\lambda)$ is invariant under W, where $\check{\delta}$ denotes the contragradient representation of δ.*

By this lemma, we can characterize the properties possessed by the Fourier coefficients of the functions belong to $\mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W$ and $\mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W$, respectively.

In order to describe these Fourier coefficients, we now introduce the following two function spaces :

$$H^C(\mathfrak{a}_c^*)_W = \{\,\{\Phi^\delta(\lambda)\}_{\delta\in\widehat{K}_M} \;:\; \Phi^\delta(\lambda) \quad \text{satisfy the conditions} \\ F(O),\ F(W) \text{ and } F(RD)\,\},$$

and

$$H^A(\mathfrak{a}_c^*)_W = \{\,\{\Phi^\delta(\lambda)\}_{\delta\in\widehat{K}_M} \;:\; \Phi^\delta(\lambda) \quad \text{satisfy the conditions} \\ F(O),\ F(W) \text{ and } F(ED)\,\}.$$

Here the conditions $F(O)$, $F(W)$, $F(RD)$ and $F(ED)$ are as follows :

$F(O)$: Φ^δ is a $\ell(\delta)\times d(\delta)$ matrix whose entries are holomorphic functions on $\mathfrak{a}_c^*$,

$F(W)$: For any $\delta\in\widehat{K}_M$,

$$Q^{\check{\delta}}(w\lambda)^{-1}\Phi^\delta(w\lambda) = Q^{\check{\delta}}(\lambda)^{-1}\Phi^\delta(\lambda)$$

for any $w\in W$ and $\lambda\in\mathfrak{a}_c^*$,

$F(RD)$: $\|\{\Phi^\delta\}\|^{r',r}$

$$= \sup_{\|\operatorname{Im}\lambda\|\le r}\ \sup_{\delta\in\widehat{K}_M} \|\Phi^\delta(\lambda)\|(1+|\delta|)^{r'}(1+|\lambda|)^{r'} < \infty$$

for any $r'\ge 0$ and $r\ge 0$

and

$F(ED)$: $\|\{\Phi^\delta\}\|^{J,r}$

$$= \sup_{\|\operatorname{Im}\lambda\|\le r}\ \sup_{\delta\in\widehat{K}_M} \|\Phi^\delta(\lambda)J_2(\delta)J_1(\lambda)\| < \infty$$

for any $J=(J_1,\,J_2)\in\widetilde{\mathcal{O}}(\mathfrak{a}_c^*)\times\widetilde{\mathcal{O}}(\mathfrak{t}_c^*)$ and $r\ge 0$,

where we put

$$\|\Phi^\delta(\lambda)\| = \max_{\substack{1\le i\le \ell(\delta)\\ 1\le j\le d(\delta)}} \|\Phi^\delta_{ij}(\lambda)\|.$$

Remark. *Let* $(\mathfrak{a}_c^*)' = \{\lambda\in\mathfrak{a}_c^* \;:\; \det Q^\delta(\lambda)\ne 0$ *for all* $\delta\in\widehat{K}_M\}$ *and* $(\mathfrak{a}_c^*)^- = \{\lambda\in\mathfrak{a}_c^* \;:\; \mathrm{Re}(i\lambda,\,\alpha)\le 0$ *for all* $\alpha\in\Sigma^+\}$. *Since it is known that* $(\mathfrak{a}_c^*)'\supset(\mathfrak{a}_c^*)^-$, *the condition* $F(W)$ *implies that, in particular,* $Q^{\check{\delta}}(\lambda)^{-1}\Phi^\delta(\lambda)$ *are entire holomorphic for* $\Phi\in H^C(\mathfrak{a}_c^*)_W$.

Theorem 3. *$F_{K/M}$ gives the following two linear topological isomorphism :*

$$F_{K/M} : \mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W \xrightarrow{\sim} H^C(\mathfrak{a}_c^*)_W,$$
$$F_{K/M} : \mathcal{Z}_*^A(\mathfrak{a}_c^* \times K/M)_W \xrightarrow{\sim} H^A(\mathfrak{a}_c^*)_W.$$

Moreover we have

$$\Phi(\lambda : kM) = \sum_{\delta \in \widehat{K}_M} d(\delta) \sum_{\substack{1 \le i \le \ell(\delta) \\ 1 \le j \le d(\delta)}} \Phi_{ij}^\delta(\lambda) f_{ij}^\delta(kM)$$

for $\mathcal{Z}_*^C(\mathfrak{a}_c^* \times K/M)_W$.

Using the similar argument that of §5 and §6, the theorem can be proved (cf.[Su]). So we omit its proof.

8. Surjectivity of invariant differential operators.

In this section, we will derive some results on the solvability of a single differential equation on G/K, as the application of the Paley-Wiener theorem.

Let $\mathbb{D}(G/K)$ be the algebra of G-invariant differential operators on G/K. It is well known that $\mathbb{D}(G/K)$ is commutative and there exists a generator system $\{\Delta_1, \ldots, \Delta_\ell\}$ ($\ell = \dim \mathfrak{a} = \operatorname{rank} G/K$) such that $\mathbb{D}(G/K) = \mathbb{C}[\Delta_1, \ldots, \Delta_\ell]$. We put $D = (\Delta_1, \ldots, \Delta_\ell)$. For any $P \in \mathbb{C}[D]$ we put

$$P(\lambda) = P(D) e^{(i\lambda - \rho)(H(x^{-1}))}|_{x=e} \qquad (\lambda \in \mathfrak{a}_c^*), \tag{8.1}$$

where e denotes the unit element of G. Of course, $P(\lambda) \in \mathbb{C}[\lambda]^W$, the algebra of W-invariant polynomials on $\mathfrak{a}_c^*$, and for any $\phi \in \mathcal{C}_*(G/K)$, it is easy to see that the following formula holds:

$$F(P(D)\phi)(\lambda : kM) = P(\lambda) F\phi(\lambda : kM). \tag{8.2}$$

Theorem 4. *Let $P(D)$ be any non-zero G-invariant differential operator on G/K. The we have*

$$P(D)\mathcal{C}'_*(G/K) = \mathcal{C}'_*(G/K),$$
$$P(D)\mathcal{A}'_*(G/K) = \mathcal{A}'_*(G/K),$$

where $\mathcal{C}'_*(G/K)$ and $\mathcal{A}'_*(G/K)$ are the strong dual spaces of $\mathcal{C}_*(G/K)$ and $\mathcal{A}_*(G/K)$, respectively.

Proof. Let $P(\lambda)$ be the polynomial on $\mathfrak{a}_c^*$ defined by (8.1). Suppose that $P(\lambda) \neq 0$. Since multiplication by $P(\lambda)$ is a homeomorphism of $\mathcal{Z}_*^C(\mathfrak{a}_c^* \times$

$K/M)_W$ (resp. $\mathcal{A}^A_*(\mathfrak{a}^*_c \times K/M)_W$) onto $P \cdot \mathcal{Z}^C_*(\mathfrak{a}^*_c \times K/M)_W$ (resp. $P \cdot \mathcal{Z}^A_*(\mathfrak{a}^*_c \times K/M)_W$), (8.2) and the Paley-Wiener theorem (Theorem 1) imply that the mapping $\phi \to P(D)\phi$ is a homeomorphism of $\mathcal{C}_*(G/K)$ (resp. $\mathcal{A}_*(G/K)$) onto $P(D)\mathcal{C}_*(G/K)$ (resp. $P(D)\mathcal{A}_*(G/K)$). Of course, if we denote by $P(D)^*$ the adjoint of $P(D)$, then the mapping $\phi \to P(D)^*\phi$ is also a homeomorphism of $\mathcal{C}_*(G/K)$ (resp. $\mathcal{A}_*(G/K)$) onto $P(D)^*\mathcal{C}_*(G/K)$ (resp. $P(D)^*\mathcal{A}_*(G/K)$).

Therefore, if $S \in \mathcal{C}'_*(G/K)$ (resp. $\mathcal{A}'_*(G/K)$) then the linear form $P(D)^*\phi \to S(\phi)$ on $P(D)^*\mathcal{C}_*(G/K)$ (resp. $P(D)^*\mathcal{A}_*(G/K)$) is continuous. By the Hahn-Banach theorem, there exists an element T of $\mathcal{C}'_*(G/K)$ (resp. $\mathcal{A}'_*(G/K)$) such that $T(P(D)^*\phi) = S(\phi)$ for arbitrary $\phi \in \mathcal{C}_*(G/K)$ (resp. $\mathcal{A}_*(G/K)$). This implies that $P(D)T = S$, so the theorem is proved.

Appendix 1.

For a positive number L, we put $U_L = \{z \in \mathbb{C}^n; |z_1| + \cdots + |z_n| < L\}$. Let $X_1, \dots, X_k$ and $Y_1, \dots, Y_m$ be differential operators of the forms

$$X_i = \sum_{j=1}^{n} a_{i,j}(z)\frac{\partial}{\partial z_j}, \tag{A1.1}$$

$$Y_i = \sum_{j=1}^{k} b_{i,j}(z)X_j + b_i(z) \tag{A1.2}$$

which satisfy

$$[X_i, X_j] = \sum_{\ell=1}^{k} c^{\ell}_{i,j}(z)X_{\ell}. \tag{A1.3}$$

Here $a_{i,j}(z), b_{i,j}(z), b_i(z)$ and $c^{\ell}_{i,j}(z) \in \mathcal{O}(U_L)$ and we assume that the rank of the matrix $\left(a_{i,j}\right)_{1 \leqq i \leqq k, 1 \leqq j \leqq n}$ equals k at a generic point of U_L. Then we have

Lemma 1. *We can uniquely write*

$$Y_1 \cdots Y_m = \sum_{|\alpha| \leqq m, \alpha \in \mathbb{N}^k} p_\alpha(z) X_1^{\alpha_1} \cdots X_k^{\alpha_k} \tag{A1.4}$$

with suitable $p_\alpha(z) \in \mathcal{O}(U_L)$. Suppose there exists a positive number M such that the absolute values of $a_{i,j}(z), b_{i,j}(z), b_i(z), c^{\ell}_{i,j}(z)$ are less *than M on U_L. Then for any positive number L' satisfying $L' < L$ we have*

$$\text{(A1.5)} \qquad |p_\alpha(z)| \leqq C^{|\alpha|+1} M^m m!/|\alpha|! \qquad \text{for any } z \in U_{L'}.$$

with a positive number C which depends only on n, L and L'.

Proof. For simplicity we will write X^α in place of $X_1^{\alpha_1} \cdots X_k^{\alpha_k}$. It is obvious that we can express $Y_1 \cdots Y_m$ in the form (A1.4). Suppose there exist $r \in \mathbb{N}$ and $q_\alpha(z) = \mathcal{O}(U_L)$ such that $\sum_{|\alpha| \leqq r} q_\alpha(z) X^\alpha = 0$ and suppose $q_\beta \neq 0$ with a suitable $\beta \in \mathbb{N}^k$ satisfying $|\beta| = r$. Fix a point $w \in U_L$ so that $q_\beta(w) \neq 0$ and moreover the rank of the matrix $\big(a_{i,j}(w)\big)$ equals k. We can find polynomial functions $f_i(z)$ of degree 1 which satisfy $f_i(z) = 0$ and $X_i(f_j)(w) = \delta_{i,j}$ for $i, j = 1, \ldots, k$. Then we have

$$\sum_{|\alpha| \leqq r} \big(q_\alpha X^\alpha (f_1^{\beta_1} \cdots f_k^{\beta_k})\big)(w) = q_\beta(w).$$

this contradicts the assumption $\sum q_\alpha X^\alpha = 0$, from which the uniqueness of the expression (A1.4) follows.

To get the estimates for p_α we employ the method of majorant. Let $\widehat{O}$ denotes the ring of formal power series of z. For $\phi = \sum C_\nu x^\nu \in \widehat{O}$ and $\phi' - \sum C'_\nu z^\nu \in \widehat{O}$ we write $\phi \ll \phi'$ if and only if $|C_\nu| \leqq C'_\nu$ for any$\nu \in \mathbb{N}^n$. For differential operators $P_i = \sum p_{i,\nu} \partial^\nu \quad (i = 1, 2, \ldots$ of finite order with $p_{i,j} \in \widehat{O}$, we denote $P_1 \ll P_2$ if and only if $pi, \nu \ll p2, \nu$ for any ν. We remark that if $P_1 \ll P_2$ and $P_3 \ll P_4$, then $P_1 P_2 \ll P_4 P_5$. Here we denote

$$z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n} \text{ and } \partial^\nu = \big(\frac{\partial}{\partial z_1}\big)^{\nu_1} \cdots \big(\frac{\partial}{\partial z_n}\big)^{\nu_n}$$

for $\nu \in \mathbb{N}^n$.

By changing z_i and X_i by $(L + L')z_i/2$ and $\min\{(L + L')/2, 1\}$ $(i = 1, \cdots, k)$, respectively, we may assume $M = 1$ and $L' < 1 < L$ to prove (A1.5). Put $t = z_1 + \cdots + z_n$ and $\phi = \sum_i t^i = (1 - t)^{-1}$. It follows from Lemma 2 that there exists $C > 1$ such that

$$a_{i,j} \ll C\phi,\ b_{i,j}(z) \ll C\phi,\ b_i(z) \ll C\phi \text{ and } c_{i,j}(z) \ll C\phi.$$

Now we define

$$X'_i = \sum_{j=1}^n C\phi^{2k-i} \frac{\partial}{\partial z_j} \qquad \text{for } i = 1, \cdots, k,$$

$$Y'_i = \sum_{j=1}^n C\phi^j X'_j + C\phi \qquad \text{for } i = 1, \cdots, m.$$

Then $X_i \ll X'_i$, and $Y_i \ll Y'i$ and

$$[X'_i, X'_j] = C^2 n(i-j)\phi^{4k-i-j+1} \sum_{\ell=1}^{n} \frac{\partial}{\partial z_\ell}$$
$$= \sum_{\ell=1}^{k} C(i-j)nk^{-1}\phi^{2k+\ell-i-j+1} X'_\ell.$$

Since $C(i-j)nk^{-1}\phi 2k+\ell-i-j+1 \ll C\phi$ if $1 \leqq j < i \leqq k$ and $1 \leqq \ell \leqq k$, it is easy to see that there exist $p'_\alpha(z) \in \widehat{O}$ which satisfy

$$\text{(A1.6)} \qquad Y'_1 \cdots Y'_m = \sum_{|\alpha| \leqq m} (X'_1)^{\alpha_1} \cdots (X'_k)^{\alpha_k}$$

and $p_\alpha(z) \ll p'_\alpha(z)$ for all $\alpha \in \mathbb{N}^k$. On the other hand, since $Y'_i = C^2 k\phi^{2k}(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n}) + C\phi$, we can uniquely write

$$\text{(A1.7)} \qquad Y'_1 \cdots Y'_m = \sum_{i \leqq m} q_i(z) \big(\sum_{j=1}^{n} \frac{\partial}{\partial z_j}\big)^i$$

with $q_i(z) \in \widetilde{\mathcal{O}}$. Combining (A1.6) and (A1.7), we have $p'_a lpha(z) \ll q_{|\alpha|}(Z)$. Put $L'' = (L'+1)/2$. Then Lemma 2 assures the existence of $C' > 1$ which satisfies $C^2 n^2 \phi^{2n} \ll C'(L''-t)^{-1}$. Define $q'_i(t) \in \widetilde{O}$ by

$$\text{(A1.8)} \qquad \big(C'(L''-t)^{-1}\frac{\partial}{\partial t} + C'(L''-t)^{-2}\big)^m = \sum_{i=0}^{m} q'_i(t)\big(\frac{\partial}{\partial t}\big)^i.$$

Then $q_i(z) \ll q'_i(z_1 + \cdots + z_n)$ and $q'_i(t) = C'^m C_{m,i}(L''-t)i - 2m$ under the notation in Lemma 3. Using the estimate for $C_{m,j}$ in Lemma 3, we have

$$\text{(A1.9)} \qquad |p_\alpha(z)| \leqq C'^m 2^{2m-|\alpha|}(L''-L')^{|\alpha|-2m} m!/|\alpha|!$$

for any $z \in U_{L'}$ because $p_\alpha(z) \ll q_{|\alpha|}(z_1 + \cdots + z_n)$. Now the proposition clear.

Lemma 2. *For positive numbers L and L' with $L > L'$, we can choose $C > 0$ such that*

$$\text{(A1.10)} \qquad \phi \ll C\big(\sup_{z \in U_L} |\phi(z)|\big) \sum_{k=0}^{\infty} (z_1 + \cdots + z_n)^k$$

for any bounded holomorphic function ϕon U_L.

Proof. (cf. [O3] Lemma 2.3) Put $M = \sup\{x \in U_L|\phi(z)|\}$ and $\phi = \sum a_\nu z^\nu$ with $a_\nu \in \mathbb{C}$. Denoting $X_i = |z_i| + \frac{1}{2n}(L - |z|)$ for $z \in U_L$, we have

$$a_\nu = (2\pi\sqrt{-1})^n \int_{|w_1|=x_1} \cdots \int_{|w_n|=x_n} \frac{1}{w_1 \cdots w_n} z^\nu w^{-\nu} \phi(w) dw_1 \cdots dw_n.$$

This proves $|a_\nu z^\nu| \leqq M$ for all $z \in U_L$. Then the rest part of the proof of the lemma is the same as that of [O3] Lemma 2.3. In fact by an estimate

$$\sup\{x^\alpha; x_1 + \cdots + x_n < L,\ x_i \geqq 0\} \geqq L^{|\alpha|}\alpha!(|\alpha|!)^{-1}(|\alpha|+1)^{1-n},$$

we can put $C = \max_{i \geqq 0}(i+1)^{n-1}(L'/L)^i$ as in [O3].

Lemma 3. *Putting $\phi = (1-t)^{-1}$ and $X = \frac{d}{dt}$, we have*

$$(\phi X + \phi^2)^m = \sum_{j=0}^{m} C_{m,j} \phi^{2m-j} X^j \tag{A1.11}$$

for any positive integer m. Here the constants $C_{m,j}$ are determined by the relation

$$\begin{cases} C_{0,0} \text{ and } C_{m,-1} = 0, \\ C_{m,j} = (2m-j-1)C_{m-1,j} + C_{m-1,j-1} \end{cases} \tag{A1.12}$$

and satisfy $0 \leqq C_{m,j} \leqq 2^{2m-j} m!/j!$.

Proof. Since the lemma is clear when $m = 1$, we will prove it by the induction on m. Then

$$\begin{aligned}
&(\phi X + \phi^2)(\phi X + \phi^2)^m = (\phi X + \phi^2) \sum C_{m,j} \phi^{2m-j} X^j \\
&\quad = \sum C_{m,j} (\phi^{2m-j+1} X^{j+1} + \phi(2m-j)\phi^{2m-j+1} X^j + \phi^{2m-j+2} X^j) \\
&\quad = \sum (C_{m,k-1} + (2m+2-k-1) C_{m,k} \phi^{2m+2-k} X^k
\end{aligned}$$

and hence we have (A1.12).

Put $a_{m,j} = C_{m,j} \prod_{i=1}^{m-j} (2i+j-1)^{-1}$. Then

$$a_{m,j} = a_{m-1,j} + (\prod_{i=1}^{m-j} \frac{2i+j-2}{2i+j-1}) a_{m-1,j-1}.$$

Hence we can prove $0 \leqq a_{m,j} \leqq 2^m$ by the induction on m and $0 \leqq C_{m,j} \leqq 2^m \prod_{i=1}^{m-j}(2i+j-1) \leqq 2^m \prod_{i=1}^{m-j} 2(i+j) = 2^{2m-j} m!/j!$.

Appendix 2.

We review well known facts and elementary lemmas on the functions and the differential operators of infra exponential type.

Let

$$J(\zeta) = \sum_{\alpha \in \mathbb{N}^m} \frac{a_\alpha}{\alpha!} \zeta^\alpha \ \in \mathcal{O}(\mathbb{C}^m) \qquad (\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_m^{\alpha_m}).$$

Then the following conditions are mutually equivalent :

(1) J is of infra exponential type;

(2) For any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that $|J(\zeta)| \leq C_\varepsilon \, e^{\varepsilon|\zeta|}$;

(3) $\overline{\lim}_{|\alpha| \to \infty} |a_\alpha|^{\frac{1}{|\alpha|}} = 0$;

(4) For any $r > 0$ there is a constant $D_r > 0$ such that $|a_\alpha| \leq D_r \, r^{-|\alpha|}$.

Of course, the product of functions of infra exponential type is also a infra exponential type. Let $\widetilde{\mathcal{O}}(\mathbb{C}^j)$ be the set of all (entire holomorphic) functions of infra exponential type on $\mathbb{C}^j$ ($j = 1, 2, \dots$). By (4) of characterizations of $\widetilde{\mathcal{O}}(\mathbb{C}^m)$, if $J(\zeta)$ belongs to $\widetilde{\mathcal{O}}(\mathbb{C}^m)$ then for any $c = (c_1, \dots, c_m) \in \mathbb{C}^m$, the new function $J_c(\zeta)$ defined by

$$J_c(\zeta) = \sum_{\alpha \in \mathbb{N}^m} \frac{a_\alpha}{\alpha!} (c\zeta)^\alpha$$

also belongs to $\widetilde{\mathcal{O}}(\mathbb{C}^m)$, where we put $c\zeta = (c_1\zeta_1, \dots, c_m\zeta_m)$. Moreover, since

$$\sum_{|\alpha|=N, \, \alpha \in \mathbb{N}^m} \frac{1}{\alpha!} = \frac{m^N}{N!},$$

if we put

$$\widetilde{a}_N = \sup_{|\alpha|=N, \, \alpha \in \mathbb{N}^m} |a_\alpha|,$$

then we observe that

$$J(t) = \sum_{\alpha \in \mathbb{N}^m} \frac{\widetilde{a}_{|\alpha|}}{\alpha!} t^{|\alpha|} \in \widetilde{\mathcal{O}}(\mathbb{C}).$$

Now we put

$$\mathcal{A}_m = \{ \, \{a_\alpha\}_{\alpha \in \mathbb{N}^m} \ : \ J(\zeta) = \sum_{\alpha \in \ N^m} \frac{a_\alpha}{\alpha!} \zeta^\alpha \in \widetilde{\mathcal{O}}(\mathbb{C}^m) \ \}.$$

Then the following lemmas can be shown immediately from the above equivalent characterizations of $\widetilde{\mathcal{O}}(\mathbb{C}^m)$.

Lemma 1. *Let $\{a_n\} \in \mathcal{A}_1$. For any positive integer k, we put*

$$b_k = \sup_{n \geq k} |a_n|^{\frac{k}{n}}.$$

Then $\{b_k\}$ is also an element of $\mathcal{A}$ and the estimate

$$b_k b_j \geq |a_{k+j}|$$

holds for any positive integers k and j.

Lemma 2. *For any C^∞ function f on G, the following conditions are equivalent :*

(1) $\sup_{x \in G} |\sum_{\alpha \in \mathbb{N}^m} \frac{a_\alpha}{\alpha!} X^\alpha f(x)| < \infty \qquad \forall \{a_\alpha\} \in \mathcal{A}_m$;
(2) $\sup_{x \in G} \sup_{\alpha \in \mathbb{N}^m} |\frac{a_\alpha}{\alpha!} X^\alpha f(x)| < \infty \qquad \forall \{a_\alpha\} \in \mathcal{A}_m$;

where we put $m = \dim G$. Moreover if G is abelian and X^α means the multiplication operator (that is to say, $X^\alpha : f(x) \longmapsto x^\alpha f(x)$), then the assertion is also true for any C^∞ function f on G.

Let $\widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$ be the set of all G-invariant differential operators of infra exponential type on G (see §2). The following important lemma can be proved by Lemma 1 in Appendix 1.

Lemma 3. *If $J_i(X) \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$ $(i = 1, 2)$ then $J_1(X)J_2(X) \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$ for any choice of the ordered basis $\{X_1, \dots, X_m\}$ on $\mathfrak{g}$. In particular $\widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$ is an $U(\mathfrak{g})$-module.*

Proof. Let $\{X_1, \dots, X_m\}$ be an ordered basis of $\mathfrak{g}$ and let

$$J_1(X) = \sum_{\alpha \in \mathbb{N}^m} \frac{a_\alpha}{\alpha!} X^\alpha, \qquad J_2(X) = \sum_{\gamma \in \mathbb{N}^m} \frac{c_\gamma}{\gamma!} X^\gamma.$$

Then

$$J_1(X)J_2(X) = \sum_{\alpha \in \mathbb{N}^m} \frac{a_\alpha}{\alpha!} X^\alpha J_2(X).$$

Put

$$X^\alpha J_2(X) = \sum_{\beta \in \mathbb{N}^m} c(\alpha, \beta) X^\beta.$$

By Lemma 1 in Appendix 1, we can express $X^\alpha X^\gamma$ as

$$X^\alpha X^\gamma = \sum_{|\beta| \leq |\alpha| + |\gamma|} p_\beta^{\alpha,\gamma} X^\beta$$

and we have the estimate

$$|p_\beta^{\alpha,\gamma}| \le C^{|\beta|+1} M^{|\alpha|+|\gamma|} \frac{(|\alpha|+|\gamma|)!}{|\beta|!}$$

M for some constants C and M.

Hence we see that

$$\begin{aligned} X^\alpha J_2(X) &= \sum_{\gamma\in\mathbb{N}^m} \frac{c_\gamma}{\gamma!}\Big(\sum_{|\beta|\le|\alpha|+|\gamma|} p_\beta^{\alpha,\gamma} X^\beta\Big) \\ &= \sum_{L=0}^{\infty} \sum_{\substack{|\beta|=L \\ \beta\in\mathbb{N}^m}} \Big(\sum_{\substack{\gamma\in\mathbb{N}^m \\ |\alpha|+|\gamma|\ge L}} \frac{c_\gamma}{\gamma!} p_\beta^{\alpha,\gamma}\Big) X^\beta. \end{aligned}$$

Suppose that $|\beta| = L$. Then we observe that

$$c(\alpha,\beta) = \sum_{\gamma\in\mathbb{N}^m,\, |\gamma|\ge L-|\alpha|} \frac{c_\gamma}{\gamma!} p_\beta^{\alpha,\gamma}.$$

Therefore we find that

$$\begin{aligned} |c(\alpha,\beta)| &\le \sum_{k=L}^{\infty} \sum_{|\gamma|=k-|\alpha|} \Big|\frac{c_\gamma}{\gamma!} p_\beta^{\alpha,\gamma}\Big| \\ &\le \frac{C^{L+1}}{L!} \sum_{k=L}^{\infty} M^k k! \sum_{|\gamma|=k-|\alpha|} \frac{|c_\gamma|}{\gamma!}. \end{aligned}$$

Since for any $r > 0$ there is a constant $d_r > 0$ such that

$$|c_\gamma| \le d_r \Big(\frac{1}{r}\Big)^{|\gamma|},$$

we have

$$\begin{aligned} |c(\alpha,\gamma)| &\le \frac{d_r C^{L+1}}{L!} \sum_{k=L}^{\infty} M^k k! \sum_{|\gamma|=k-|\alpha|} \frac{1}{\gamma!}\Big(\frac{1}{r}\Big)^{|\gamma|} \\ &= \frac{d_r C^{L+1}}{L!} \sum_{k=L}^{\infty} M^k k! \frac{m^{k-|\alpha|}}{(k-|\alpha|)!}\Big(\frac{1}{r}\Big)^{k-|\alpha|} \qquad \text{(A2.1)} \\ &= \frac{d_r C^{L+1}}{L!} \Big(\frac{r}{m}\Big)^{|\alpha|} \sum_{k=L}^{\infty} \frac{k!}{(k-|\alpha|)!}\Big(\frac{mM}{r}\Big)^k. \end{aligned}$$

On the other hand, we have

$$J_1(X)J_2(X) = \sum_\alpha \frac{a_\alpha}{\alpha!}\sum_\beta c(\alpha,\beta)X^\beta$$

(A2.2)

$$= \sum_\beta(\sum_\alpha \frac{a_\alpha}{\alpha!}c(\alpha,\beta))X^\beta.$$

Since there is also a constant $\widetilde{d}_r$ such that

$$|a_\alpha| \leq \widetilde{d}_r(\frac{1}{r})^{|\alpha|},$$

if $|\beta| = L$ then by (A2.1) we have

$$\begin{aligned}
&|\sum_\alpha \frac{a_\alpha}{\alpha!}c(\alpha,\beta)| \\
&\leq \sum_{\ell=0}^{\infty}\sum_{|\alpha|=\ell}\frac{|a_\alpha|}{\alpha!}\frac{d_rC^{L+1}}{L!}(\frac{r}{m})^\ell\sum_{k=L}^{\infty}\frac{k!}{(k-\ell)!}(\frac{mM}{r})^k \\
&\leq \frac{d_r\widetilde{d}_rC^{L+1}}{L!}\sum_{\ell=0}^{\infty}\frac{m^\ell}{\ell!}(\frac{1}{r})^\ell(\frac{r}{m})^\ell\sum_{k=L}^{\infty}k^\ell(\frac{mM}{r})^k \\
&= \frac{d_r\widetilde{d}_rC^{L+1}}{L!}\sum_{k=L}^{\infty}\sum_{\ell=0}^{\infty}\frac{k^\ell}{\ell!}(\frac{mM}{r})^k \\
&= \frac{d_r\widetilde{d}_rC^{L+1}}{L!}\sum_{k=L}^{\infty}(\frac{emM}{r})^k
\end{aligned}$$

Hence, if r is sufficiently large number we have

$$\begin{aligned}
|\sum_\alpha \frac{a_\alpha}{\alpha!}c(\alpha,\beta)| &\leq \frac{d_r\widetilde{d}_rC^{L+1}}{L!}\frac{(\frac{emM}{r})^L}{1-\frac{emM}{r}} \\
&\leq \frac{d_r\widetilde{d}_rC}{L!}(\frac{CemM}{r})^L.
\end{aligned}$$

By (A2.2), this proves the assertion.

We denote by $\mathcal{O}_2(\mathbb{C})$ the set of all functions $J(z) \in \mathcal{O}(\mathbb{C})$ such that $J(z^2)$ are of the infra exponential type. The following lemma can be proved similar way as above lemma using Lemma 1 in Appendix 1. So we leave the proof.

Lemma 4. *Let Δ be a Casimir operator of $\mathfrak{g}$. Suppose that $J(z) \in \mathcal{O}_2(\mathbb{C})$. For any ordered basis $\{X_1, \dots, X_m\}$ of $\mathfrak{g}$, put*

$$\widetilde{J}(X) = \widetilde{J}(X_1, \dots, X_m) = J(\Delta).$$

Then we have $\widetilde{J}(X) \in \widetilde{\mathcal{O}}(\mathfrak{g}_c^*)$.

The following similar lemma is now also obvious.

Lemma 5. *Let $J \in \mathcal{O}(\mathbb{C})$. Suppose that $J(t^M)$ is of the infra exponential type for some positive integer M. If we put*

$$\widetilde{J}(x_1, \dots, x_m) = J(\sum_{i=1}^{m} x_i^M \ + \textit{ lower terms in } \ x_i\text{'s})$$

then $\widetilde{J} \in \widetilde{\mathcal{O}}(\mathbb{C}^m)$.

Moreover, by Lemma 8.1.6 in [Kan], we have the following estimate.

Lemma 6. *For any $J \in \widetilde{\mathcal{O}}(\mathbb{C}^m)$, $r \leq 0$ and $a_i \in \mathbb{C}\,(i = 1, \dots, m)$, there exists $\widetilde{J} \in \mathcal{O}_2(\mathbb{C})$ such that*

$$|J(\zeta_1, \dots, \zeta_m)| \leq |\widetilde{J}(\sum_{i=1}^{m} \zeta_i^2 + \sum_{i=1}^{m} a_i \zeta_i)| \qquad \textit{for} \qquad |\operatorname{Im} \zeta| \leq r.$$

References

[B] J.-E. Björk, *Rings of Differential Operators*, North-Holland, Amsterdam, 1979.

[Eg1] M. Eguchi, *Asymptotic expansions of Eisenstein integrals and Fourier transform on symmetric spaces*, J. Funct. anal. **34** (1979), 167–216.

[Eg2] ———, *An application of topological Paley-Wiener theorems to invariant differential equations on symmetric spaces*, Analyse Harmonique sur les Groupes de Lie II, Lect. Notes in Math. **739** (1979), 193–206.

[EK] M. Eguchi and A. Kowata, *On the Fourier transform of rapidly decreasing function of L^p type on a symmetric space*, Hiroshima Math. J. **6** (1976), 143–158.

[EW] M. Eguchi and M. Wakayama, *An elementary proof of the Trombi theorem for Fourier transform of $C^p(G;F)$*, Hiroshima Math. J. **17** (1987), 471–487.

[Eh] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley-Interscience, New York, 1970.

[FJ] M. Flensted-Jensen, *Analysis on non-Riemannian Symmetric Spaces*, Regional conference series in math., 61, AMS, 1986.

[GS] I. M. Gel'fand and G. E. Shilov, *Generalized Functions 2*, Academic Press, New York, 1968.

[HC1] Harish-Chandra, *Spherical functions on a semisimple Lie group I, II*, Amer. J. Math. **80** (1958), 241–310, 553–613.

[HC2] ———, *Invariant eigendistributions on a semisimple Lie group*, Amer. Math. Soc. **119** (1965), 457–508.

[HC3] ———, *Discrete series for semisimple Lie groups II*, Acta Math. **116** (1966), 1–111.

[Ha] M. Hasumi, *Note on the n-dimensional tempered ultra-distributions*, Tohoku Math. J. **13** (1961), 94–104.

[He1] S. Helgason, *The surjectivity of invariant differential operators on symmetric spaces I*, Ann of Math. **98** (1973), 451–479.

[He2] ———, *A duality for symmetric spaces with applications to group representations, II, Differential equations and Eigenspace representations*, Advances in Math. **22** (1976), 187–219.

[He3] ———, *Invariant differential equations on homogeneous manifolds*, Bull. Amer. math. Soc. **83** (1977), 751–744.

[He4] ———, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New york, 1978.

[He5] ———, *A duality for symmetric spaces with applications to group representations, III, Tangent space analysis*, Advances in Math. **36** (1980), 297–323.

[He6] ———, *Groups and Geometric Analysis*, Academic Press, New York, 1984.

[K–] M. Kashiwara. A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math. **107** (1978), 1–39.

[Kan] A. Kaneko, *Introduction to Hyperfunctions II*, in Japanese, Press Univ. Tokyo, Tokyo, 1982.

[Kaw] T. Kawai, *On the theory of Fourier hyperfunctions and its application to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. tokyo Sect. IA **17** (1970), 467–517.

[Kom] H. Komatsu, *A proof of Kotaké and Narashimhan's theorem*, Proc. Japan Acad. **38** (1962), 615–618.

[Kos1] B. Kostant, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. **75** (1969), 627–642.

[Kos2] ———, *On the existence and irreducibility of certain series of representations*, Lie Groups and Their Representations, (ed. by I. M. Gel'fand), Halsted, New York, 1975., pp. 231–329.

[M] M. Morimoto, *Fourier transform and hyperfunctions*, Sem. Notes in Math.,**2**, in Japanese, Sophia Univ., Tokyo, 1978.

[N] S. Nagamachi, *The theory of vector valued Fourier hyperfunctions of mixed type, I, II*, Publ. RIMS Kyoto Univ. **17** (1981), 25–93.

[O1] T. Oshima, *A realization of Riemannian symmetric spaces*, J. Math. Soc. Japan **30** (1980), 117–132.

[O2] ———, *The boundary value problems of the differential equations with regular singularities and the theory of representations*, Sem. notes in Math., **2**, in Japanese, Sophia Univ., Tokyo, 1979.

[O3] ———, *A definition of boundary values of solutions of partial differential equations with regular singularities*, Publ. RIMS Kyoto Univ. **19** (1983), 1203–1230.

[OSW1] T. Oshima, Y. Saburi and M. Wakayama, *A note on Ehrenpreis' fundamental principle on a symmetric space*, Algebraic Analysis, (ed. by T. Kawai and M. Kashiwara), Academic Press, New York, 1989.

[OSW2] ———, *Ehrenpreis' fundamental principle on a symmetric space*, to appear.

[OS] T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators on an affine symmetric spaces*, Invent. Math. **57** (1980), 1–81.

[Sab] Y. Saburi, *Fundamental properties of modified Fourier hyperfunctions*, Tokyo J. Math. **8** (1985), 231–273.

[Sat] M. Sato, *Theory of hyperfunctions, I*, J. Fac. Univ. Tokyo, Sect. I **8** (1958), 139–193; *II*, ibid **8** (1960), 387–437.

[Sc] H. Schlichtkrull, *Hyperfunction and harmonic Analysis on Symmetric Spaces*, Birkhäuser, Boston, 1984.

[SS] J. Sebastiaõe Silva, *Les fonctions analitiques comme ultra-distributions dans le calcul opérationnel*, Math. Ann. **136** (1958), 58–96.

[Su] M. Sugiura, *Fourier series of smooth functions on a compact Lie group*, Osaka J. Math. **8** (1971), 33–47.

[TB] P. C. Trombi and V. S. Varadarajan, *Spherical transforms on semisimple Lie groups*, Ann. Math. **94** (1971), 246–303.

[Wal1] N. R. Wallach, *Asympototic expansion of generalized matrix entries of representations of reductive group Representations I*, Lect. Notes in Math., Springer **1024** (1983), 287–361.

[Wal2] ———, *P^γ and Q^γ matrices and intertwining operators*, Harmonic Analysis on Homogeneous Spaces, Proceeding of the symposium in pure mathematics of American Math. Soc., Vol. XXVI (ed. by Calvin C. Moore), 1973, pp. 269–273.

[War] G. Warner, *Harmonic Analysis on Semisimple Lie Groups I, II*, Springer-Verlag, New York, 1972.

[Z] V. V. Zharinov, *The Laplace transform of Fourier hyperfunctions and other similar cases of analytic functions I*, Teoret. Math. Fiz. **33** (1977), 291–309; *II*, ibid **37** (1978), 12–29.