Semilocal monodromy of rigid local systems

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Abstract The rigid local system on $\mathbb{P}^1 \setminus S$ with a set S of finite points is realized as a rigid Fuchsian differential equation $\mathscr{M}$ of Schlesinger canonical form. Here "rigid" means that the equation is uniquely determined by the equivalence classes of residue matrices of $\mathscr{M}$ at the points in S. The *semilocal monodromy* in this paper is the conjugacy class of the monodromy matrix obtained by the analytic continuation of the solutions of $\mathscr{M}$ along an oriented *simple* closed curve γ on $\mathbb{C} \setminus S$. Since it corresponds to the sum of residue matrices at the singular points surrounded by γ and the equation $\mathscr{M}$ is obtained by applying additions and middle convolutions to the trivial equation, we study the application of the middle convolution to the sums of residue matrices. In this way we give an algorithm calculating this semilocal monodromy, which also gives the local monodromy at the irregular singular point obtained by the confluence of these points.

1 Introduction

The global theory of Fuchsian differential equations has been greatly developed after the work of Katz [3] on rigid local systems, which we will shortly explain.

Fuchsian differential equation of Schlesinger canonical form is

$$\mathscr{M} : \frac{du}{dx} = \sum_{i=1}^{p} \frac{A_i}{x - c_i} u \tag{1}$$

with $A_i \in M(n, \mathbb{C})$. Here n is the rank of the equation, u is a column vector of n unknown functions, $M(n, \mathbb{C})$ denotes the set of square matrices of size N with components in $\mathbb{C}$, A_i is called the *residue matrix* at $x = c_i$ and the residue matrix at $x = \infty$

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equals $-(A_1+\cdots+A_p)$ which we denote by A_{p+1}. The equation $\mathscr{M}$ is called *irreducible* in Schlesinger canonical form if there exists no non-trivial proper subspace $V\subset\mathbb{C}^n$ satisfying $A_iV\subset V$ for $i=1,\dots,p$.

Definition 1. For a matrix $A\in M(n,\mathbb{C})$ we put $\{\mu_1,\dots,\mu_r\}=\{\mu\in\mathbb{C}\mid \operatorname{rank}(A-\mu)<n\}$. Then there exist positive numbers n_j and $m_{j,\nu}$ for $1\le j\le r$ and $1\le\nu\le n_j$ such that

$$\operatorname{corank}(A-\mu_j)^\nu=m_{j,1}+\cdots+m_{j,\nu}\quad(\nu=1,\dots,n_j),\tag{2}$$

$$\operatorname{rank}(A-\mu_j)^{n_j}=\operatorname{rank}(A-\mu_j)^{n_j+1}.\tag{3}$$

Here $\sum_{j=1}^r\sum_{\nu=1}^{n_j}m_{j,\nu}=n$. Note that the set $\{[\mu_j]_{m_{j,\nu}}\mid 1\le\nu\le n_j \text{ and } 1\le j\le r\}$, which we call the *eigenvalue class* of A and write by (EC) of A for simplicity, determines the conjugacy class of matrices containing A. The matrix A is semisimple if and only if $n_1=\cdots=n_r=1$. Here we may simply write μ_j in place of $[\mu_j]_1$.

Let $\{[\lambda_{i,\nu}]_{m_{i,\nu}}\mid 1\le\nu\le n_i\}$ be the eigenvalue classes of A_i for $i=1,\dots,p+1$, respectively. The *index of rigidity* of $\mathscr{M}$ defined by Katz [3] equals

$$\operatorname{idx}\mathscr{M}:=\sum_{i=1}^{p+1}\sum_{\nu=1}^{n_i}m_{i,\nu}^2-(p-1)n^2.$$

An irreducible equation $\mathscr{M}$ is called *rigid* if the conjugacy classes of A_i for $i=1,\dots,p+1$ uniquely determine the simultaneous conjugacy class of $(A_1,\dots,A_{p+1})$, which means that the local structure of $\mathscr{M}$ at singular points uniquely determines the global structure of $\mathscr{M}$. Katz [3] proved that an irreducible equation $\mathscr{M}$ is rigid if and only if $\operatorname{idx}\mathscr{M}=2$ by introducing two types of operations of the equations. They are additions and middle convolutions and keep the irreducibility and the index of rigidity. The addition $\operatorname{Ad}\bigl((x-c_k)^{\lambda_k}\bigr)$ is defined by the transformation $A_i\mapsto A_i+\lambda_k\delta_{i,k}$ $(i=1,\dots,p)$ with $\lambda_k\in\mathbb{C}$ and $1\le k\le p$, which corresponds to the transformation $u\mapsto(x-c_k)^{\lambda_k}u$. The middle convolution mc_μ corresponds to the fractional derivation $u\mapsto\bigl(\frac{d}{dx}\bigr)^{-\mu}u$ with $\mu\in\mathbb{C}$, which will be explained in the next section.

Katz [3] proved that any rigid local system is transformed into the trivial equation $u'=0$ of rank 1 by successive applications of additions and middle convolutions. Since these operations are invertible, any rigid local system is constructed and realized in the form (1) from the trivial equation by successive applications of these operations.

The author [4] interpreted the middle convolution for linear ordinary differential equations with polynomial coefficients, reduced various analysis of rigid Fuchsian ordinary differential equations to the study of solutions under the middle convolution and got many general results for solutions of rigid Fuchsian differential equations, such as their integral representations, connection formulas, series expansions, irreducibility of monodromy groups, contiguous relations etc. Note that any rigid local system is uniquely realized by a rigid single Fuchsian differential equation without an apparent singularity (cf. [6, Lemma 2.1]).

Dettweiler–Reiter [1] interpreted the middle convolution mc_μ introduced by Katz into an operation of the tuple of residue matrices $(A_1,\dots,A_{p+1})$. In fact, they explicitly gave the conjugacy classes of residue matrices $\bar{A}_j$ in terms of those of $A_1,\dots,A_{p+1}$. Here $(\bar{A}_1,\dots,\bar{A}_{p+1})$ is the tuple of residue matrices of $\mathrm{mc}_\mu\mathscr{M}$.

Let I be a subset of $\{1,\dots,p\}$ and put $A_I=\sum_{i\in I}A_i$. We show that the residue class of $\bar{A}_I=\sum_{i\in I}\bar{A}_i$ is explicitly determined by the residue classes of $\bar{A}_1,\dots,\bar{A}_{p+1}$ and $\bar{A}_I$, which is a generalization of a result in [1] and the main purpose of this paper.

Definition 2. Let γ be an oriented simple closed curve γ in $\mathbb{C}\setminus\{c_1,\dots,c_p\}$. We may assume

$$\frac{1}{2\pi\sqrt{-1}}\int_\gamma\frac{dz}{x-c_i}=\begin{cases}1 & (i\in I)\\ 0 & (i\notin I)\end{cases}\tag{4}$$

with a subset $I\subset\{1,\dots,p\}$. The *semilocal monodromy* of $\mathscr{M}$ for $\{c_i\mid i\in I\}$ is the conjugacy class of the monodromy matrix M of the solutions of $\mathscr{M}$ along the path γ. The semilocal monodromy of $\mathscr{M}$ for $\{c_i\mid i\in\{1,\dots,p+1\}\setminus I\}$ is the conjugacy class of the matrix M^{-1}.

Suppose $\mathscr{M}$ is rigid. Then the semilocal monodromy does not depend on the positions of c_i if (4) is valid. Hence it is the class containing $e^{2\pi\sqrt{-1}\bar{A}_I}$ if any difference of eigenvalues of A_I is not a non-zero integer. Note that it follows from Corollary 1 that any eigenvalue of the semilocal monodromy of $\mathscr{M}$ is a certain product of integer powers of eigenvalues of local monodromies at singular points. This is not valid when γ is not simple as is given in the first example in §4.

Suppose the points c_i for $i\in I$ coalesce into one conflunet point c_I and the rigid equation $\mathscr{M}$ is changed into an equation $\mathscr{M}'$ with an irregular singular point $c_I\in\mathbb{C}$. We may assume that the semilocal monodromy does not change in the confluence and then we get the local monodromy of $\mathscr{M}'$ at c_I. This is the same for the confluence of the points c_i for $i\in\{1,\dots,p+1\}\setminus I$.

2 Middle convolution of a sum of residue matrices

The convolution $\tilde{A}_k$ of the residue matrices A_k of $\mathscr{M}$ is given by

$$\tilde{A}_k=\begin{array}{c}\\ \\ k\\ \\ \\ \end{array}\!\!\begin{array}{c}\quad\quad\quad k\quad\quad\quad\\ \begin{pmatrix}0&\cdots&0&\cdots&0\\ \vdots&\cdots&\vdots&\cdots&\vdots\\ A_1&\cdots&A_k+\mu&\cdots&A_p\\ \vdots&\cdots&\vdots&\cdots&\vdots\\ 0&\cdots&0&\cdots&0\end{pmatrix}\end{array}\in M(pn,\mathbb{C})\qquad(1\le k\le p)\tag{5}$$

$$= \Big((A_j+\mu\delta_{i,j})\delta_{i,k}\Big)_{\substack{1\le i\le p\\ 1\le j\le p}}$$

Here $\tilde{A}_k$ are block matrices of size p whose entries are square matrices of size n and $\tilde{A}_{p+1} = -(\tilde{A}_1+\cdots+\tilde{A}_p)$. Let $\mu\in\mathbb{C}$ with $\mu\ne 0$. Then the subspaces

$$\mathscr{K}_j := \begin{array}{c} \\ \\ j \\ \\ \\ \end{array}\!\!\begin{pmatrix} 0\\ \vdots\\ \operatorname{Ker}A_j\\ 0\\ \vdots \end{pmatrix} \simeq \operatorname{Ker}A_j \quad (j=1,\dots,p),$$

$$\mathscr{K}_{p+1} := \left\{\begin{pmatrix} v\\ \vdots\\ v\end{pmatrix} \mid A_{p+1}v = \mu v\right\} \simeq \operatorname{Ker}(A_{p+1}-\mu) \ \text{ and } \ \mathscr{K} := \bigoplus_{j=1}^{p+1}\mathscr{K}_j$$

of $\mathbb{C}^{pn}$ are invariant under the linear transformations defined by $\tilde{A}_j$ for $j=1,\dots,p$. Then $\tilde{A}_j$ induce linear transformations of $\mathbb{C}^{pn}/\mathscr{K}$ and the corresponding matrices with respect to a base of $\mathbb{C}^{pn}/\mathscr{K}$ are denoted by $\bar{A}_j$, respectively. Then the equation

$$\bar{\mathscr{M}} : \frac{d\bar{u}}{dx} = \sum_{i=1}^{p}\frac{\bar{A}_i}{x-c_i}\bar{u}$$

is the middle convolution $\mathrm{mc}_\mu\,\mathscr{M}$ of $\mathscr{M}$ and the tuple of matrices $(\bar{A}_1,\dots,\bar{A}_p,\bar{A}_{p+1})$ is the middle convolution of the tuple $(A_1,\dots,A_{p+1})$. Here $\bar{A}_{p+1} = -(\bar{A}_1+\cdots+\bar{A}_p)$.

Put $\bar{A}_I := \sum_{i\in I}\bar{A}_i$, $\tilde{A}_I := \sum_{i\in I}\tilde{A}_i$ and

$$\iota_j(v) := \begin{array}{c} \\ \\ \\ j \\ \\ \\ \end{array}\!\!\begin{pmatrix} 0\\ \vdots\\ v\\ 0\\ \vdots\end{pmatrix} \quad (v\in\mathbb{C}^n,\ 1\le j\le p).$$

For simplicity, we assume $I=\{1,\dots,k\}$ with $1\le k\le p$. Then

$$\tilde{A}_I = \begin{array}{c} \\ \\ \\ \\ k \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} & & & k & & & \end{array} \\ \begin{pmatrix} A_1+\mu & A_2 & \cdots & A_k & A_{k+1} & \cdots & A_p\\ A_1 & A_2+\mu & \cdots & A_k & A_{k+1} & \cdots & A_p\\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots\\ A_1 & A_2 & \cdots & A_k+\mu & A_{k+1} & \cdots & A_p\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\end{array} \in M(pn,\mathbb{C}). \tag{6}$$

By the linear automorphism on $\mathbb{C}^{pn}$ defined by the matrix

$$P = \begin{array}{cc} & \begin{array}{cccccc} & & & k & & \end{array} \\ \begin{array}{c} \\ \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} I_n & & & & & \\ -I_n & I_n & & & & \\ \vdots & & \ddots & & & \\ -I_n & & & I_n & & \\ & & & & I_n & \\ & & & & & \ddots \end{pmatrix} \end{array} \in M(pn,\mathbb{C}),$$

the linear transformation $\tilde{A}_I$ on $\mathbb{C}^{pn}$ and the subspaces $\mathscr{K}_j$ are changed into

$$\tilde{A}_I' := P\tilde{A}_I P^{-1} = \begin{pmatrix} A_1+\cdots+A_k+\mu & A_2 & \cdots & A_k & A_{k+1} & \cdots & A_n \\ & \mu & & & & & \\ & & \ddots & & & & \\ & & & \mu & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{7}$$

$$\mathscr{K}_1' := \left\{ \begin{array}{c} \\ \\ k \\ \\ \\ \end{array} \begin{pmatrix} v \\ -v \\ \vdots \\ -v \\ 0 \\ \vdots \end{pmatrix} \middle| \, v \in \mathscr{K}_1 \right\}, \quad \mathscr{K}_{p+1}' := \left\{ \begin{array}{c} \\ \\ \\ k \\ \\ \\ \end{array} \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \\ v \\ \vdots \end{pmatrix} \middle| \, v \in \mathscr{K}_{p+1} \right\},$$

$$\mathscr{K}_j' := \mathscr{K}_j \quad (2 \le j \le p) \text{ and } \quad \mathscr{K}' := \bigoplus_{j=1}^{p+1} \mathscr{K}_j'.$$

Here we note that

$$\begin{aligned} (\tilde{A}_I' - \lambda)^\nu \iota_1(w) &= \iota_1\big((A_I+\mu-\lambda)^\nu w\big) \qquad (w \in \mathbb{C}^n,\ \nu = 1,2,\ldots), \\ \operatorname{corank}(\tilde{A}_I'-\lambda)^\nu|_{\mathbb{C}^{pn}/\mathscr{K}'} &= \operatorname{corank}(A_I+\mu-\lambda)^\nu \quad (\lambda \in \mathbb{C}\setminus\{0,\mu\},\ \nu = 1,2,\ldots) \\ \operatorname{corank}(\tilde{A}_I'-\mu)^{pn}|_{\mathbb{C}^{pn}/\mathscr{K}'} &= \dim\operatorname{Ker} A_I^n + (k-1)n - \sum_{i=1}^k \dim \mathscr{K}_i, \\ \operatorname{corank}(\tilde{A}_I'-0)^{pn}|_{\mathbb{C}^{pn}/\mathscr{K}'} &= \dim\operatorname{Ker}(A_I+\mu)^n + (p-k)n - \sum_{j=k+1}^p \dim \mathscr{K}_j. \end{aligned}$$

Since (EC) of $(\tilde{A}_I'-\lambda)^\nu|_{\mathbb{C}^{pn}/\mathscr{K}'}$ equals (EC) of $\overline{A}_I$, we have the following theorem by the above expression.

Theorem 1. *Retain the assumption* $\mu \neq 0$ *and* $I \subset \{1,\ldots,p\}$. *We have*

$$\begin{cases} \dim \mathrm{Ker}(\bar{A}_I - \lambda)^\nu = \dim \mathrm{Ker}(A_I + \mu - \lambda)^\nu \quad (\forall \lambda \in \mathbb{C} \setminus \{0, \mu\},\ \nu = 1, 2, \ldots), \\ \dim \mathrm{Ker}(\bar{A}_I - \mu)^{pn} = \dim \mathrm{Ker} A_I^n + (k-1)n - \sum_{i=1}^{k} \dim \mathscr{K}_i, \\ \dim \mathrm{Ker}(\bar{A}_I - 0)^{pn} = \dim \mathrm{Ker}\,(A_I + \mu)^n + (p-k)n - \sum_{i=k+1}^{p+1} \dim \mathscr{K}_i. \end{cases}$$

Suppose

$$\mathrm{Ker} A_I \subset \mathrm{Ker} A_1 \cap \cdots \cap \mathrm{Ker} A_k \tag{8}$$

and

$$\mathrm{Ker}\,(A_I + \mu) = \{0\} \ \ \textit{or} \quad k = p. \tag{9}$$

Then if A_I is semisimple, so is $\bar{A}_I$.

Proof. Note that the assumption (8) implies $\iota_1(\mathrm{Ker} A_I) \subset \bigoplus_{i=1}^k \mathscr{K}_i'$. Then the claims in the theorem are clear by the argument just before the theorem. □

Remark 1. (i) If a subset $J \subset \{0, \ldots, p+1\}$ contains $p+1$, we have a similar result for $\bar{A}_J = \sum_{j \in J} A_j$ by the fact $\bar{A}_{\{0,\ldots,p\} \setminus J} + \bar{A}_J = 0$.

(ii) The condition (8) in the theorem is valid if

$$\dim \mathrm{Ker} A_I \le \max\Big\{0, n - \sum_{i=1}^{k} \mathrm{codim}\, \mathrm{Ker} A_i\Big\}.$$

(iii) Dettweiler-Reiter [1] obtained (EC) of A_I when $\#I = 1$. Theorem 1 is a generalization of their result. As is given in [1] a multiplicative version of Theorem 1 may be possible.

(iv) Haraoka [2] showed that the rigid equation $\mathscr{M}$ can be extended to a KZ equation

$$\frac{\partial \tilde{u}}{\partial x_i} = \sum_{\substack{0 \le \nu \le p \\ \nu \ne i}} \frac{A_{i,\nu}}{x_i - x_\nu} \tilde{u} \quad (0 \le i \le p)$$

with $x_0 = x$, $x_j = c_j$ and $A_{0,j} = A_j$ $(j = 1, \ldots, p)$. Here $A_{i,j} = A_{j,i}$ and $A_{i,i} = 0$. Put $A_{i,p+1} := -(A_{i,0} + \cdots + A_{i,p})$ and $\tilde{A}_I := \sum_{1 \le \nu < \nu' \le k} A_{i_\nu, i_{\nu'}}$ for $I = \{i_1, \ldots, i_k\} \subset \{0, 1, \ldots, p+1\}$. Then the author [7] studied the simultaneous conjugacy class of the tuple $(\tilde{A}_I, \tilde{A}_J)$ when $I \cap J = \emptyset$ or $I \subset J$ which assures $[\tilde{A}_I, \tilde{A}_J] = 0$. Since $A_{\{1,\ldots,k\}} = \tilde{A}_J - \tilde{A}_I$ with $I = \{1, \ldots, k\}$ and $J = \{0, \ldots, k\}$, we have (EC) of $A_{\{1,\ldots,k\}}$ by this simultaneous conjugacy class. In fact, this is the original idea of this paper.

Corollary 1. *Let $\mathscr{L}$ be the integer lattice spanned by the eigenvalues of the residue matrices $A_1, \ldots, A_{p+1}$ of a rigid Fuchsian equation $\mathscr{M}$. Then any eigenvalue of $A_I = \sum_{i \in I} A_i$ is in $\mathscr{L}$ for any $I \subset \{1, \ldots, p+1\}$.*

Proof. We can reduce $\mathscr{M}$ to the trivial equation and construct it from the trivial equation by applying suitable operations $\mathrm{Ad}\big((x-x_j)^{\lambda_j}\big)$ and mc_μ with $\lambda_j,\,\mu\in\mathscr{L}$, which is explained in the next section. Hence the corollary follows from the theorem.

3 Semilocal monodromy

Let $\{[\lambda_{j,\nu}]_{n_j}\mid \nu=1,\dots,m_{j,\nu}\}$ be the eigenvalue classes of the residue matrices A_j of $\mathscr{M}$ given in (1) for $j=1,\dots,p+1$. Then the *generalized Riemann scheme* of $\mathscr{M}$ is defined by

$$\{\lambda_{\mathbf{m}}\}=\begin{Bmatrix} x=c_1 & \dots & c_p & \infty \\ [\lambda_{1,1}]_{m_{1,1}} & \cdots & [\lambda_{p,1}]_{m_{p,1}} & [\lambda_{p+1,1}]_{m_{p+1,1}} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{1,n_1}]_{m_{1,n_1}} & \cdots & [\lambda_{p,n_p}]_{m_{p,n_p}} & [\lambda_{p+1,n_{p+1}}]_{m_{p+1,n_{p+1}}} \end{Bmatrix} \tag{10}$$

and

$$\mathbf{m}=m_{1,1}\cdots m_{1,n_1},m_{2,1}\cdots m_{2,n_2},\cdots,m_{p+1,1}\cdots m_{p+1,n_{p+1}}$$

which express the $(p+1)$ tuples of partitions of n

$$n=m_{j,1}+\cdots+m_{j,n_j}\quad (j=1,\dots,p+1) \tag{11}$$

and is called the *spectral type* of $\mathscr{M}$. We put define $\operatorname{rank}\mathbf{m}=n$

The spectral type $\mathbf{m}$ is *ordered* if

$$m_{j,1}\ge m_{j,2}\ge\cdots\ge m_{j,n_j}\quad (j=1,\dots,p+1).$$

For a given spectral type $\mathbf{m}$, $s\mathbf{m}$ denotes the corresponding ordered spectral type.

For an ordered spectral type $\mathbf{m}$ we define

$$d(\mathbf{m}):=\sum_{j=1}^{p+1}m_{j,1}-(p-1)\operatorname{rank}\mathbf{m}, \tag{12}$$

$$\begin{aligned}&\partial\mathbf{m}:=\mathbf{m}'=\big(m'_{j,\nu}\big)_{\substack{\nu=1,\dots,n_j\\ j=1,\dots,p+1}}\quad\text{with}\\ &m'_{j,\nu}=m_{j,\nu}-d(\mathbf{m})\delta_{\nu,1}\quad(\nu=1,\dots,n_j,\ j=1,\dots,p+1).\end{aligned} \tag{13}$$

Here some $m'_{j,\nu}$ may be zero. Then such $m'_{j,\nu}$ are omitted and n_j may be decreased.

It is proved by Katz [3] that $\mathscr{M}$ is rigid if there exists a non-negative integer r such that $(\partial s)^\nu\mathbf{m}$ are tuples of partitions of positive integers for $\nu\in\{1,\dots,r\}$ and moreover

$$\operatorname{rank}\mathbf{m}>\operatorname{rank}\partial s\mathbf{m}>\operatorname{rank}(\partial s)^2\mathbf{m}>\cdots>\operatorname{rank}(\partial s)^r\mathbf{m}=1. \tag{14}$$

Here $(p+1)$ tuples of partition $\mathbf{m}$ mean that $m_{j,\nu}$ are non-negative integer in (11).

Suppose (10) is the generalized Riemann scheme of $\mathscr{M}$. Suppose moreover $\operatorname{rank}\mathscr{M} > 1$ and $\mathbf{m}$ is ordered by replacing $\mathbf{m}$ by $s\mathbf{m}$ if necessary. Applying $\prod_{j=1}^{p} \operatorname{Ad}\big((x-c_j)^{-\lambda_{j,1}}\big)$ to $\mathscr{M}$, we may assume $\lambda_{j,1} = \cdots = \lambda_{j,p} = 0$. Then we apply $\mathrm{mc}_{\lambda_{p+1}}$ to the system, we get a rigid Fuchsian equation with the spectral type $\partial\mathbf{m}$. The sequence (14) of spectral types corresponds to this procedure.

Katz [3] moreover showed that if $\mathbf{m}$ are tuples of partitions with this property (14), then for any $\lambda_{j,\nu}$ satisfying Fuchs condition

$$\sum_{j=1}^{p+1}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} = 0,$$

there exists a Fuchsian equation $\mathscr{M}$ with the generalized Riemann scheme (10), which is rigid for a generic $\lambda_{j,\nu}$. This follows from the fact $\operatorname{Ad}\big((x-c_j)^{\lambda}\big)\circ\operatorname{Ad}\big((x-c_j)^{-\lambda}\big) = \mathrm{mc}_{-\mu}\circ\mathrm{mc}_{\mu} = \mathrm{id}$.

The necessary and sufficient condition for the irreducibility of the monodromy group of the solutions of $\mathscr{M}$ is explicitly given (cf. [4, Proposition 10.16] and [6]). Then (EC) of the local monodromy matrix at $x = c_j$ is given by

$$\big\{[e^{2\pi\sqrt{-1}\lambda_{j,\nu}}]_{m_{j,\nu}} \mid \nu = 1,\ldots,n_j\big\}$$

if $\mathscr{M}$ is rigid, which is given in [4, Remark 10.11 iii)]. This is not obvious when there exist $\nu < \nu'$ with $\lambda_{j,\nu} - \lambda_{j,\nu'} \in \mathbb{Z}\setminus\{0\}$ but this is proved as follows.

If $\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z}$ for any ν and ν' with $1 \le \nu < \nu' \le n_j$, the claim is obvious.

Suppose (EC) of a matrix $A(t)$ with the continuous parameter $t \in [0,1]$ is given by $\{[\lambda_\nu(t)]_{m_\nu} \mid \nu = 1,\ldots,r\}$ for $t \in (0,1]$. We may assume $\lambda_\nu(t)$ are continuous functions on $[0,1]$. Then (EC) of a matrix $A(0)$ weakly equals $\{[\lambda_\nu(0)]_{m_\nu} \mid \nu = 1,\ldots,r\}$ (cf. [5, Proposition 3.3]). Here "weakly" means that the condition (2) is replaced by

$$\operatorname{corank}(A-\mu_j)^\nu \ge m_{j,1}+\cdots+m_{j,\nu} \quad (\nu = 1,\ldots,n_j) \tag{15}$$

in Definition 1. Then the index of rigidity with respect to the local monodromy matrices implies the above statement.

Proposition 1. *Let $\mathscr{M}$ in* (1) *be a rigid Fuchsian differential equation and for $I \subset \{1,\ldots,p\}$, let $\big\{[\lambda_\nu]_{m_\nu} \mid \nu = 1,\ldots,r\big\}$ be (EC) of $A_I = \sum_{i\in I} A_i$. Suppose $\lambda_\nu - \lambda_{\nu'} \notin \mathbb{Z}\setminus\{0\}$ for $1 \le \nu < \nu' \le r$. Then (EC) of the semilocal monodromy of $\mathscr{M}$ for $\{c_i \mid i \in I\}$ equals $\big\{[e^{2\pi\sqrt{-1}\lambda_\nu}]_{m_\nu} \mid \nu = 1,\ldots,r\big\}$.*

Proof. Since the equation is rigid, the semilocal monodromy does not depend on the points c_i and we may choose points c_i $(i \in I)$ as a single point, which implies the proposition. There may be a better understanding of this proof if we consider c_j as variables (cf. Remark 1 (iv)). □

Remark 2. (i) We expect that the semilocal monodromy for a rigid spectral type $\mathbf{m}$ with a generalized Riemann scheme (10) is semisimple if the exponents $\lambda_{j,\nu}$ are generic under the Fuchs condition. Note that the semisimlicity of local monodromies do not assure that of a semilocal monodromy (cf. (16) with $\lambda_1+\cdots+\lambda_k+\mu=0$).

We also expect that by the continuation of parameters $\lambda_{j,\nu}$ with the rigidity, we also determine a semilocal monodromy even if it is not semisimple as in the case of the local monodromy.

(ii) The algorithm calculating (EC) of A_I given in this paper is implemented in a computer algebra, which is contained in [8].

4 Examples

We start with Gauss hypergeometric equation, which is characterized by the spectral type $11,11,11$. Applying the operation $\mathrm{mc}_\gamma\circ\mathrm{Ad}\big((x-1)^\beta\big)\circ\mathrm{Ad}\big(x^\alpha\big)$ to the trivial equation, we get

$$\frac{du}{dx}=\left(\frac{\begin{pmatrix}\alpha+\gamma & \beta\\ 0 & 0\end{pmatrix}}{x}+\frac{\begin{pmatrix}0 & 0\\ \alpha & \beta+\gamma\end{pmatrix}}{x-1}\right)u$$

with the Riemann scheme

$$\begin{Bmatrix} x=0 & 1 & \infty\\ 0 & 0 & -\gamma\\ \alpha+\gamma & \beta+\gamma & -\alpha-\beta-\gamma \end{Bmatrix}.$$

Under a suitable base of solutions the local monodromy matrices M_0 at $x=0$ and M_1 at $x=1$ are given by

$$M_0=\begin{pmatrix} ac & (b-1)c\\ 0 & 1\end{pmatrix},\quad M_1=\begin{pmatrix} 1 & 0\\ a-1 & bc\end{pmatrix}$$

with $a=e^{2\pi\sqrt{-1}\alpha}$, $b=e^{2\pi\sqrt{-1}\beta}$ and $c=e^{2\pi\sqrt{-1}\gamma}$.

The monodromy matrix corresponding to a simple closed curve $|z|=2$ is given by M_1M_0 and (EC) of M_1M_0 is $\{c,abc\}$ if the equation is irreducible.

The monodromy matrix corresponding to a closed curve C with $\frac{1}{2\pi\sqrt{-1}}\int_C\frac{dz}{z}=-1$ and $\frac{1}{2\pi\sqrt{-1}}\int_C\frac{dz}{z-1}=1$ is given by $M_1M_0^{-1}$. The eigenvalue of $M_1M_0^{-1}$ is not a rational function of a, b and c. For example, if $a=c=-1$, the eigenvalue t satisfies $t^2+3(b-1)t-b=0$.

Applying $\mathrm{mc}_\mu\circ\prod_{j=1}^p\mathrm{Ad}\big((x-c_j)^{\lambda_j}\big)$ to the trivial equation, we get Jordan-Pochhammer equation $\mathscr{M}$ with the generalized Riemann scheme

$$\begin{Bmatrix} x=c_1 & \cdots & c_p & \infty\\ [0]_{p-1} & \cdots & [0]_{p-1} & [-\mu]_{p-1}\\ \lambda_1+\mu & \cdots & \lambda_p+\mu & -\lambda_1-\cdots-\lambda_p-\mu \end{Bmatrix}.$$

This equation is characterized by the spectral type $\overbrace{(p-1)1,(p-1)1,\cdots,(p-1)1}^{p+1}$. The monodromy group of this equation is irreducible if and only if any one of the $(p+2)$ numbers $\lambda_1,\cdots,\lambda_p,\mu,\lambda_1+\cdots+\lambda_p+\mu$ is not an integer (cf. [4, §13.3]).

Then (EC) of $A_{1,\dots,k}$ with $1\le k\le p$ equals

$$\bigl\{\lambda_1+\cdots+\lambda_k+\mu,\ [0]_{p-k},\ [\mu]_{k-1}\bigr\}$$

and (EC) of the semilocal monodromy for $\{c_1,\dots,c_k\}$ equals

$$\bigl\{e^{2\pi\sqrt{-1}(\lambda_1+\cdots+\lambda_k+\mu)},\ [1]_{p-k},\ [e^{2\pi\sqrt{-1}\mu}]_{k-1}\bigr\} \tag{16}$$

if the equation has an irreducible monodromy (cf. (7)). Replacing

$$c_j \text{ by } \frac{1}{\tilde c_j} \text{ and } \lambda_j \text{ by } \sum_{i=j}^{p}\frac{\tilde\lambda_i}{\tilde c_j\prod\limits_{k+1\le\nu\le i,\,\nu\ne j}(\tilde c_j-\tilde c_\nu)} \text{ for } j=k+1,\dots,p,$$

we get an irregular singularity at $x=\infty$ by the confluence given by $\tilde c_j\to 0$ for $j=k+1,\dots,p$ which corresponds to a *versal addition* defined in [4, §2.3] (cf. [4, §13.3]). This versal addition depends holomorphically on $\tilde c_j$ and equals $\mathrm{Ad}\bigl(e^{-\tilde\lambda_{k+1}x-\frac{\tilde\lambda_{k+2}}{2}x^2-\cdots-\frac{\tilde\lambda_p}{p-k}x^{p-k}}\prod_{j=1}^{k}(x-c_j)^{\lambda_j}\bigr)$ when $\tilde c_{k+1}=\cdots=\tilde c_p=0$. Then the conjugacy class of the semilocal monodromy matrix for $\{c_1,\dots,c_k\}$ is kept invariant under the confluence and (EC) of the inverse of the local monodromy matrix at the irregular singular point equals (16).

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