Generalized hypergeometric functions with several variables*

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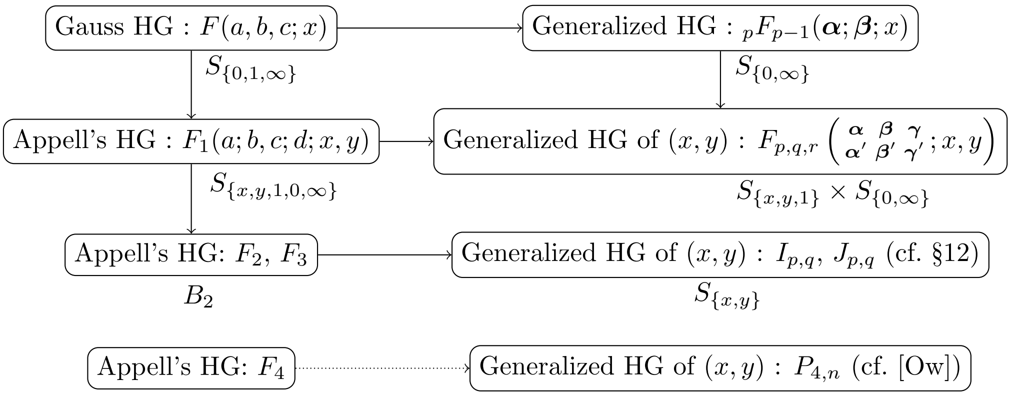
1 Generalized Hypergeometric Series of Two Variables

We show some relations between well-known hypergeometric functions and the hypergeometric functions denoted by $F^{p,q,r}_{p',q',r'}(x,y)$, which we will study and were essentially introduced by J. Kampé de Fériet [AK].



$$F^{p,q,r}_{p',q',r'}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma}\\ \boldsymbol{\alpha}' & \boldsymbol{\beta}' & \boldsymbol{\gamma}'\end{smallmatrix};x,y\right) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_m(\boldsymbol{\beta})_n(\boldsymbol{\gamma})_{m+n}}{(1-\boldsymbol{\alpha}')_m(1-\boldsymbol{\beta}')_n(1-\boldsymbol{\gamma}')_{m+n}}x^m y^n$$

$$\boldsymbol{\alpha}\in\mathbb{C}^p,\ \boldsymbol{\beta}\in\mathbb{C}^q,\ \boldsymbol{\gamma}\in\mathbb{C}^r,\ \boldsymbol{\alpha}'\in\mathbb{C}^{p'},\ \boldsymbol{\beta}'\in\mathbb{C}^{q'},\ \boldsymbol{\gamma}'\in\mathbb{C}^{r'},$$

$$(\boldsymbol{\alpha})_m=\tfrac{\Gamma(\boldsymbol{\alpha}+m)}{\Gamma(\boldsymbol{\alpha})}:=(\alpha_1)_m(\alpha_2)_m\cdots(\alpha_p)_m,\quad \Gamma(\boldsymbol{\alpha})=\Gamma(\alpha_1)\cdots\Gamma(\alpha_p),$$

$$c-\boldsymbol{\alpha}=(c-\alpha_1,\dots,c-\alpha_p),\quad |\boldsymbol{\alpha}|=\alpha_1+\cdots+\alpha_p,\quad (1-a)_{-m}=\tfrac{(-1)^m}{(a)_m}.$$

2 Generalized Hypergeometric Functions (one variable)

We review the generalized hypergeometric function ${}_pF_{p-1}(x)$ of one variable (for example, see [§13.4, Ow]).

$$F_p\left(\begin{smallmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\alpha}'\end{smallmatrix};x\right):=\sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_n}{(1-\boldsymbol{\alpha}')_n}x^n\quad (\boldsymbol{\alpha},\,\boldsymbol{\alpha}'\in\mathbb{C}^p),$$

$$\prod_{i=1}^{p}(\vartheta-\alpha_i')u\equiv x\prod_{i=1}^{p}(\vartheta+\alpha_i)u\pmod{\mathbb{C}},\qquad \vartheta:=x\tfrac{d}{dx}.$$

Here we have

$$u=\sum_{n=0}^{\infty}c_nx^n\Rightarrow\prod_{i=1}^{p}(n-\alpha_i')c_nx^n=x\prod_{i=1}^{p}(n-1+\alpha_i)c_{n-1}x^{n-1}\quad(n>0)$$

$$\Rightarrow c_n=\frac{\prod_{i=1}^{p}(n-1+\alpha_i)}{\prod_{i'=1}^{p'}(n-\alpha_i')}c_{n-1}=\cdots=\frac{(\boldsymbol{\alpha})_n}{(1-\boldsymbol{\alpha}')_n}c_0.$$

*This work is a joint work with Matsubara-Heo ([MO]).

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Note that $(1)_n = n!$ and if $(\boldsymbol{\alpha}')_1\,(= \alpha'_1 \cdots \alpha'_{p'}) = 0$, then the above "$\equiv$" can be replaced by "$=$". Since

$$\mathrm{Ad}(x^\lambda)h(\vartheta) := x^{-\lambda} \circ h(\vartheta) \circ x^\lambda = h(\vartheta + \lambda)$$

for a polynomial $f(\vartheta)$ of ϑ, the functions

$$\begin{aligned} F_p\left(\begin{smallmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\alpha}'\end{smallmatrix}; \alpha'_i; \pm; x\right) &:= (\pm x)^{\alpha'_i} F_p\left(\begin{smallmatrix}\boldsymbol{\alpha}+\alpha'_i\\ \boldsymbol{\alpha}'-\alpha'_i\end{smallmatrix}; x\right) \qquad (i = 1, \dots, p) \\ &= (\pm x)^{\alpha'_i}{}_pF_{p-1}\big(\boldsymbol{\alpha} + \alpha'_i; \{1 - \alpha'_\nu + \alpha'_i : \nu \neq i\}; x\big) \end{aligned}$$

give p solutions of the generalized HG equation

$$\mathcal{M}_p\,:\, \prod_{i=1}^{p}(\vartheta - \alpha'_i) = x \prod_{i=1}^{p}(\vartheta + \alpha_i)u.$$

Moreover we have the following.

Generalized Riemann scheme ($p = 2$: Gauss' HG):

$$\begin{Bmatrix} x = 0 & x = 1 & x = \infty \\ \alpha'_1 & [0]_{(p-1)} & \alpha_1 \\ \vdots & & \vdots \\ \alpha'_p & \delta & \alpha_p \end{Bmatrix},\ [0]_{(p-1)} := \begin{pmatrix} 0 \\ \vdots \\ p-1 \end{pmatrix} \qquad \Big(\sum_{\nu=1}^{p}\alpha_\nu + \sum_{\nu=1}^{p}\alpha'_\nu + \delta = p - 1\Big)$$

Irreducibility $\Leftrightarrow$ $\alpha'_i + \alpha_j \notin \mathbb{Z}$.
Solutions at $x = \infty$:

$$F_p\left(\begin{smallmatrix}\boldsymbol{\alpha}'\\ \boldsymbol{\alpha}\end{smallmatrix}; \alpha_j; \pm; \tfrac{1}{x}\right) = (\pm\tfrac{1}{x})^{\alpha_j}{}_pF_{p-1}(\boldsymbol{\alpha}' + \alpha_j, \{1 - \alpha_\nu + \alpha_j : \nu \neq j\}; \tfrac{1}{x})$$

Connection coefficients:

$$\begin{aligned} F_p\left(\begin{smallmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\alpha}'\end{smallmatrix}; \alpha'_i; -; x\right) &= \sum_{j=1}^{p} c(0 : \alpha'_i \rightsquigarrow \infty : \alpha_j) F_p\left(\begin{smallmatrix}\boldsymbol{\alpha}'\\ \boldsymbol{\alpha}\end{smallmatrix}; \alpha_j; -; \tfrac{1}{x}\right) \quad (x \in \mathbb{C} \setminus [0, \infty]), \\ c(0 : \alpha'_i \rightsquigarrow \infty : \alpha_j) &= \prod_{\substack{\nu \neq i \\ 1 \le \nu \le p}} \frac{\Gamma(1 + \alpha'_i - \alpha'_\nu)}{\Gamma(1 - \alpha'_\nu - \alpha_j)} \prod_{\substack{\nu \neq j \\ 1 \le \nu \le p}} \frac{\Gamma(\alpha_\nu - \alpha_j)}{\Gamma(\alpha'_i + \alpha_\nu)}. \end{aligned}$$

3 Equations satisfied by Hypergeometric Series

Recall the power series introduced in §1. Then as in the last section it is easy to see that the $p'q'$ functions

$$\begin{aligned} &F^{p,q,r}_{p',q',r'}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma}\\ \boldsymbol{\alpha}' & \boldsymbol{\beta}' & \boldsymbol{\gamma}'\end{smallmatrix}; \alpha'_i, \beta'_j; x, y\right) \qquad (i = 1, \dots, p',\ j = 1, \dots, q') \\ &\quad := x^{\alpha'_i} y^{\beta'_j} F^{p,q,r}_{p',q',r'}\left(\begin{smallmatrix}\boldsymbol{\alpha}+\alpha'_i & \boldsymbol{\beta}+\beta'_j & \boldsymbol{\gamma}+\alpha'_i+\beta'_j\\ \boldsymbol{\alpha}'-\alpha'_i & \boldsymbol{\beta}'-\beta'_j & \boldsymbol{\gamma}'-\alpha'_i-\beta'_j\end{smallmatrix}; \alpha'_i, \beta'_j; x, y\right) \end{aligned}$$

are solutions of the system

$$\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'} : \begin{cases} \displaystyle\prod_{i=1}^{p'}(\vartheta_x - \alpha'_i) \prod_{k=1}^{r'}(\vartheta_x + \vartheta_y - \gamma'_k)u = x \prod_{i=1}^{p}(\vartheta_x + \alpha_i) \prod_{k=1}^{r}(\vartheta_x + \vartheta_y + \gamma_k)u, \\ \displaystyle\prod_{j=1}^{q'}(\vartheta_y - \beta'_i) \prod_{k=1}^{r'}(\vartheta_x + \vartheta_y - \gamma'_k)u = y \prod_{j=1}^{q}(\vartheta_y + \beta_j) \prod_{k=1}^{r}(\vartheta_x + \vartheta_y + \gamma_k)u, \\ \displaystyle x\prod_{i=1}^{p}(\vartheta_x + \alpha_i) \prod_{j=1}^{q'}(\vartheta_y - \beta'_i)u = y \prod_{j=1}^{q}(\vartheta_y + \beta_j) \prod_{i=1}^{p'}(\vartheta_x - \alpha'_i)u. \end{cases}$$

4 Some Conditions

Consider the following conditions for $F^{p,q,r}_{p',q',r'}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma} \\ \boldsymbol{\alpha}' & \boldsymbol{\beta}' & \boldsymbol{\gamma}'\end{smallmatrix}; x, y\right)$ and $\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'}$.

$$p + r = p' + r', \quad q + r = q + r' \tag{F}$$

$$p = p',\ q = q',\ r = r' \tag{F1}$$

$$p = p' - 1,\ q = q' - 1,\ r = 1,\ r' = 0 \tag{F2}$$

$$p' = p - 1,\ q' = q - 1,\ r = 0,\ r' = 1 \tag{F3}$$

We always assume the condition (F) in this note.

(F) $\Leftrightarrow$ $F^{p,q,r}_{p',q',r'}(x, 0)$ and $F^{p,q,r}_{p',q',r}(0, y)$ satisfy a Fuchsian differential equation

$\Rightarrow$ $\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'}$: Fuchsian

$\Rightarrow$ $F^{p,q,r}_{p',q',r'}(x, y)$ has an integral representation given by [Oi] (cf. §6).

Note that (F1) or (F2) or (F3) implies (F).

5 Examples: Known Series

We introduce another function $G^{p,q,r,r'}_{p',q'}(x, y)$. Appell's hypergeometric functions and some Horn's hypergeometric functions (cf. [Er]) are examples of the functions we introduced:

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = F^{1,1,1}_{1,1,1}\left(\begin{smallmatrix}\beta & \beta' & \alpha \\ 0 & 0 & 1-\gamma\end{smallmatrix}; x, y\right)$$

$$F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = F^{1,1,1}_{2,2,0}\left(\begin{smallmatrix}\beta & \beta' & \alpha \\ 0,1-\gamma & 0,1-\gamma' & \end{smallmatrix}; x, y\right)$$

$$F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = F^{2,2,0}_{1,1,1}\left(\begin{smallmatrix}\alpha,\alpha' & \beta,\beta' & \\ 0 & 0 & 1-\gamma\end{smallmatrix}; x, y\right)$$

$$F_4(\alpha; \beta; \gamma, \gamma'; x, y) = F^{0,0,2}_{2,2,0}\left(\begin{smallmatrix}\emptyset & \emptyset & \alpha,\beta \\ 0,1-\gamma & 0,1-\gamma' & \emptyset\end{smallmatrix}; x, y\right)$$

$$G_2(\alpha, \alpha'; \beta, \beta'; x, y) = G^{1,1,1,1}_{1,1}\left(\begin{smallmatrix}\alpha & \alpha' & \beta' & \beta \\ 0 & 0 & & \end{smallmatrix}; x, y\right)$$

$$H_2(\alpha, \beta, \gamma, \delta, \epsilon; x, y) = G^{1,2,1,0}_{2,1}\left(\begin{smallmatrix}\beta & \gamma,\delta & \alpha \\ 0,1-\epsilon & 0 & \end{smallmatrix}; x, y\right)$$

$$F^{p,q,r}_{p',q',r'}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma} \\ \boldsymbol{\alpha}' & \boldsymbol{\beta}' & \boldsymbol{\gamma}'\end{smallmatrix}; x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha})_m(\boldsymbol{\beta})_n(\boldsymbol{\gamma})_{m+n}}{(1-\boldsymbol{\alpha}')_m(1-\boldsymbol{\beta}')_n(1-\boldsymbol{\gamma}')_{m+n}} x^m y^n$$

$$G^{p,q,r,r'}_{p',q'}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma} & \boldsymbol{\gamma}' \\ \boldsymbol{\alpha}' & \boldsymbol{\beta}' & & \end{smallmatrix}; x, y\right) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha})_m(\boldsymbol{\beta})_n(\boldsymbol{\gamma})_{m-n}(\boldsymbol{\gamma}')_{n-m}}{(1-\boldsymbol{\alpha}')_m(1-\boldsymbol{\beta}')_n} x^m y^n$$

$$G^{p,q,r}_{p',q',r'}(x, y) := G^{p,q,r,r'}_{p',q'}((-1)^{r'}x, (-1)^{r'}y), \quad (\boldsymbol{\gamma}')_{n-m}x^m y^n = \frac{((-1)^{r'}x)^m((-1)^{r'}y)^n}{(1-\boldsymbol{\gamma}')_{m-n}}$$

6 Integral Representation

$$K^{\mu,\boldsymbol{\lambda}}_x u := \frac{\Gamma(|\boldsymbol{\lambda}| + \mu)}{\Gamma(\mu)\Gamma(\boldsymbol{\lambda})} \int_{\substack{t_1>0,\dots,t_n>0 \\ t_1+\cdots+t_n<1}} t^{\boldsymbol{\lambda}-1}(1 - |\mathbf{t}|)^{\mu-1} u(t_1x_1, \dots, t_nx_n)\, dt_1 \dots dt_n$$

$$L^{\mu,\boldsymbol{\lambda}}_x u := \frac{\Gamma(\mu + n)\Gamma(\boldsymbol{\lambda})}{(2\pi i)^n\Gamma(|\boldsymbol{\lambda}| + \mu)} \int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty} \cdots \int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty} t^{1-\boldsymbol{\lambda}}(1 - |\mathbf{t}|)^{-\mu-n} u(\tfrac{x_1}{t_1}, \dots, \tfrac{x_n}{t_n}) \tfrac{dt_1}{t_1} \cdots \tfrac{dt_n}{t_n}$$

$$(T_{x\to R(x)}u)(x) := u(R(x)) \quad \text{(coordinate transformation)}$$

These and the following transformation of convergent power series are introduced by [Oi].

$$\tilde{K}_x^{\mu,\lambda} := T^{-1}_{\mathbf{x}\mapsto(x_1,\frac{x_1}{x_2},\ldots,\frac{x_1}{x_n})} \circ K^{\mu,\lambda}_{x_1} \circ T_{\mathbf{x}\mapsto(x_1,\frac{x_1}{x_2},\ldots,\frac{x_1}{x_n})}$$

$$(\tilde{K}^{\mu,\lambda}_{\mathbf{x}} u)(x) = \frac{\Gamma(\mu+\lambda)}{\Gamma(\mu)\Gamma(\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\mu-1} u(tx_1,\ldots,tx_n)dt$$

These transformations have the following properties:

	$K^{\mu-\alpha-\beta,(\alpha,\beta)}_{x,y}$	$L^{\mu-\alpha-\beta,(\alpha,\beta)}_{x,y}$	$\tilde{K}^{\mu-\gamma,\gamma}_{x,y}$	$K^{\mu-\alpha,\alpha}_{x}$
$x^m y^n \mapsto$	$\frac{(\alpha)_m(\beta)_n}{(\mu)_{m+n}} x^m y^n$	$\frac{(\mu)_{m+n}}{(\alpha)_m(\beta)_n} x^m y^n$	$\frac{(\gamma)_{m+n}}{(\mu)_{m+n}} x^m y^n$	$\frac{(\alpha)_m}{(\mu)_m} x^m y^n$

$$F^{p,q,r}_{p,q,r}(x,y) = \prod_{i=1}^{p-1} K_x^{1-\alpha'_i-\alpha_i,\alpha_i} \prod_{j=1}^{q-1} K_y^{1-\beta'_j-\beta_j,\beta_j} \prod_{r=1}^{r} \tilde{K}_{x,y}^{1-\gamma'_k-\gamma_k,\gamma_k} (1-x)^{1-\alpha_p}(1-y)^{-\beta_q}$$

$$F^{p-1,q-1,r+1}_{p,q,r}(x,y) = \prod_{i=1}^{p-1} K_x^{1-\alpha'_i-\alpha_i,\alpha_i} \prod_{j=1}^{q-1} K_y^{1-\beta'_j-\beta_j,\beta_j} \prod_{k=1}^{r} \tilde{K}_{x,y}^{1-\gamma'_k-\gamma_k,\gamma_k} (1-x-y)^{-\gamma_{r+1}}$$

etc. with $\alpha'_{p'} = \beta'_{q'} = 0$.

7 Transformation of Hypergeometric Equations

By a fractional linear transformation of $\mathbb{P}^1_{\mathbb{C}}$, 5 points x_0, x_1, x_2, x_3 and x_4 in $\mathbb{P}^1_{\mathbb{C}}$ can be uniquely transformed to 5 points x, y, 1, 0 and ∞ . Some hypergeometric functions with the variables (x,y) has a symmetry corresponding to the permutation of 5 points x_0, x_1, x_2, x_3 and x_4 under this correspondence. This permutation group isomorphic to S_5 is realized by a group of coordinate transformations of (x,y) generated by 4 elements:

$$\mathbb{P}^1_{\mathbb{C}} \ni (x_0,x_1,x_2,x_3,x_4) \mapsto (x,y,1,0,\infty) \qquad (\Leftarrow \text{KZ-type})$$

$$\begin{array}{ccccc} S_5 \ni x_0 \leftrightarrow x_1 & x_1 \leftrightarrow x_2 & x_2 \leftrightarrow x_3 & x_3 \leftrightarrow x_4 & \\ \circ\!\!-\!\!-\!\!-\!\!-\!\!-\!\!- & \circ\!\!-\!\!-\!\!-\!\!-\!\!-\!\!- & \bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!- & \circ & : S_3 \times S_2 \\ (x,y) \mapsto (y,x) & (\frac{x}{y},\frac{1}{y}) & (1-x,1-y) & (\frac{1}{x},\frac{1}{y}) & \end{array}$$

The subgroup generated by the transformations $(x,y) \mapsto (y,x)$, $(\frac{x}{y},\frac{1}{y})$ and $(\frac{1}{x},\frac{1}{y})$ define transformations of the equation $\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'}$ as follows. Note that the subgroup is isomorphic to $S_3 \times S_2$ with 12 elements.

Equations satisfied by (X,Y) with $X = X(x,y)$, $Y = Y(x,y)$ (under (F)):

(X,Y)	(x,y)	$(\epsilon'\frac{x}{y},\frac{1}{y})$	$(\frac{1}{y},\epsilon'\frac{x}{y})$	$(\frac{1}{x},\epsilon\frac{y}{x})$	$(\epsilon\frac{y}{x},\frac{1}{x})$	(y,x)	$(\frac{1}{x},\frac{1}{y})$
(x,y)	(X,Y)	$(\epsilon'\frac{X}{Y},\frac{1}{Y})$	$(\epsilon\frac{Y}{X},\frac{1}{X})$	$(\frac{1}{X},\epsilon\frac{Y}{X})$	$(\frac{1}{Y},\epsilon'\frac{X}{Y})$	(Y,X)	$(\frac{1}{X},\frac{1}{Y})$
$\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'}$	$(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ $(\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}')$	$(\boldsymbol{\alpha},\boldsymbol{\gamma}',\boldsymbol{\beta}')$ $(\boldsymbol{\alpha}',\boldsymbol{\gamma},\boldsymbol{\beta})$	$(\boldsymbol{\gamma}',\boldsymbol{\alpha},\boldsymbol{\beta}')$ $(\boldsymbol{\gamma},\boldsymbol{\alpha}',\boldsymbol{\beta})$	$(\boldsymbol{\gamma}',\boldsymbol{\beta},\boldsymbol{\alpha}')$ $(\boldsymbol{\gamma},\boldsymbol{\beta}',\boldsymbol{\alpha})$	$(\boldsymbol{\beta},\boldsymbol{\gamma}',\boldsymbol{\alpha}')$ $(\boldsymbol{\beta}',\boldsymbol{\gamma},\boldsymbol{\alpha})$	$(\boldsymbol{\beta},\boldsymbol{\alpha},\boldsymbol{\gamma})$ $(\boldsymbol{\beta}',\boldsymbol{\alpha}',\boldsymbol{\gamma}')$	$(\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}')$ $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$
	$(0,0) \leftarrow$	$y=\infty$	$y=\infty$	$x=\infty$	$x=\infty$	$(0,0)$	(∞,∞)
	$(0,\infty) \leftarrow$	$(0^2,0)$	(∞^2,∞)	(∞,∞^2)	$(0,0^2)$	$(\infty,0)$	$(\infty,0)$

$\epsilon = (-1)^{p-p'}$, $\epsilon' = (-1)^{q-q'}$

Consider another transformation $(x,y) \mapsto (X,Y) = (x,\frac{1}{y})$.

Then $(\vartheta_x,\vartheta_y) = (\vartheta_X,-\vartheta_Y)$ and the equation $\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'}$ in (X,Y) coordinate is

$$\begin{cases} \prod_{i=1}^{p'}(\vartheta_X-\alpha'_i)\prod_{k=1}^{r'}(\vartheta_X-\vartheta_Y-\gamma'_k)u = X\prod_{i=1}^{p}(\vartheta_X+\alpha_i)\prod_{k=1}^{r}(\vartheta_X-\vartheta_Y+\gamma_k)u, \\ \prod_{j=1}^{q}(\vartheta_Y-\beta_j)\prod_{k=1}^{r}(\vartheta_X-\vartheta_Y-\gamma_k)u = (-1)^{q'-q}Y\prod_{j=1}^{q'}(\vartheta_Y-\beta'_j)\prod_{k=1}^{r'}(\vartheta_X-\vartheta_Y-\gamma'_k)u, \\ \prod_{i=1}^{p'}(\vartheta_X-\alpha'_i)\prod_{j=1}^{q}(\vartheta_Y-\beta_j)u = XY(-1)^{q'-q}\prod_{i=1}^{p}(\vartheta_X+\alpha_i)\prod_{i=1}^{q'}(\vartheta_Y-\beta_j)u. \end{cases}$$

The local solution of this system in a neighborhood of the origin in (X,Y) coordinate corresponds to a local solution in a neighborhood of $(0,\infty) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. In fact we have $p'q$ solutions corresponding to the solutions $G^{p,q',r'}_{p',q,r}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta}' & \boldsymbol{\gamma} \\ \boldsymbol{\alpha}' & \boldsymbol{\beta} & \boldsymbol{\gamma}'\end{smallmatrix};\alpha'_i,\beta_j;X,\epsilon Y\right)$ (cf. §6):

$$G^{p,q',r'}_{p',q,r}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta}' & \boldsymbol{\gamma}\ \boldsymbol{\gamma}' \\ \boldsymbol{\alpha}' & \boldsymbol{\beta} & \end{smallmatrix};\alpha'_i,\beta_j;x,\epsilon\frac{1}{y}\right) : p'q \text{ solutions of } \mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'} \text{ at } (0,\infty) \quad \text{with} \quad \epsilon = (-1)^{r-r'}.$$

8 Solutions

Owing to the coordinate transformations in §7, we have solutions of $\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'}$ (12 types):

$$p'q' : F\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}; \alpha'_i, \beta'_j; x, y\Big) \quad \text{around } (0,0) \tag{1}$$

$$p'q : G\begin{pmatrix} \alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma' \end{pmatrix}; \alpha'_i, \beta_j; x, \epsilon\tfrac{1}{y}\Big) \quad \text{at } (0,\infty) \tag{2}$$

$$p'r : F\begin{pmatrix} \alpha & \gamma' & \beta' \\ \alpha' & \gamma & \beta \end{pmatrix}; \alpha'_i, \gamma_k; \epsilon\tfrac{x}{y}, \tfrac{1}{y}\Big) \quad \text{along } y=\infty \ni (\infty,\infty^2) \tag{3}$$

$$q'r : F\begin{pmatrix} \gamma' & \beta & \alpha' \\ \gamma & \beta' & \alpha \end{pmatrix}; \gamma_k, \beta'_j; \tfrac{1}{x}, \epsilon\tfrac{y}{x}\Big) \quad \text{along } x=\infty \ni (\infty^2,\infty) \tag{4}$$

$$qr : G\begin{pmatrix} \gamma' & \beta' & \alpha' \\ \gamma & \beta & \alpha \end{pmatrix}; \gamma_k, \beta_j; \tfrac{1}{x}, \tfrac{x}{y}\Big) \quad \text{at } (\infty,\infty^2) \tag{5}$$

$$pq : F\begin{pmatrix} \alpha' & \beta' & \gamma' \\ \alpha & \beta & \gamma \end{pmatrix}; \alpha_i, \beta_j; \tfrac{1}{x}, \tfrac{1}{y}\Big) \quad \text{around } (\infty,\infty) \tag{6}$$

$$qr' : F\begin{pmatrix} \gamma & \beta' & \alpha \\ \gamma' & \beta & \alpha' \end{pmatrix}; \gamma'_k, \beta_j; x, \epsilon\tfrac{x}{y}\Big) \quad \text{along } x=0 \ni (0^2,0) \tag{7}$$

$$p'r' : G\begin{pmatrix} \alpha & \gamma & \beta' \\ \alpha' & \gamma & \beta \end{pmatrix}; \alpha'_o, \gamma_k; \epsilon\tfrac{x}{y}, \epsilon y\Big) \quad \text{at } (0^2,0) \tag{8}$$

(F1): $(p,q,r;p',q',r') \mapsto (p,q,r;p,q,r)$, $F_1 : (1,1,1;1,1,1)$
(F2): $(p,q,r;p',q',r') \mapsto (p-1,q-1,1;p,q,0)$, $F_2 : (1,1,1;2,2,0)$, $F_3 : (2,2,0;1,1,1)$
(F4): $(p,q,r,p',q',r') \mapsto (p-2,q-2,2,p,q,0)$, $F_4 : (0,0,2,2,2,0)$

	solutions	# solutions	(F1): $pq{+}qr{+}rp$	(F2) : pq	(F4): pq
(∞,∞^2)	(3)(5)(6)	$p'r{+}qr{+}pq$	$pr+qr+pq$	$p{+}(q{-}1){+}(p{-}1)(q{-}1)$	$2p{+}2(q{-}2){+}(p{-}2)(q{-}2)$
$(0,\infty)$	(2)(3)(7)	$p'q{+}p'r{+}qr'$	$pq+pr+qr$	$p(q-1)+p+0$	$p(q-2)+2p+0$
$(0^2,0)$	(1)(7)(8)	$p'q'{+}qr'{+}p'r'$	$pq+qr+pr$	$pq+0+0$	$pq+0+0$

(F) $\Rightarrow$ $p'r+qr+pq = p'q+p'r+qr' = p'q'+qr'+p'r' =$ rank (Theorem).

We consider $\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'}$ on the space $\tilde{X}$ which is constructed from $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ by blowing up at $(0,0)$ and (∞,∞). The local coordinates are $(\frac{x}{y}, y)$ and $(x, \frac{y}{x})$ at $(0,0)$ and they are $(\frac{y}{x}, \frac{1}{y})$ and $(\frac{1}{x}, \frac{x}{y})$ at (∞,∞). Then for example, $(0^2,0)$ represents the point of $\tilde{X}$ corresponding to the origin under the coordinate $(\frac{x}{y}, y)$.

Theorem. In a neighborhood of any one of 6 points $(0^2,0)$, $(0,\infty)$, (∞,∞^2), (∞^2,∞), $(\infty,0)$ and $(0,0^2)$, we have $pq+qr+rp+(p+q+r)(r-r')+(r-r')^2$ local solutions which span the space of local solutions.

Note that the 6 points in the theorem is in the orbit of $(0^2,0)$ under the group of coordinate transformations in §7 which is isomorphic to $S_3 \times S_2$.

9 Connection Problem

We study the connection problem among the local solutions given in the last section

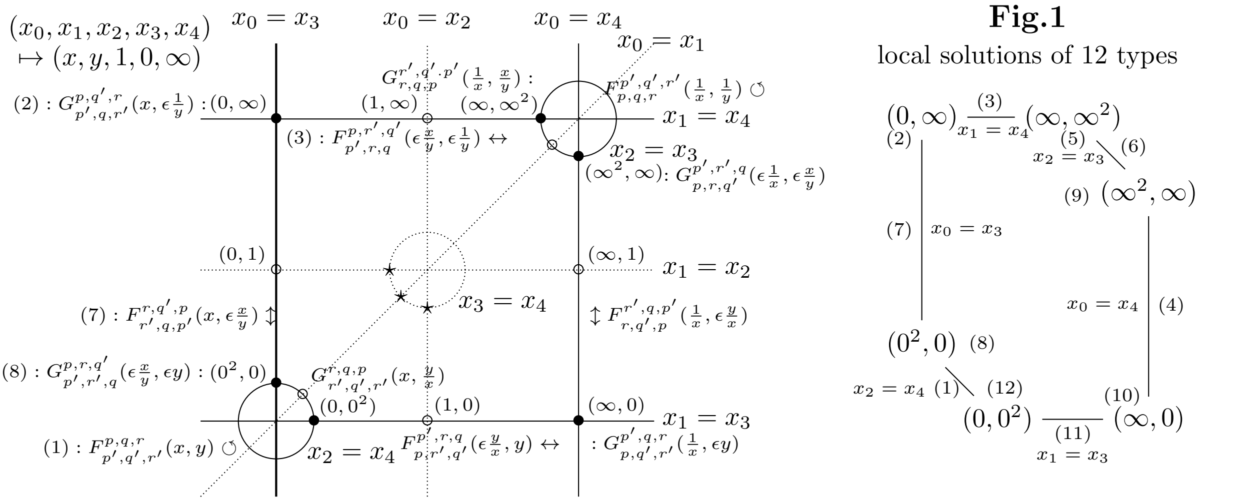


Fig.1
local solutions of 12 types

We solve the connection problem along the path $\{(0,-t)\mid t\in(0,\infty)\}$:

$(0^2,0)\leftrightarrow(0,\infty):(1)F[\boldsymbol{\alpha}':\boldsymbol{\beta}']+(8)G[\boldsymbol{\alpha}':\boldsymbol{\gamma}']\leftrightarrow(2)G[\boldsymbol{\alpha}':\boldsymbol{\beta}]+(3)F[\boldsymbol{\alpha}':\boldsymbol{\gamma}]$

$x^{\alpha'_i}:F_{q+r}\left({\boldsymbol{\beta},\boldsymbol{\gamma}+\alpha'_i \atop \boldsymbol{\beta}',\boldsymbol{\gamma}'-\alpha'_i};y\right):0\leftrightarrow\infty\ \Rightarrow$ connection problem (7)+(8) $\leftrightarrow$ (2)+(3)

$$c(F_{q+r}:{\mathbf{a}\atop\mathbf{b}};0:b_i\rightsquigarrow\infty:a_j)=\frac{\prod_{\nu\neq j}\Gamma(1+b_i-b_\nu)\prod_{\nu\neq i}\Gamma(a_\nu-a_j)}{\prod_{\nu\neq j}\Gamma(b_i+a_\nu)\prod_{\nu\neq i}\Gamma(1-b_\nu-a_j)}\quad(\mathbf{a},\mathbf{b}\in\mathbb{C}^{q+r}).$$

Note that $q+r=q'+r'$ and

$$(1):F\left({\alpha\ \beta\ \gamma\atop\alpha'\ \beta'\ \gamma'};\alpha'_i,\beta'_j;+,-;x,y\right)\quad\text{around }(0,0)\qquad(1\le i\le p',1\le j\le q')$$

$$=x^{\alpha'_i}(-y)^{\beta'_j}F\left({\boldsymbol{\alpha}+\alpha'_i\ \ \boldsymbol{\beta}+\beta'_j\ \ \boldsymbol{\gamma}+\alpha'_i+\beta'_j\atop\boldsymbol{\alpha}'-\alpha'_i\ \ \boldsymbol{\beta}'-\beta'_j\ \ \boldsymbol{\gamma}-\alpha'_i-\beta'_j};x,y\right)$$

$$\overset{x\to0}{\sim}x^{\alpha'_i}F^{q+r}_{q'+r'}\left({\boldsymbol{\beta},\boldsymbol{\gamma}+\alpha'_i\atop\boldsymbol{\beta}',\boldsymbol{\gamma}'-\alpha'_i};\beta'_j;-;y\right),$$

$$(8):G\left({\boldsymbol{\alpha}\ \boldsymbol{\gamma}\ \boldsymbol{\beta}'\atop\boldsymbol{\alpha}'\ \boldsymbol{\gamma}'\ \boldsymbol{\beta}};\alpha'_i,\gamma'_k;-,-;(-1)^{q-q'}\tfrac{x}{y},(-1)^{q-q'}y\right)\quad\text{at }(0^2,0)\qquad(1\le i\le p',1\le k\le r')$$

$$=(-\tfrac{x}{y})^{\alpha'_i}(-y)^{\gamma'_k}G\left({\boldsymbol{\alpha}+\alpha'_i\ \ \boldsymbol{\gamma}+\gamma'_k\ \ \boldsymbol{\beta}'+\alpha'_i-\gamma'_k\atop\boldsymbol{\alpha}'-\alpha'_i\ \ \boldsymbol{\gamma}'-\gamma'_k\ \ \boldsymbol{\beta}-\alpha'_i+\gamma'_k};(-1)^{q-q'}\tfrac{x}{y},(-1)^{q-q'}y\right)$$

$$\overset{x\to0}{\sim}x^{\alpha'_i}(-y)^{\lambda'_k-\alpha'_i}F^{q+r}_{q'+r'}\left({\boldsymbol{\gamma}+\gamma'_k,\boldsymbol{\beta}-\alpha'_i+\gamma'_k\atop\boldsymbol{\gamma}'-\gamma'_k,\boldsymbol{\beta}'+\alpha'_i-\gamma'_k};y\right)=x^{\alpha'_i}F^{q+r}_{q'+r'}\left({\boldsymbol{\beta},\boldsymbol{\gamma}+\alpha'_i\atop\boldsymbol{\beta}',\boldsymbol{\gamma}'-\alpha'_i};\gamma'_k-\alpha'_i;-;y\right),$$

$$(3):F\left({\boldsymbol{\alpha}\ \boldsymbol{\gamma}'\ \boldsymbol{\beta}'\atop\alpha'\ \gamma\ \beta};\alpha'_i,\gamma_k;-\epsilon,-1;(-1)^{q-q'}\tfrac{x}{y},\tfrac{1}{y}\right)\quad\text{along }y=\infty\qquad(1\le i\le p',1\le k\le r)$$

$$=(-\tfrac{x}{y})^{\alpha'_i}(-\tfrac{1}{y})^{\gamma_k}F\left({\boldsymbol{\alpha}\ \boldsymbol{\gamma}'\ \boldsymbol{\beta}'\atop\boldsymbol{\alpha}'\ \boldsymbol{\gamma}\ \boldsymbol{\beta}};\alpha'_i,\gamma_k;-\epsilon,-;(-1)^{q-q'}\tfrac{x}{y},\tfrac{1}{y}\right)$$

$$\overset{x\to0}{\sim}x^{\alpha'_i}F^{q'+r'}_{q+r}\left({\boldsymbol{\beta}',\boldsymbol{\gamma}'-\alpha'_i\atop\boldsymbol{\beta},\boldsymbol{\gamma}+\alpha'_i};\gamma_k+\alpha'_i;-;\tfrac{1}{y}\right)$$

$$(2):G\left({\alpha\ \beta'\ \gamma\atop\alpha'\ \beta\ \gamma'};\alpha'_i,\beta_j;+,-\epsilon;x,\epsilon\tfrac{1}{y}\right)\quad\text{at }(0,\infty)\qquad(1\le i\le p',1\le j\le q)$$

$$=x^{\alpha'_i}(-\tfrac{1}{y})^{\beta_j}G\left({\alpha\ \beta'\ \gamma\atop\alpha'\ \beta\ \gamma'};\alpha'_i,\beta_j;+,-\epsilon;x,\epsilon\tfrac{1}{y}\right)$$

$$\overset{x\to0}{\sim}x^{\alpha'_i}F^{q'+r'}_{q+r}\left({\boldsymbol{\beta}',\boldsymbol{\gamma}'-\alpha'_i\atop\boldsymbol{\beta},\boldsymbol{\gamma}+\alpha'_i};\beta_j;-;\tfrac{1}{y}\right).$$

Owing to the coordinate transformations in §7, we have solutions of other connection problems.

10 Singularities

type	Singularities
$\left({p\ q\ r\atop p\ q\ r}\right)$	$x,y\in\{0,1,\infty\},\ x=y$ (KZ type)
$\left({p\quad q\quad 1\atop p+1\ q+1\ 0}\right)$	$x,y\in\{0,1,\infty\},\ x+y=1$ (KZ type)
$\left({p\quad q\quad r+1\atop p+1\ q+1\ r}\right)$	$x,y\in\{0,1,\infty\},\ x+y=0,\ x+y=1$
$\left({0\ 0\ r\atop r\ r\ 0}\right),\ r>1$	$x,y\in\{0,\infty\},\ f_r(x,y)=0$ $(r=2:F_4)$
$\left({0\ 0\ s+L\atop L\ L\ s}\right)$	$x,y\in\{0,\infty\},\ x+(-1)^Ly=0,\ f_L(x,y)=0$
$\left({p\quad 0\ L\atop p+L\ L\ 0}\right)$	$x=0,\infty,\ y=0,1,\infty,\ f_L(x,y)=0$
$\left({p\quad q\quad s+L\atop p+L\ q+L\ s}\right)$	$x,y\in\{0,1,\infty\},\ x+(-1)^Ly=0,\ f_L(x,y)=0$
$\left({p+r\ q+L\ s\atop p\quad q\quad s+L}\right)$	$x,y\in\{0,1,\infty\},\ x+(-1)^Ly=0,\ f_L(\frac{1}{x},\frac{1}{y})=0$
$F_A:\left({1\ 1\ 1\ 1\atop 2\ 2\ 2\ 0}\right)$	$x,y,z\in\{0,1,\infty\},\ x+y,x+z,y+z,x+y+z=1$
$\left({1\ 1\ 1\ 2\atop 2\ 2\ 2\ 1}\right)$	$x,y,z\in\{0,1,\infty\},\ x+y,x+z,y+z,x+y+z\in\{0,1\}$
$F_D:\left({p\ q\ r\ s\atop p\ q\ r\ s}\right)$	$x,y,z\in\{0,1,\infty\},\ x=y,\ y=z,\ z=x$ (KZ type)

p, q, r, s, L : positive integer

$$f_1(x,y)=1-x-y$$

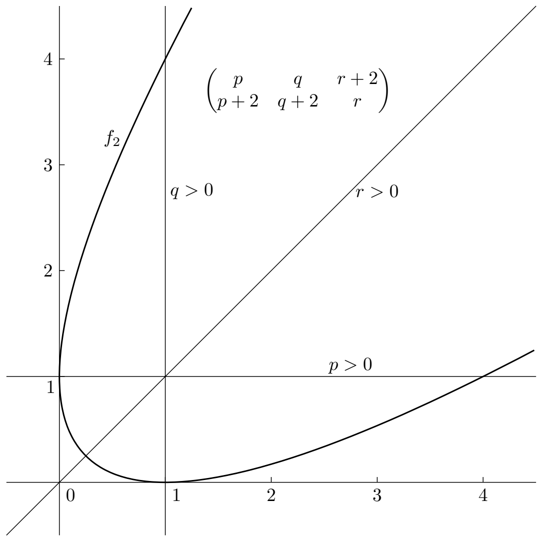
$$f_2(x,y)=(1-x-y)^2-4xy$$

$$f_3(x,y)=(1-x-y)^3-27xy$$

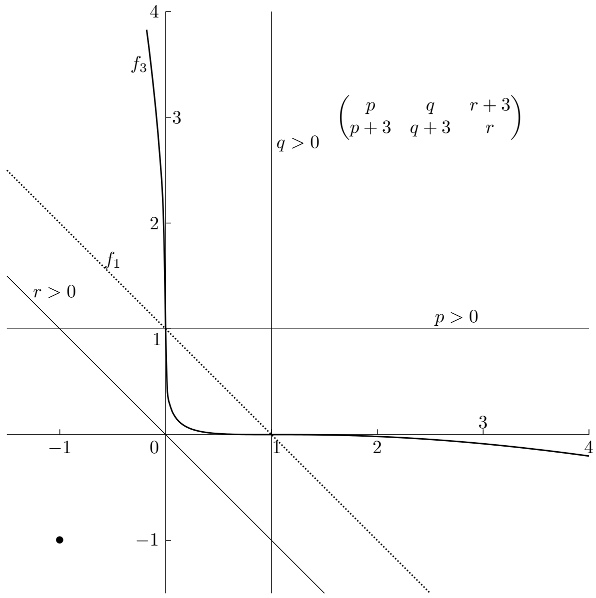
$$f_4(x,y)=(1-x-y)^4-8xy\big(x^2+y^2+14(x+y)+17\big)$$

$$f_5(x,y)=(1-x-y)^5-625xy\big(x^2+y^2-3(xy-x-y)+1\big)$$

$$
\begin{aligned}
f_L(x,y) &= \text{resultant of polynomials } x(1+t)^L-1 \text{ and } y(1+t)^L-t^L \text{ of } t \quad (L>0)\\
&= \prod_{\omega,\,\omega'\in U_L}(1-\omega x^{\frac{1}{L}}-\omega' y^{\frac{l}{L}}),\quad U_L:=\{\omega\in\mathbb{C}\mid \omega^L=1\}.
\end{aligned}
$$



Singularities of $\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'}$



$f_L(0,y)=0\Leftrightarrow y=1$

$f_L(\frac{1}{2^L},\frac{1}{2^L})=f_L(\frac{1}{(1+t)^L},\frac{t^L}{(1+t)^L})=0\quad(t\in\mathbb{C}\cup\{\infty\})$

•

11 Example: Generalization of Appell's F_1

Case (F1) : $F_{p,q,r}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma}\\ \boldsymbol{\alpha}' & \boldsymbol{\beta}' & \boldsymbol{\gamma}'\end{smallmatrix};x,y\right):=F^{p,q,r}_{p,q,r}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma}\\ \boldsymbol{\alpha}' & \boldsymbol{\beta}' & \boldsymbol{\gamma}'\end{smallmatrix};x,y\right)$

The spectral type and the Riemann scheme of the corresponding KZ type equation (cf. [Oi]) in this case are as follows. For a given (p,q,r) the calculation of them is supported by [Or].

Here $(q+r)^p1^{qr}$ means the partition of $pq+qr+rp$: $\overbrace{q+r,\cdots,q+r}^{p},\overbrace{1,\dots,1}^{qr}$

	$x_0=x$	$x_1=y$	$x_2=1$	$x_3=0$	$x_4=\infty$
x_0		$(R-r)r$	$(R-q)q$	$(q+r)^p1^{qr}$	$(q+r)^p1^{qr}$
x_1	$(R-r)r$		$(R-p)p$	$(p+r)^q1^{pr}$	$(p+r)^q1^{pr}$
x_2	$(R-q)q$	$(R-p)p$		$(p+q)^r1^{pq}$	$(p+q)^r1^{pq}$
x_3	$(q+r)^p1^{qr}$	$(p+r)^q1^{pr}$	$(p+q)^r1^{pq}$		S
x_4	$(q+r)^p1^{qr}$	$(p+r)^q1^{pr}$	$(p+q)^r1^{pq}$	S	

rank : $R:=pq+qr+rp,\ S:=(R-p-q-r+1)(p-1)(q-1)(r-1)2$

$$
\left\{\begin{matrix}
A_{01} & A_{02} & A_{03} & A_{04} & A_{12}\\
[0]_{pq+(p+q-1)r} & [0]_{pr+(p+r-1)q} & [\alpha'_i]_{q+r} & [\alpha_i]_{q+r} & [0]_{qr+(q+r-1)p}\\
[-\alpha''-\beta'']_r & [-\alpha''-\gamma'']_q & \beta_j+\gamma'_k & \beta'_j+\gamma_k & [-\beta''-\gamma'']_p
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
A_{13} & A_{23} & A_{14} & A_{24} & A_{34}\\
[\beta'_j]_{p+r} & [\gamma_k]_{p+q} & [\beta_j]_{p+r} & [\gamma'_k]_{p+q} & [0]_{pq+qr+rp-(p+q+r)+1}\\
\alpha_i+\gamma'_k & \alpha_i+\beta_j & \alpha'_i+\gamma_k & \alpha'_i+\beta'_j & [-\alpha''-\beta''-\gamma'']_2\\
 & & & & [-\alpha''-\beta'']_{r-1}\\
 & & & & [-\beta''-\gamma'']_{p-1}\\
 & & & & [-\alpha''-\gamma'']_{q-1}
\end{matrix}\right\}
$$

$$
\alpha'':=\sum_{i=1}^{p}(\alpha_i+\alpha'_i),\ \beta'':=\sum_{j=1}^{q}(\beta_j+\beta'_j),\ \gamma'':=\sum_{k=1}^{r}(\gamma_k+\gamma'_k),\ 1\le i\le p,\ 1\le j\le q,\ 1\le k\le r.
$$

$F_1 : p = q = r = 1$ (rank: 3)

	x_0	x_1	x_2	x_3	x_4	idx
x_0		21	21	21	21	2
x_1	21		21	21	21	2
x_2	21	21		21	21	2
x_3	21	21	21		21	2
x_4	21	21	21	21		2

$p = q = r = 2$ (rank: 12)

	x_0	x_1	x_2	x_3	x_4	idx
x_0		$(10)2$	$(10)2$	4^21^4	4^21^4	-8
x_1	$(10)2$		$(10)2$	4^21^4	4^21^4	-8
x_2	$(10)2$	$(10)2$		4^21^4	4^21^4	-8
x_3	4^21^4	4^21^4	4^21^4		721^3	-124
x_4	4^21^4	4^21^4	4^21^4	721^3		-124

$$\text{Special symmetry : } F_1 : \mathcal{M}^{\alpha,\beta,\gamma}_{0,0,\gamma'} \xrightarrow{(x,y)\mapsto(1-x,1-y)} \mathcal{M}^{\alpha,\beta,\gamma}_{0,0,1-\alpha-\beta-\gamma-\gamma'}$$

Symmetry : $S_{\{x,y,1\}} \times S_{\{0,\infty\}} \quad (S_{\{x,y,1,0,\infty\}} \Leftarrow p = q = r = 1)$

Rank : $pq + qr + rp$

$\mathrm{idx}_{x_0} = (R-q)^2 + q^2 + (R-r)^2 + r^2 + 2(p(q+r)^2 + qr) - 2R^2$
$\quad = 2 - 2(q-1)(r-1)(q+r+1)$

Irreducibility : $\alpha_i + \alpha'_{i'},\ \beta_j + \beta'_{j'},\ \gamma_k + \gamma'_{k'},\ \alpha_i + \beta_j + \gamma'_k,\ \alpha'_i + \beta'_j + \gamma_k \notin \mathbb{Z}$
(#: $p^2 + q^2 + r^2 + 2pqr$)

$d(\mathbf{m}_x) = (R-r) + (R-q) + (q+r) + (q+r) - 2R = q + r$
$\Rightarrow$ Katz-Haraoka reduction : $p \mapsto p-1 \to \cdots \to p = 1$
$\underline{(R-r)}r, \underline{(R-q)}r, \underline{(q+r)^p}1^{qr}, \underline{(q+r)^p}1^{qr} \qquad R = (q+r)p + qr$

12 Example: Generalization of Appell's F_2 and F_3

$$\text{Case (F2)} : I_{p,q}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \gamma\\ \boldsymbol{\alpha'} & \boldsymbol{\beta'} & \end{smallmatrix};x,y\right) := F^{p-1,q-1,1}_{p,q,0}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \gamma\\ \boldsymbol{\alpha'} & \boldsymbol{\beta'} & \end{smallmatrix};x,y\right)$$
$$\text{Case (F3)} : J_{p,q}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \\ \boldsymbol{\alpha'} & \boldsymbol{\beta'} & \gamma\end{smallmatrix};x,y\right) := F^{p,q,0}_{p-1,q-1,1}\left(\begin{smallmatrix}\boldsymbol{\alpha} & \boldsymbol{\beta} & \\ \boldsymbol{\alpha'} & \boldsymbol{\beta'} & \gamma\end{smallmatrix};x,y\right)$$

Note that the transformation $(x,y) \mapsto (\frac{1}{x}, \frac{1}{y})$ induces $\mathcal{M}^{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}_{\boldsymbol{\alpha'},\boldsymbol{\beta'},\boldsymbol{\gamma'}} \mapsto \mathcal{M}^{\boldsymbol{\alpha'},\boldsymbol{\beta'},\boldsymbol{\gamma'}}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$.

Spectral type of $I_{p,q}$

	$x_0 = x$	$x_1 = 1-y$	$x_2 = 1$	$x_3 = 0$	$x_4 = \infty$	idx
x_0		$(pq-1)1$	$(pq-q+1)1^{q-1}$	q^p	$q^{p-1}1^q$	2
x_1	$(pq-1)1$		p^q	$(pq-p+1)1^{p-1}$	$p^{q-1}1^p$	2
x_2	$(pq-q+1)1^{q-1}$	p^q		$(p+q-1)1^{(p-1)(q-1)}$	$(q-1)^pp$	
x_3	q^p	$(pq-p+1)1^{p-1}$	$(p+q-1)1^{(p-1)(q-1)}$		$(p-1)^qq$	
x_4	$q^{p-1}1^q$	$q^{p-1}1^q$	$(q-1)^pp$	$(p-1)^qq$		

$$\left\{\begin{array}{ccccc} A_{01} & A_{02} & A_{03} & A_{04} & A_{12}\\ {[0]}_{pq-1} & [0]_{pq-q+1} & [\alpha'_i]_q & [\alpha_i]_q & [\beta'_j]_p\\ -\alpha''-\beta''-\gamma & \beta_j-\alpha''-\gamma & & \beta'_j+\gamma & \end{array}\right.$$
$$\left.\begin{array}{ccccc} A_{13} & A_{23} & A_{14} & A_{24} & A_{34}\\ {[0]}_{pq-p+1} & [\gamma]_{p+q-1} & [\beta_j]_p & [\alpha'_i]_{q-1} & [\beta'_j]_{p-1}\\ \alpha_i-\beta''-\gamma & \alpha_i+\beta_j & \alpha'_i+\gamma & [-\beta''-\gamma]_p & [-\alpha''-\gamma]_q \end{array}\right\}$$

$F_2 : p = q = 2$ (rank: 4)

	x_0	x_1	x_2	x_3	x_4	idx
x_0		31	31	22	211	2
x_1	31		22	31	211	2
x_2	31	22		31	211	2
x_3	22	31	31		211	2
x_4	211	211	211	211		-8

$I_{4,3} : p = 4,\ q = 3$ (rank: 12)

	$x_0 = x$	$x_1 = 1-y$	$x_2 = 1$	$x_3 = 0$	$x_4 = \infty$	idx
x_0		$(11)1$	$(10)1^2$	3^4	3^31^3	2
x_1	$(11)1$		4^3	91^3	4^21^3	2
x_2	$(10)1^2$	4^3		61^6	42^4	-64
x_3	3^4	91^3	61^6		3^4	-90
x_4	3^31^3	4^21^4	42^4	3^4		-154

Rank : pq

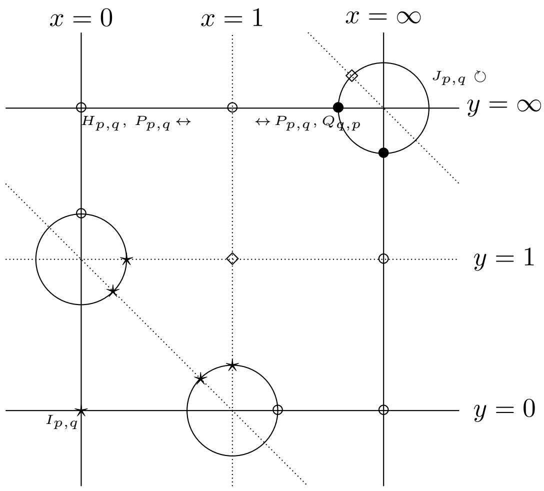
Symmetry : $S_{\{x,y\}} = \langle(0\ 1)(2\ 3)\rangle$
($B_2 = \langle(0\ 1)(2\ 3), (1\ 2)\rangle$ which keeps $\big\{\{0,3\},\{1,2\}\big\} \Leftarrow p = q = 2$)

Irreducibility : $\alpha_i + \alpha'_{i'},\ \beta_j + \beta'_{j'},\ \alpha'_i + \beta'_j + \gamma\ \notin \mathbb{Z}$ ($\Leftarrow$ [Oir]), $\quad \# = p^2 + q^2 + pq - p - q$

Rigid decomposition:

$$\begin{aligned}
&(pq-1)1, ((p-1)q+1)1^{q-1}, q^p, q^{p-1}1^q \\
&\quad = (p(q-1)-1)1, ((p-1)(q-1)+1)1^{q-1}, (q-1)^p, (q-1)^{p-1}1^{q-1} \\
&\qquad \oplus p0, (p-1)1, 1^p, 1^{p-1}1 \\
&\quad = 10, 10, 1, 01 \oplus (pq-2)1, ((p-1)q)1^{q-1}, (q-1)q^{p-1}, q^{p-2}1^{q-1} \\
&\quad = q(10, 10, 1, 10) \oplus ((p-1)q-1)1, ((p-2)q+1)1^{q-1}, q^{p-1}, q^{p-2}1^q
\end{aligned}$$

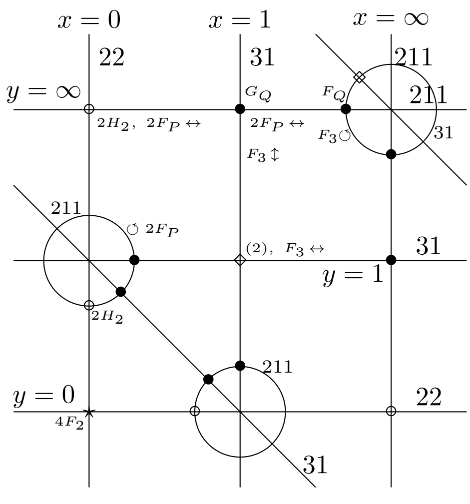
$d(\mathbf{m}_x) = (pq-1) + (pq-q+1) + q + q - 2pq = q \Rightarrow$ reduction : $p \mapsto p-1$
$\mathbf{m}_x \Rightarrow p = 1 : \underline{(q-1)}1, \underline{1}1^{q-1}, \underline{0}q, \underline{0}1^q : {}_qF_{q-1}$



$$I_{p,q} = F^{p-1,q-1,1}_{p,q,0}(x,y)$$
$$J_{p,q} = F^{p,q,0}_{p-1,q-1,1}(\tfrac{1}{x}, \tfrac{1}{y})$$
$$H_{p,q} = G^{p-1,q,1}_{p,q-1,0}(x, -\tfrac{1}{y})$$
$$P_{p,q} = F^{p-1,0,q}_{p,1,q-1}(-\tfrac{x}{y}, \tfrac{1}{y})$$
$$Q_{q,p} = G^{0,q,p}_{1,q-1,p-1}(\tfrac{1}{x}, \tfrac{x}{y})$$

Special symmetry : $P_{2,2} = F^{1,0,2}_{2,1,1} : \mathcal{M}^{\alpha,\emptyset,(\gamma_1,\gamma_2)}_{(\alpha',0),0,\gamma'} \xrightarrow{(x,y)\to(-x,1-y)} \mathcal{M}^{\alpha,\emptyset,(\gamma_1,\gamma_2)}_{(\alpha',0),0,1-\gamma_1-\gamma_2-\gamma'}$

$F_2(a;b_1;b_2;1-c_1',1-c_2';x,y)$ ↻B_2



$$\begin{cases}
1 : \star 22 \wedge 22 : 4F_2 \\
4 : \circ 22 \wedge 211 : 2H_2 + 2F_P \\
8 : \bullet 31 \wedge 211 : G_Q + 2F_P + F_3 \\
2 : \diamond 31 \wedge 31 : 2F_3 + (2) \\
4 : F_2 = F^{1,1,1}_{2,2,0} \\
1 : F_3 = F^{2,2,0}_{1,1,1} \\
2 : F_P = F^{1,0,2}_{2,1,1} \\
1 : G_Q = G^{0,2,2}_{1,1,1} \\
2 : H_2 = G^{1,2,1}_{2,1,0}
\end{cases}$$

$$F_P(a,b_1,b_2,1-c_1',1-c_2';x,y) = y^{-a}F^{1,0,2}_{2,1,1}\left(\begin{smallmatrix} b_1 & \emptyset & a,b_2 \\ c_1',0 & 0 & 1-a-b_2-c_2' \end{smallmatrix}; \frac{x}{y}, 1-\frac{1}{y}\right) \text{ around } (0,1).$$

Here the function F_P is introduced by Olsson [Ol].

B_2 symmetry for F_2 $\Rightarrow$

$$\begin{aligned}
F_2(a;b,b';c,c';x,y) &= (1-y)^{-a}F_2\Big(a;b,c'-b';c,c';\frac{x}{1-y},\frac{y}{y-1}\Big) \\
&= (1-x-y)^{-a}F_2\Big(a;c-b;c-b';c,c';\frac{x}{x+y-1},\frac{y}{x+y-1}\Big).
\end{aligned}$$

13 Generalization to many variables

We generalize the functions $F^{p,q,r}_{p',q',r'}(x,y)$ and $G^{p.q.r}_{p'q'r'}(x,y)$ and the results stated in this note are extended as follows.

$$F^{p_1,\dots,p_n,r}_{p'_1,\dots,p'_n,r'}\left(\begin{smallmatrix}\boldsymbol{\alpha}_1 & \cdots & \boldsymbol{\alpha}_n & \boldsymbol{\gamma}\\ \boldsymbol{\alpha}'_1 & \cdots & \boldsymbol{\alpha}'_n & \boldsymbol{\gamma}'\end{smallmatrix};x\right) := \sum_{\mathbf{m}\in\mathbb{Z}^n_{\ge 0}} \frac{(\boldsymbol{\alpha}_1)_{m_1}\cdots(\boldsymbol{\alpha}_n)_{m_n}(\boldsymbol{\gamma})_{|\mathbf{m}|}}{(1-\boldsymbol{\alpha}'_1)_{m_1}\cdots(1-\boldsymbol{\alpha}'_n)_{m_n}(1-\boldsymbol{\gamma}')_{|\mathbf{m}|}} x_1^{m_1}\cdots x_n^{m_n}$$

$$G^{p_1,\dots,p_n,r}_{p'_1,\dots,p'_n,r',\epsilon}\left(\begin{smallmatrix}\boldsymbol{\alpha}_1 & \cdots & \boldsymbol{\alpha}_n & \boldsymbol{\gamma}\\ \boldsymbol{\alpha}'_1 & \cdots & \boldsymbol{\alpha}'_n & \boldsymbol{\gamma}'\end{smallmatrix}\,\epsilon;x\right) := \sum_{\mathbf{m}\in\mathbb{Z}^n_{\ge 0}} \frac{(\boldsymbol{\alpha}_1)_{m_1}\cdots(\boldsymbol{\alpha}_n)_{m_n}(\boldsymbol{\gamma})_{\epsilon\mathbf{m}}}{(1-\boldsymbol{\alpha}'_1)_{m_1}\cdots(1-\boldsymbol{\alpha}'_n)_{m_n}(1-\boldsymbol{\gamma}')_{\epsilon\mathbf{m}}} x_1^{m_1}\cdots x_n^{m_n}$$

$$\boldsymbol{\alpha}_j\in\mathbb{C}^{p_j},\ \boldsymbol{\alpha}'_j\in\mathbb{C}^{p'_j},\ \boldsymbol{\gamma}\in\mathbb{C}^r,\ \boldsymbol{\gamma}'\in\mathbb{C}^{r'},\ \alpha'_{1,p'_1}=\cdots=\alpha'_{n,p'_n}=0\ \ (j=1,\dots,n)$$

$$\epsilon\in\{1,-1\}^n\simeq\{+,-\}^n,\quad p_j+\epsilon_j r=p'_j+\epsilon_j r'\quad (j=1,\dots,n)$$

- $(n+1)!$ normally crossing singular points with free multiplicity $\Leftarrow$ blowing up
- Rank of $F^{p_1,\dots,p_n,r}_{p'_1,\dots,p'_n,r'}(x)$:

$$\begin{cases}\frac{1}{r-r'}(p'_1\cdots p'_n r-p_1\cdots p_n r') & (r\ne r')\\ p_1\cdots p_n+\sum_{i=1}^n p_1\cdots p_{i-1}p_{i+1}\cdots p_n r & (r=r')\end{cases}$$

$$F_A:\left(\begin{smallmatrix}1&\cdots&1&1\\2&\cdots&2&0\end{smallmatrix}\right):2^n,\ \ F_B:\left(\begin{smallmatrix}2&\cdots&2&0\\1&\cdots&1&1\end{smallmatrix}\right):2^n,\ \ F_C:\left(\begin{smallmatrix}0&\cdots&0&2\\2&\cdots&2&0\end{smallmatrix}\right):2^n,\ \ F_D:\left(\begin{smallmatrix}1&\cdots&1&1\\1&\cdots&1&1\end{smallmatrix}\right):n+1$$

- Irreducibility $\Leftrightarrow$ $\alpha_{i,\nu}+\alpha'_{i,\nu'},\ \ \lambda_\nu+\lambda'_{\nu'},\ \ \sum_{i=1}^n\alpha_{i,\nu_i}+\gamma'_\nu,\ \ \sum_{i=1}^n\alpha'_{i,\nu_i}+\gamma_\nu\notin\mathbb{Z}$.

$$\#=p_1p'_1+\cdots+p_np'_n+rr'+p_1p_2\cdots p_nr'+p'_1p'_2\cdots p'_nr$$

Remark. Tsuda [Ts] studied the case $p_1=\cdots=p_n=q_1=\cdots=q_n=1$ and $r=r'$.

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