Dynkin Diagram

– Modern Mathematics –

Toshio Oshima (大島利雄)
Center for Mathematics and Data Science
https://www.ms.u-tokyo.ac.jp/~oshima

National University of Mongolia

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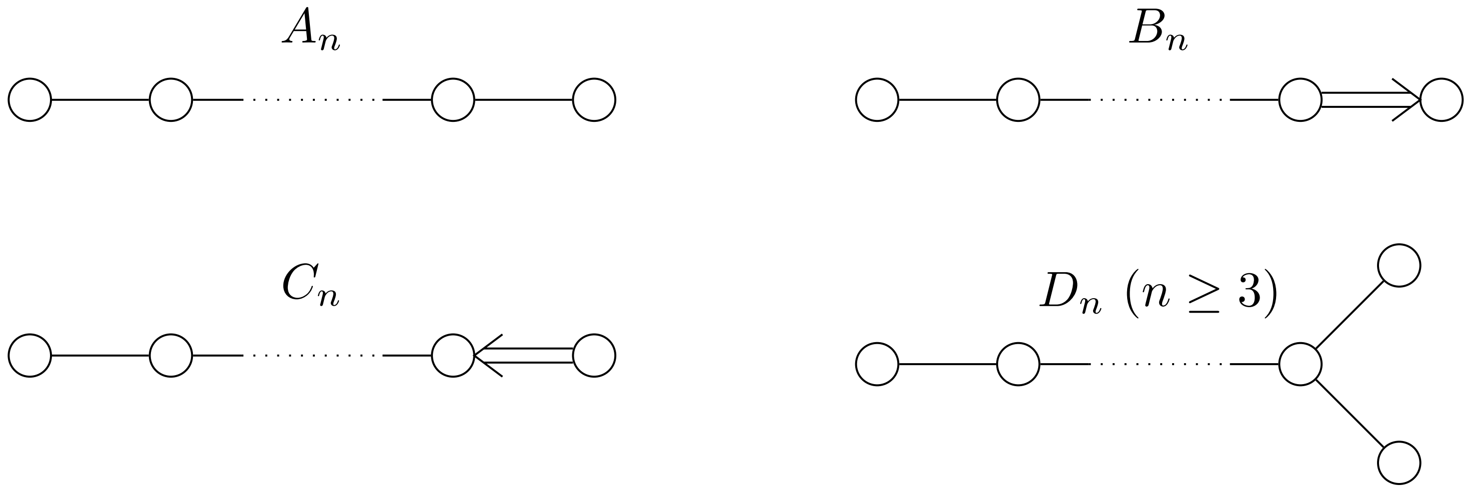
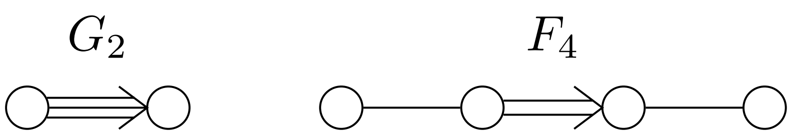
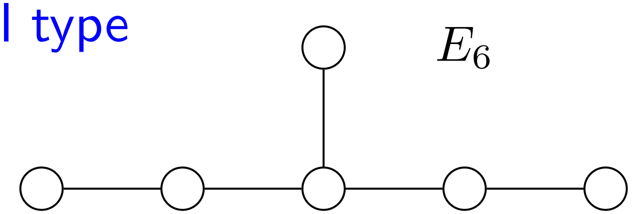
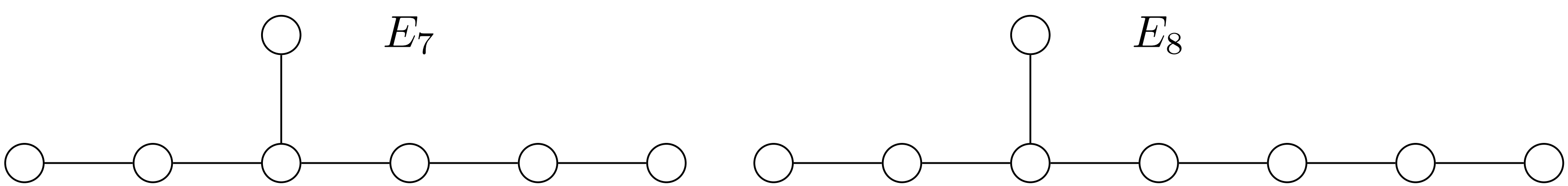
Classical type ($A_1 = B_1 = C_1$, $B_2 = C_2$, $A_3 = D_3$, $D_2 = A_1 + A_1$)



Exceptional type







○ : Linearly independent vectors, a line shows the relation between two vectors not connected by a line segment $\Leftrightarrow$ two vectors are orthogonal

Eugene Dynkin (1924〜2014)

A Russian Jew, Moscow University

Exempted from military service due to an eye problem,
disadvantaged position

Gelfand (1903〜2009) Gelfand seminar (algebra)

1947 “Structure of semisimple Lie group” Dynkin diag.

1952 “Structure of semisimple Lie algebra

After that he studied Probability Theory

Supported by his supervisor Kolmogorov (1903〜1987)

1952 ”Theory of Markov process”

1962 ICM (Invited)

1977〜 Cornell Univ. (USA)

In Russia, It is nor called by “Dynkin diagram”

2009 September : I received a mil from Dynkin

Reflection transformation

$x=(x_1,x_2)$ is a point in a plane, $x=(x_1,x_2,x_3)$ is a point in a space

$x=(x_1,\ldots,x_n)$: coordinate system of n-dimensional Euclidean space

$O:=(0,\ldots,0)$: Origin

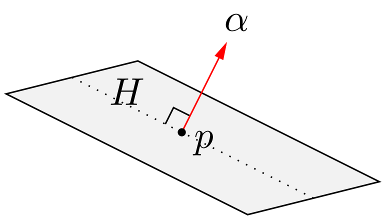
$|x-y|=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$: distance between two points x, y

$(x,y)=x_1y_1+\cdots+x_ny_n=|x|\cdot|y|\cdot\cos\angle xOy$: inner produce of x and y

$$\vec{x}\perp\vec{y} \quad \Leftrightarrow \quad (x,y)=0$$

hyperplane H : determined by α with $\alpha\perp H$ and $p\in H$

$n=2 \Rightarrow$ line $\qquad n=3 \Rightarrow$ plane

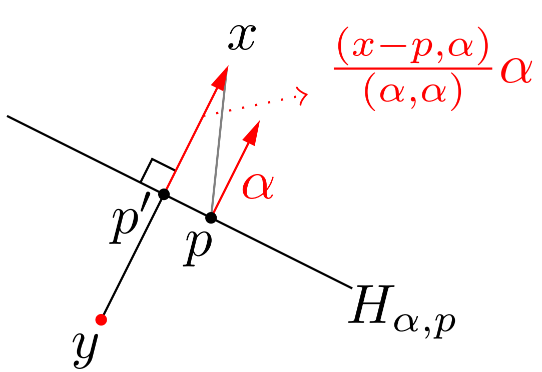


$$H_{\alpha,p}=\{x \mid (x-p,\alpha)=0\}$$

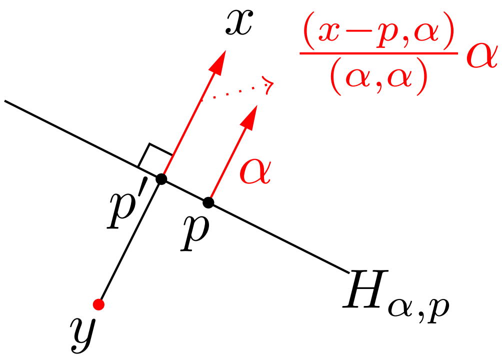
x, y are mirror images of each other wrt H : $\frac{x+y}{2}\in H_{\alpha,p}$ and $(x-y)\perp H_{\alpha,p}$

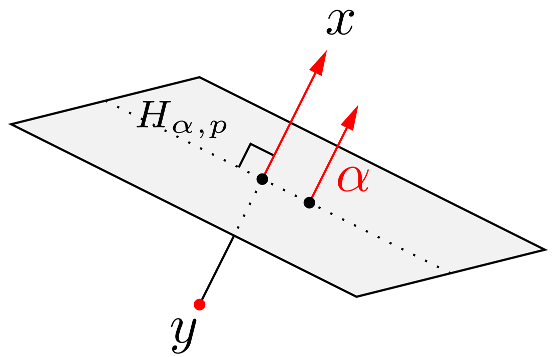
$s_{H_{\alpha,p}}(x):=y$: reflection with respect to

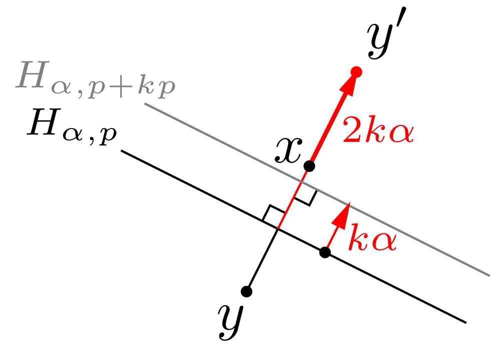
$$s_{H_{\alpha,p}}(x)=x-2\frac{(x-p,\alpha)}{(\alpha,\alpha)}\alpha$$



$\because\ (x-p,\alpha)=(x-p',\alpha),\ \ x-p'=c\alpha$

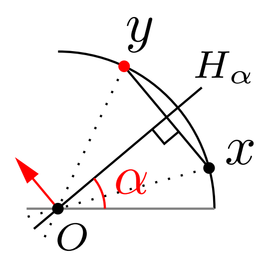






$s_{H_{\alpha,p}} \circ s_{H_{\alpha,p}} = id \ (= 1)$

$s_{H_{\alpha,p+k\alpha}} \circ s_{H_{\alpha,p}}(x) = x + 2k\alpha$: translation



$H_\alpha := H_{\alpha,0}$: Hyperplane $\ni 0$, $s_\alpha := s_{H_{\alpha,0}}$

$s_{(-\sin\alpha,\cos\alpha)}(x)$: θ :=argument of $x \ \mapsto 2\alpha - \theta$

$s_{(-\sin\alpha,\cos\alpha)} \circ s_{(-\sin\beta,\cos\beta)}(x)$: $2(\alpha-\beta)$ rotation about the origin

Isometry of Euclidean space = composition of reflections

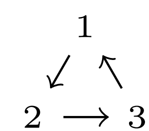
finite reflection group : $G = \langle s_{\alpha_1}, \ldots, s_{\alpha_n} \rangle$ $\qquad \#G < \infty$ ($\Rightarrow$ classified)

$S_3 := \langle g_1, \ g_2 \rangle$: $g_1^2 = g_2^2 = (g_1 g_2)^3 = 1$: permutation gp, of degree 3

$$1 = g_1g_2g_1g_2g_1g_2 \ \Rightarrow \ g_1g_2g_1 = g_1g_2g_1 \cdot g_1g_2g_1g_2g_1g_2 = g_2g_1g_2$$

$S_3 = \{1, \ g_1, \ g_2, \ g_1g_2, \ g_2g_1, \ g_1g_2g_1\}$, $\#S_3 = 6$

: permutations of $\{1,2,3\}$ $\quad g_1 : \begin{cases} 1\mapsto 2 \\ 2\mapsto 1 \\ 3\mapsto 3 \end{cases}$ $\quad g_2 : \begin{cases} 1\mapsto 1 \\ 2\mapsto 3 \\ 3\mapsto 2 \end{cases}$ $\quad g_1g_2 : \begin{cases} 1\mapsto 2 \\ 2\mapsto 3 \\ 3\mapsto 1 \end{cases}$



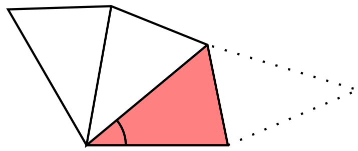
Tiling of the plane by reflections of a triangle

Folding back and forth around one interior angles of a triangle until it returns

$\Rightarrow$ interior angle : $\frac{360^\circ}{p}$ $\quad p$: odd $\Rightarrow$ isosceles triangle

interior angles : $\frac{360^\circ}{p}, \frac{360^\circ}{q}, \frac{360^\circ}{r}$ $\quad$ (sum $= 180^\circ$)

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2} \quad (3 \le p \le q \le r)$$



$\frac{1}{2} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le \frac{3}{p} \Rightarrow p \le 6$

$\underline{p = 6} \Rightarrow q = r = 6 \Rightarrow$ equilateral triangle $\underline{p = 5} \Rightarrow \frac{2}{q} = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$ $\times$, ...

$\Rightarrow (p, q, r) : (6,6,6) \quad (4,8,8) \quad (4,6,12) \quad (3,12,12)$ $\Rightarrow$ OK

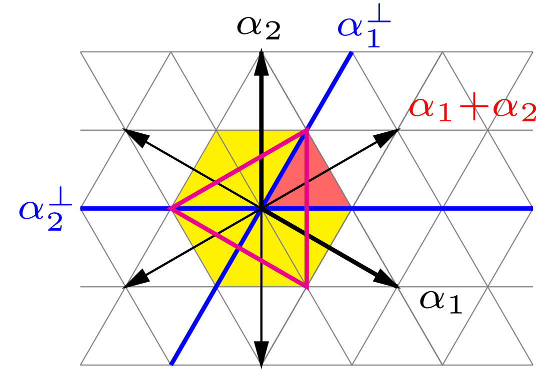
The tiled shape is divided by lines, invariant under reflections for the lines

Origin : one of the points where lines with different directions meet

Lines are given by finite vectors $\alpha \in \Sigma$ (root system) in a plane

$$L_{\alpha,k} := \left\{ x \in \mathbb{R}^2 \mid 2\frac{(x,\alpha)}{(\alpha,\alpha)} = k \right\} \qquad (k \in \mathbb{Z},\ \alpha \in \Sigma)$$

Root system of rank 2 and (extended) Dynkin diagram



A_2 : $(60°, 60°, 60°)$ $I_2(3)$

$\Sigma^+ = \{\alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2\}$: positive roots

$\Sigma^- := -\Sigma^- = \{-\alpha \mid \alpha \in \Sigma^+\}$: negative roots

$\Sigma = \Sigma^+ \cup \Sigma^-$: total roots

$\Psi = \{\alpha_1,\ \alpha_2\}$: fundamental system of roots

$\angle\alpha_1\alpha_2 = 120°$, $|\alpha_1| = |\alpha_2|$ $\left(\Rightarrow\ 2\frac{(\alpha_1,\alpha_2)}{(\alpha_1,\alpha_1)} = -1\right)$ Good realization $\Rightarrow$

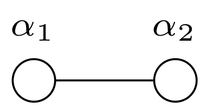
$\Psi = \{\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3\} \subset \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1+x_2+x_3 = 0\}$

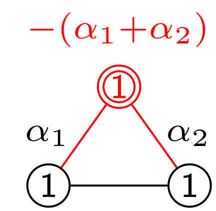
$e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1),\ \alpha_1 = e_1 - e_2 = (1,-1,0)$

$s_1(x) := s_{\alpha_1}((x_1,x_2,x_3)) = (x_1,x_2,x_3) - 2\frac{(x,\alpha_1)}{(\alpha_1,\alpha_1)}\alpha_1 = (x_2,x_1,x_3)$

$-2\frac{(x,\alpha_1)}{(\alpha_1,\alpha_1)}\alpha_1 = -2\frac{x_1-x_2}{2}(1,-1,0) = (x_2-x_1, x_1-x_2, 0)$

$s_2(x) = (x_1,x_3,x_2)$, $s_1 s_2$: $120°$-rotation $\Rightarrow$ $(s_1 s_2)^3 = 1$

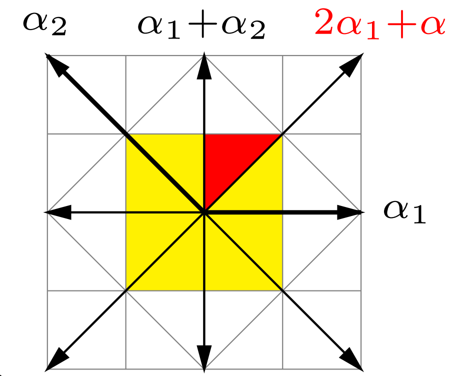
Dynkin diagram : 

Extended Dynkin diagram : 

$\{$mirror planes$\} = \bigcup_{\alpha\in\Sigma^+}\{x = (x_1,x_2,x_3) \mid 2\frac{(x,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}\}$

$W := \langle s_1, s_2\rangle$: Weyl group (permutation group of $\{1,2,3\}$, $\#W = 6$)

$B_2 = C_2 \quad (90^\circ, 45^\circ, 45^\circ) \quad I_2(4)$



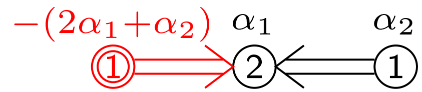
$\Psi = \{\alpha_1,\, \alpha_2\} = \{e_1,\, e_2 - e_1\}$

$\Sigma^+ = \{\alpha_1,\, \alpha_2,\, \alpha_1 + \alpha_2,\, 2\alpha_1 + \alpha_2\}$

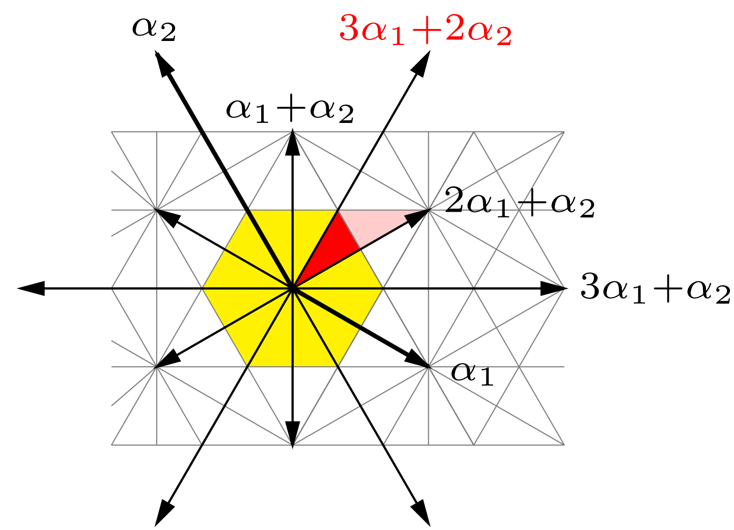
$\angle\alpha_1\alpha_2 = 135^\circ,\ \alpha_2 = \sqrt{2}\alpha_1,\ (s_1 s_2)^4 = 1$

$s_1(x_1, x_2) = (-x_1, x_2),\ \ s_2(x_1, x_2) = (x_2, x_1)$

$\#W = 8$ Extended Dynkin diagram :



$G_2 \quad (30^\circ, 60^\circ, 90^\circ),\ (30^\circ, 30^\circ, 120^\circ) \quad I_2(6)$



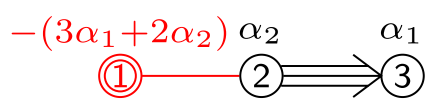
$\Psi = \{\alpha_1 = e_1 - e_2,\, \alpha_2 = -2e_1 + e_2 + e_3\}$

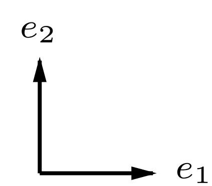
$\Sigma^+ = \{\alpha_1,\, \alpha_2,\, \alpha_1 + \alpha_2,\, 2\alpha_1 + \alpha_2,$

$3\alpha_1 + \alpha_2,\, 3\alpha_1 + 2\alpha_2\}$

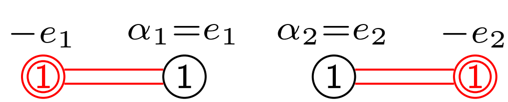
$\angle\alpha_1\alpha_2 = 135^\circ,\ \alpha_2 = \sqrt{3}\alpha_1,\ (s_1 s_2)^6 = 1$

$\#W = 12$





$A_1 + A_1$



$(s_1 s_2)^2 = 1$

Classification of root systems (Dynkin diagrams)

Σ : root system (rank n) $(\#\Sigma < \infty)$

(1) They span $\mathbb{R}^n$

(2) $\alpha \in \Sigma,\ k\alpha \in \Sigma \ \Rightarrow\ k = \pm 1$

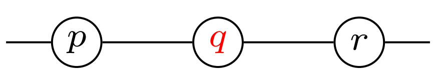
(3) $s_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Sigma \quad (\forall \alpha,\ \beta \in \Sigma)$

(4) $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \quad (\forall \alpha,\ \beta \in \Sigma) \quad \Rightarrow \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \cdot \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \{0,1,2,3\} \quad (\alpha \neq \pm\beta)$

$\Sigma \supset \Psi$: fundamental: (1), $\#\Psi = n$, $\Psi \supset \forall\{\alpha,\beta\}$: $(\mathbb{R}\alpha + \mathbb{R}\beta) \cap \Sigma$ is fund.

Dynkin diagram : element of Ψ by ○ and relation of two by Dynkin diagram

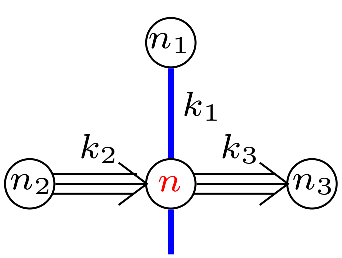
Extended Dynkin diag. : add negative of the maximal root to Ψ and put the coefficients with respect to Ψ (put 1 in added ○)



$\Rightarrow\ 2q = p + r$

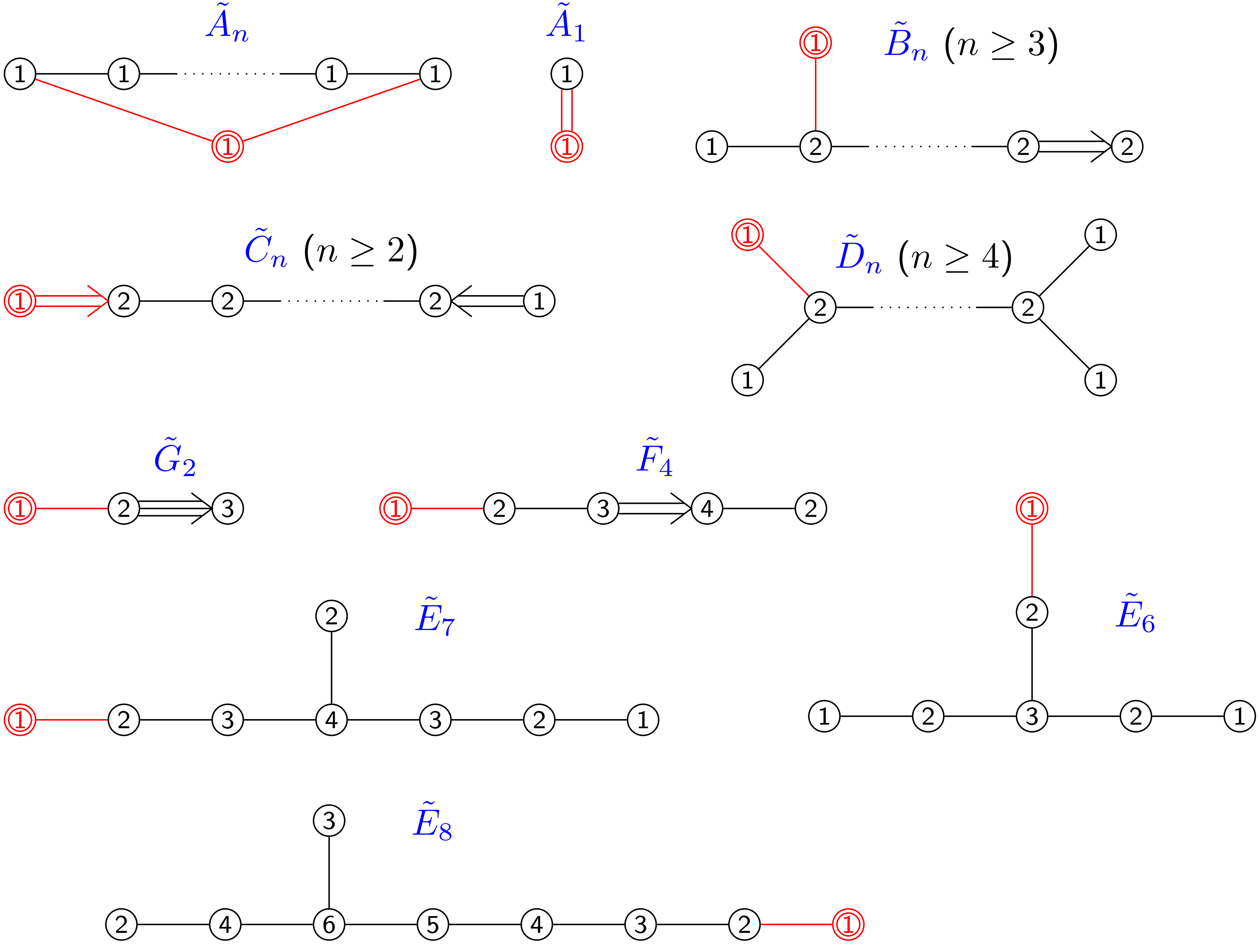
$\left(2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right)_{\substack{1\le i\le n\\ 1\le j\le n}}$: Cartan matrix

$\Psi = \{\alpha_1, \ldots, \alpha_n\}$



$\Rightarrow\ 2n = k_1 n_1 + k_2 n_2 + n_3 + \cdots$

Extended Dynkin diagram



$$\Psi = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_n = e_n - e_{n+1}\}$$

$$\Sigma = \{\pm(e_i - e_j) \mid 1 \le i < j \le n+1\}$$

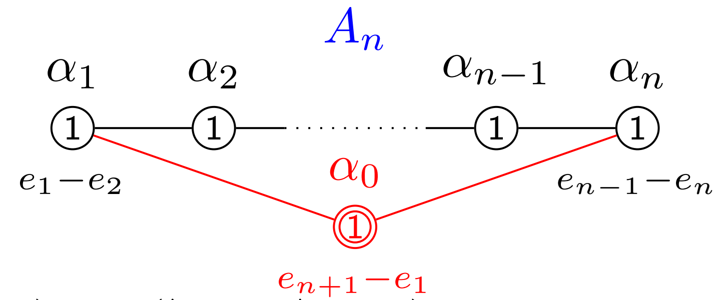
$$s_j := s_{e_j - e_{j+1}} \ : \ x_j \leftrightarrow x_{j+1} \ (j = 1, \ldots, n)$$

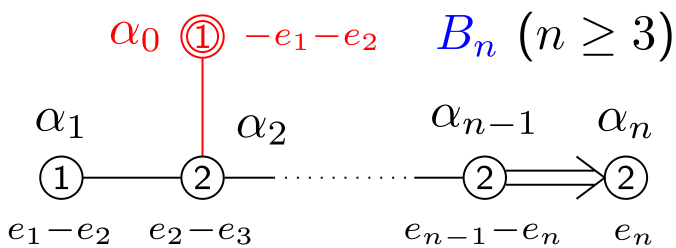
$$(s_i s_j)^{n_{i,j}} = 1 \quad : \ n_{i,j} = 1 \ (i = j), \ 2 \ (|i-j| > 1), \ 3 \ (|i-j| = 1)$$

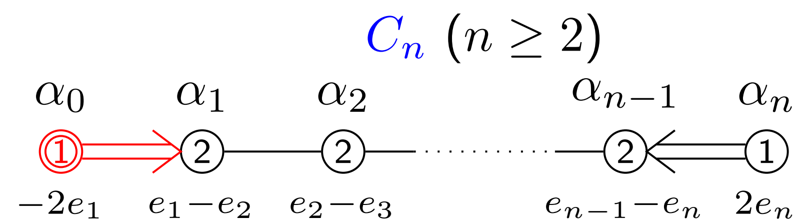
W : permutation group of $\{1, 2, \ldots, n+1\}$ (of degree $n+1$)

$$\#W = (n+1)!, \ \ \#\Sigma = n(n+1)$$

W-invariants : symmetric polynomial, elementary symmetric polynomial, $\cdots$

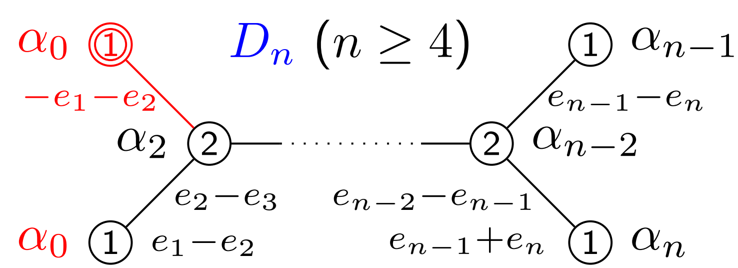






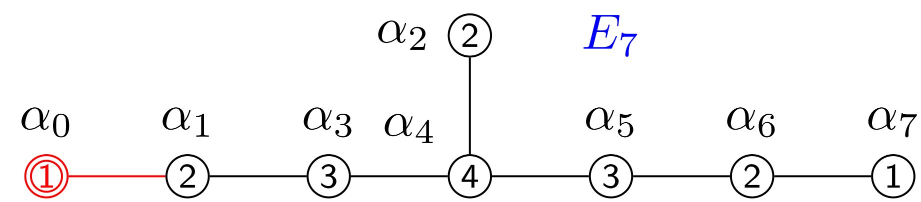
$$s_j \ : \ x_j \leftrightarrow x_{j+1}$$

$$s_n \ : \begin{cases} x_n \mapsto -x_n & (B_n, \ C_n) \\ (x_{n-1}, x_n) \mapsto (-x_{n-1}, -x_n) & (D_n) \end{cases}$$

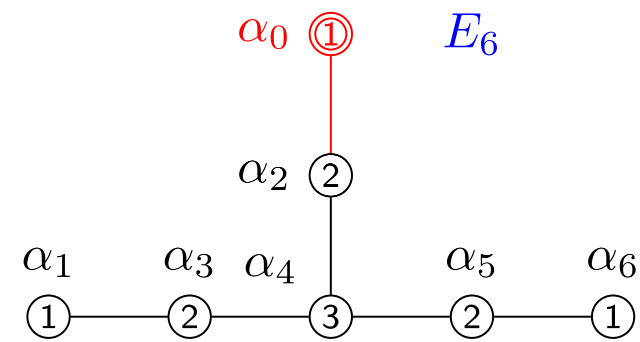


$$B_n, C_n \ : \ \#\Sigma = 2n^2, \ \ \#W = 2^n n!$$

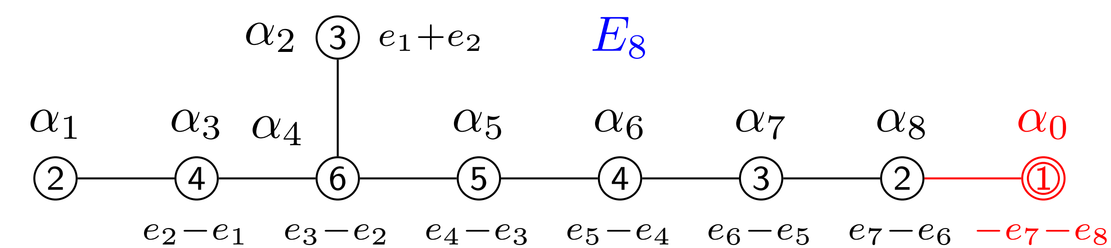
$$D_n \qquad : \ \#\Sigma = 2n(n-1), \ \ \#W = 2^{n-1} n!$$



$$\#\Sigma = 126, \;\; \#W = 2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 72903040$$



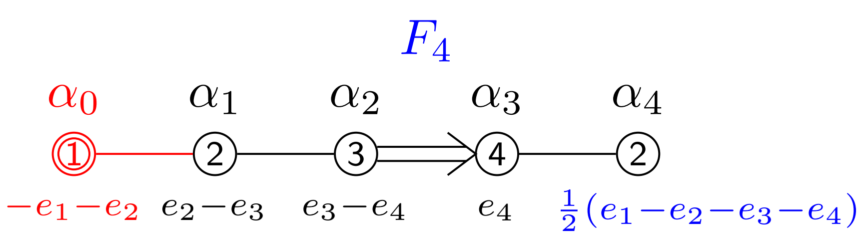
$$\#\Sigma = 72, \;\; \#W = 51840$$

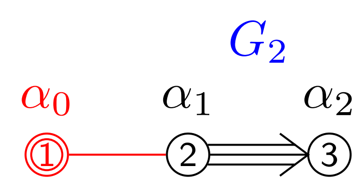


$$\alpha_1 = \tfrac{1}{2}(e_1 + e_8) - \tfrac{1}{2}(e_2 + e_3 + \cdots + e_7)$$

$$\Sigma = \{\pm(e_i - e_j), \;\pm(e_i + e_j), \;\pm e_1 \pm \cdots \pm e_8 \;(\#- : \text{ even}) \mid 1 \le i < j \le 8\}$$

$$\#\Sigma = 240, \; \#W = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600$$



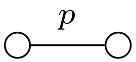


$$\Sigma = \{\pm e_i \pm e_j, \;\pm e_k, \;\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \mid 1 \le i < j \le 4, \, 1 \le k \le 4\}$$

$\#\Sigma = 48, \;\; \#W = 2^7 \cdot 3^2 = 1152$ 24-cell group

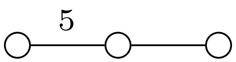
Finie reflection group

Other than the Weyl group corresponding to root spaces

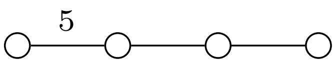


$I_2(p)$: Dihedral group of order p, $p = 5, 7, 8, \ldots$

$\#G = 2p \quad s_1^2 = s_2^2 = (s_1 s_2)^p = 1$



H_3 : icosahedral group, $\quad \#G = 120$



H_4 : 120 (600)-cell group (regular polytope) $\quad \#G = 14400$

W_{A_3} : regular tetrahedron

$W_{B_3} = W_{C_3}$: regular hexahedron, regular octahedron

Coxeter system, Kac-Moody root system

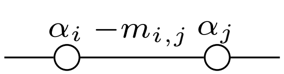
Coxeter group : $\langle s_i \mid s_i^2 = 1,\ (s_i s_j)^{m_{i,j}} = 1 \rangle$

$m_{i,i} = 2,\ m_{i,j} = 0, -1, -2, \ldots$

finite Coxeter group = (finite) reflection group

Kac-Moody root system : $s_{\alpha_i}(\alpha_j) = \alpha_j - m_{i,j}\alpha_i$ ($m_{i,j}$:integer)

$(m_{i,j})$: Cartan matrix $\quad A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad B_2 = C_2 : \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad G_2 : \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$



$m_{i,j} = m_{j,i}$: symmetric

Simple Lie algebra, simple Lie group, their classification

G : (connected) complex simple Lie group) $\subset M(N,\mathbb{C})$: $N \times N$ matrices

$g_1,\ g_2 \in G \Rightarrow g_1^{-1} g_2 \in G,\ {}^t g_1 \in G$

simple : cannot be constructed from smaller Lie groups

A_n : $SL(n+1,\mathbb{C}) := \{g \in M(n+1,\mathbb{C}) \mid |g| = 1\}$

B_n : $SO(2n+1,\mathbb{C}) := \{g \in SL(2n+1,\mathbb{C}) \mid {}^t g g = I_{2n+1}$: identity matrix $\}$

C_n : $Sp(2n,\mathbb{C}) := \{g \in SL(2n,\mathbb{C}) \mid {}^t g J_n g = J_n\}, \quad J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

D_n : $SO(2n,\mathbb{C}) := \{g \in SL(2n,\mathbb{C}) \mid {}^t g g = I_{2n}\}$

finite group : determined by generators and their relations

Lie group is "determined" by its Lie algebra (generators/relations)

$\mathrm{Lie}(G) := \{X \in M(N,\mathbb{C}) \mid e^{tX} \in G\ (|t| < 1)\}$: linear space

$$e^X = I_n + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots$$

$$e^{tX} e^{tY} e^{-tX} e^{-tY} = (I_n + tX + \frac{t^2}{2} X^2 + \cdots) \cdots (I_n - tY + \frac{t^2}{2} Y^2 + \cdots)$$

$$= I_n + \frac{t^2}{2}(XY - YX) + o(t^2)$$

$[X,Y] := XY - YX$ (Lie algebra represents non-commutativity of Lie group)

$G = SL(n+1,\mathbb{C}) \ \Rightarrow \mathrm{Lie}(G) = \{X \in M(n+1,\mathbb{C}) \mid \mathrm{trace}\, X = 0\}$

$E_{k,\ell} = \left(\delta_{i,k}\delta_{j,\ell}\right)_{\substack{1\le n+1,\\ j\le n+1}} \in M(n+1,\mathbb{C})$

$D(e_1,\dots,e_{n+1}) := e_1E_{11} + e_2E_{22} + \cdots + e_{n+1}E_{n+1n+1}$: diagonal matrix

$\mathfrak{a} := \{D(e_1,\dots,e_{n+1}) \mid e_1+\cdots+e_{n+1} = 0\} \in \mathrm{Lie}(G)$ (maximal commutative)

$[D(e_1,\dots,e_{n+1}), E_{i,j}] = (e_i - e_j)E_{i,j}$, $\{e_i - e_j\}$: root system of type A_n

$D(e_1,\dots,e_{n+1})E_{i,j} = e_iE_{i,j}$, $E_{i,j}D(e_1,\dots,e_{n+1}) = e_jE_{i,j}$

Dynkin diagram gives the structure of Lie group (skeleton)

isomorphism of Dynkin diagram : outer automorphism of Lie gp. (alg.)

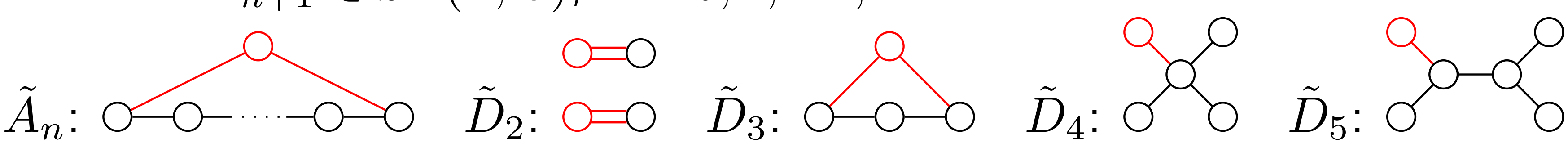
($n > 2 \Rightarrow 2$ case) $\left(g \mapsto {}^tg^{-1},\ n = 2 \Rightarrow {}^tg^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} g \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}\right)$

Isomorphism of extended Dynkin : Global structure of Lie gp.

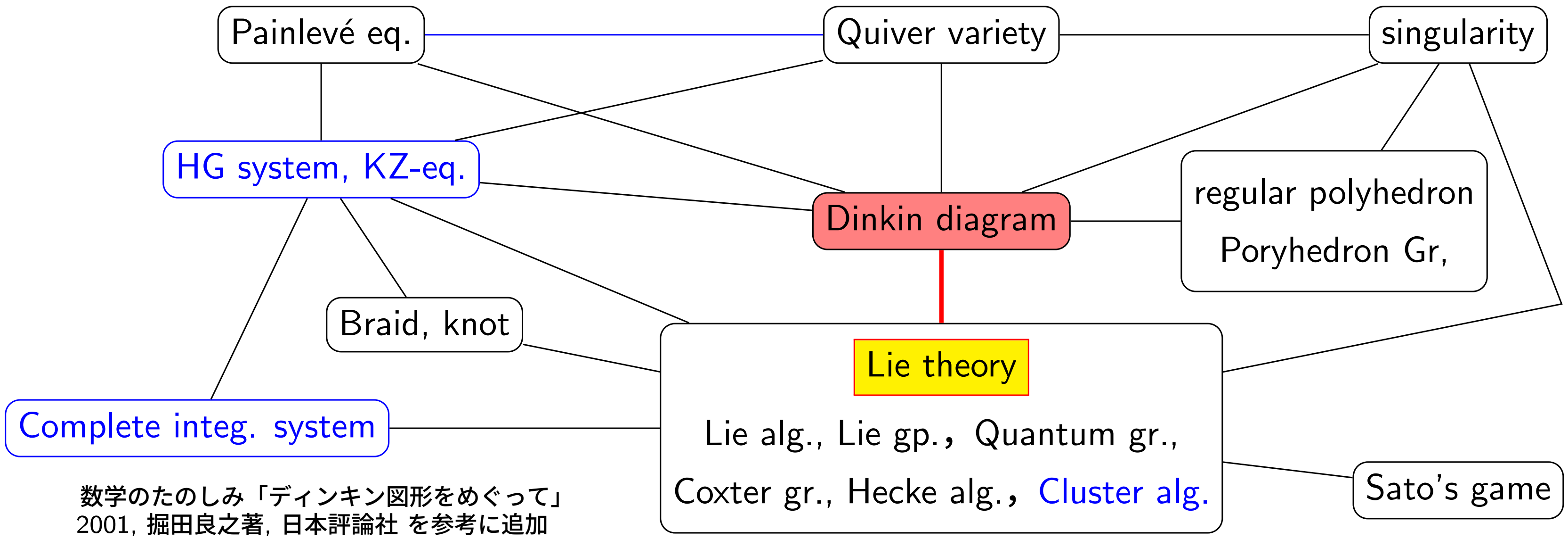
$Spin(m,\mathbb{C})$: simply connected $\quad B_n,\ D_n \quad SO(n,\mathbb{C}) \simeq Spin(n,\mathbb{C})/\{\pm 1\}$

$PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm1\} \simeq SO(3,\mathbb{C}),\quad SU(2)/\{\pm1\} \simeq SO(3)$

$e^{\frac{2k\pi\sqrt{-1}}{n+1}} I_{n+1} \in SL(n,\mathbb{C}),\ k = 0,1,\dots,n$



Connections to Dynkin diagram and root systems



数学のたのしみ「ディンキン図形をめぐって」
2001, 堀田良之著, 日本評論社 を参考に追加

real simple Lie gp. : Satake diag. (Dynkin diag. + an involution)

completely integral quantum system : $P_1 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - V(x)$

$\overset{\text{def}}{\Leftrightarrow} \exists P_i$ with $P_i P_j = P_j P_i \quad (1 \le i < j \le n)$

$P_i u = \lambda_i u \quad (i = 1, \ldots, n)$ has a meaning

$\Rightarrow$ orthogonal polynomial, spherical expansion, harmonic analysis,

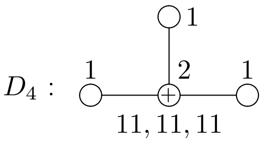
Ex. $V(x) = C \sum_{\alpha \in \Sigma} u\big((\alpha, x)\big)$, $u(t) = \frac{1}{t^2}, \frac{1}{\sin^2 t}, \wp(t)$

$V(x)$: singular at $x_1 = 0 \Rightarrow V(x) \sim \frac{C}{(e_1, x)^2}$ and invariant by reflection s_{e_1}

singular at $x_1 = c_j \Rightarrow$ periodicity: trigonometric/elliptic fn. $\frac{1}{\sin^2 t} = \sum_n \frac{1}{(t - n\pi)^2}$

Differential Eq. (Hypergeometric, Painlevé)

Gauss HG $\Rightarrow$ rigid : real root$(\alpha,\alpha) = 2$



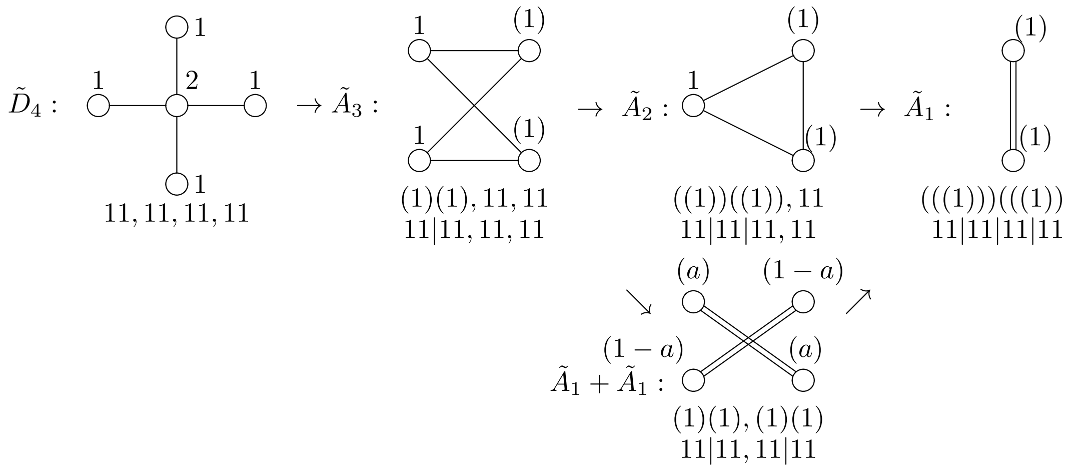
$$x(1-x)u'' - \big(\gamma - (\alpha+\beta+1)x\big)u' + \alpha\beta u = 0$$

Versal Huen $\Rightarrow$ affine root : $(\alpha,\alpha) = 0$

2 accessory parameters $\Rightarrow$ deformation with preserving monodromy $\Rightarrow$ Painlevé

$$\frac{d\vec{u}}{dx} = \frac{A_1 u}{1-c_1x} + \frac{A_2 x\vec{u}}{(1-c_1x)(1-c_2x)} + \frac{A_3x^2\vec{u}}{(1-c_1x)(1-c_2x)(1-c_3x)} \quad (A_i \in M(2,\mathbb{C}))$$

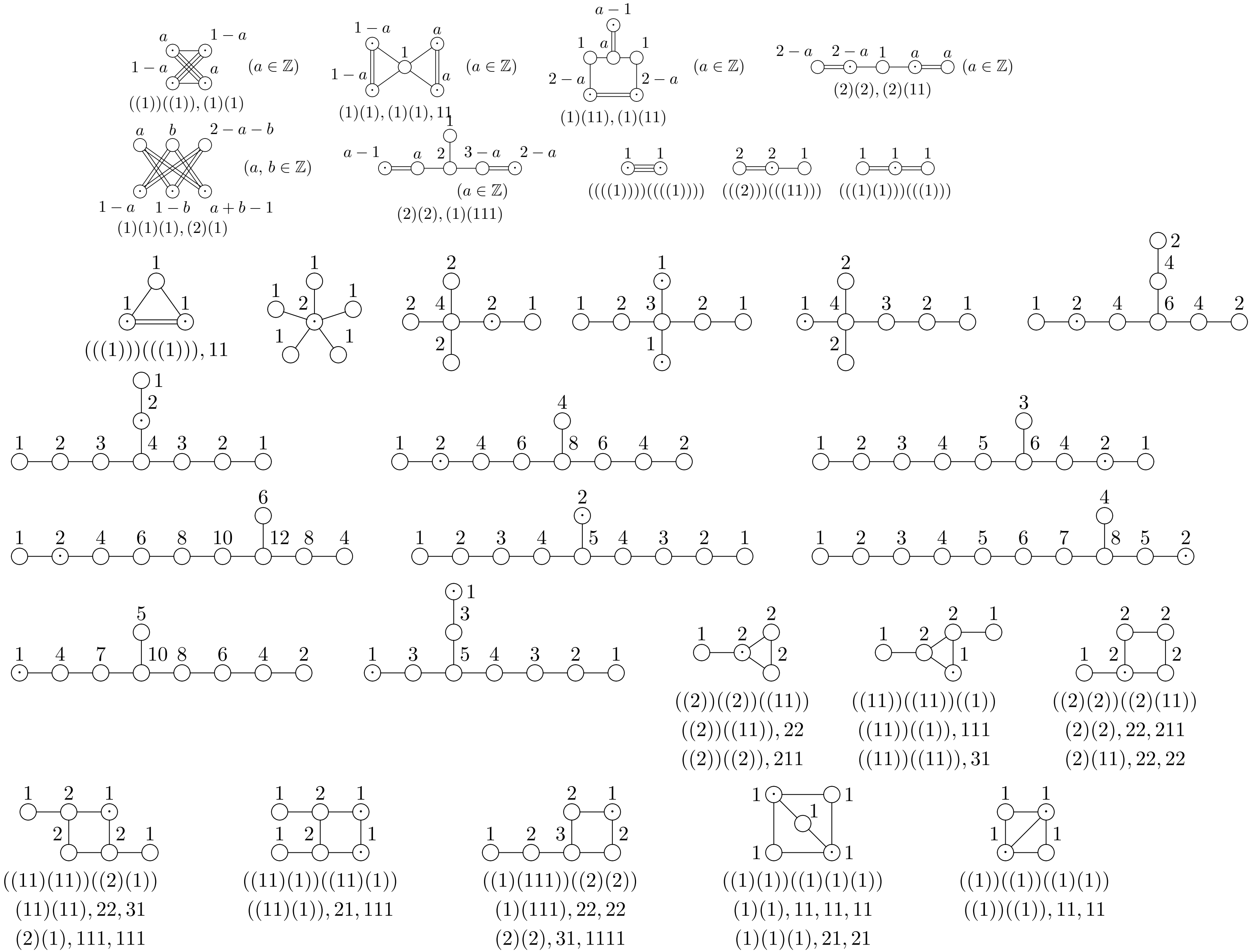
confluent $\rightarrow$



{Sol. of Painlevé eq.} $\simeq$ (Okamoto) initial value space : alg. surface (classified)

symmetry	$\tilde{E}_8$	$\tilde{E}_7$	$\tilde{E}_6$	$\tilde{D}_4$	$\tilde{A}_3$	$2\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_1$	$\tilde{A}_1$	$\tilde{A}_0$	$\tilde{A}_0$
initial val. sp.	$\tilde{A}_0$	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{D}_4$	$\tilde{D}_5$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_7$	$\tilde{E}_7$	$\tilde{D}_8$	$\tilde{E}_8$

Hiroe − O (2012) $(\alpha, \alpha) = -2 \Rightarrow$ HG, 4-dim Painlevé eq.



Thank your for your attention