Japanese Theorem and Catalan number

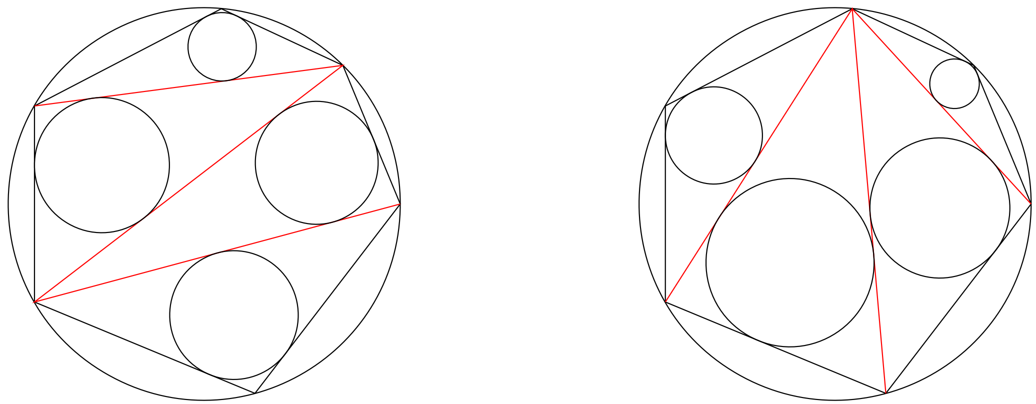
Toshio Oshima (大島利雄)
Center for Mathematics and Data Science
https://www.ms.u-tokyo.ac.jp/~oshima

National University of Mongolia

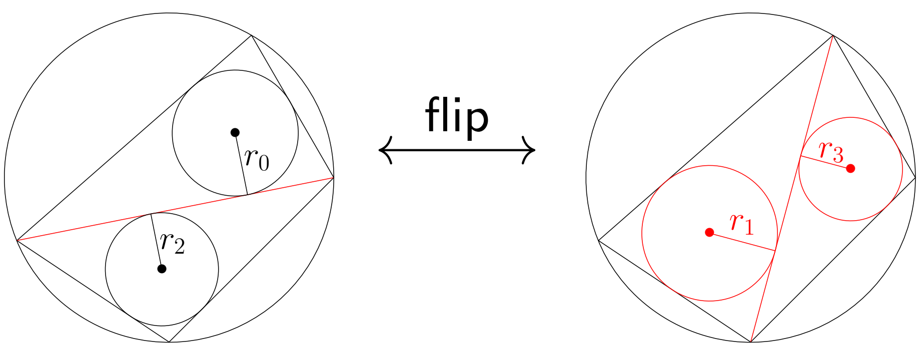
September 19, 2024

§ Japanese Theorem

Theorem. No matter how one triangulates a cyclic polygon, the sum of the radii of all the incircles of triangles is constant.



Theorem [Ryokan Maruyama(Sangaku, 1800)]. For quadrilaterals.



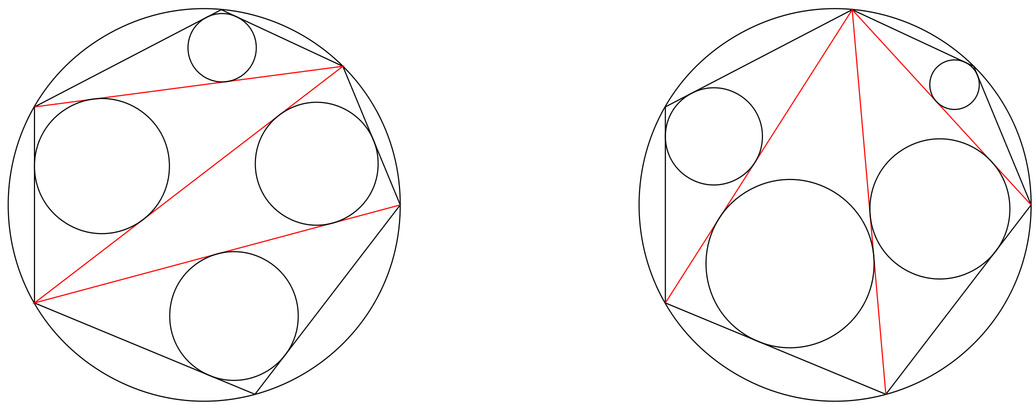
Mathematical Tablet

Dedicated to a shrine in Yamagata

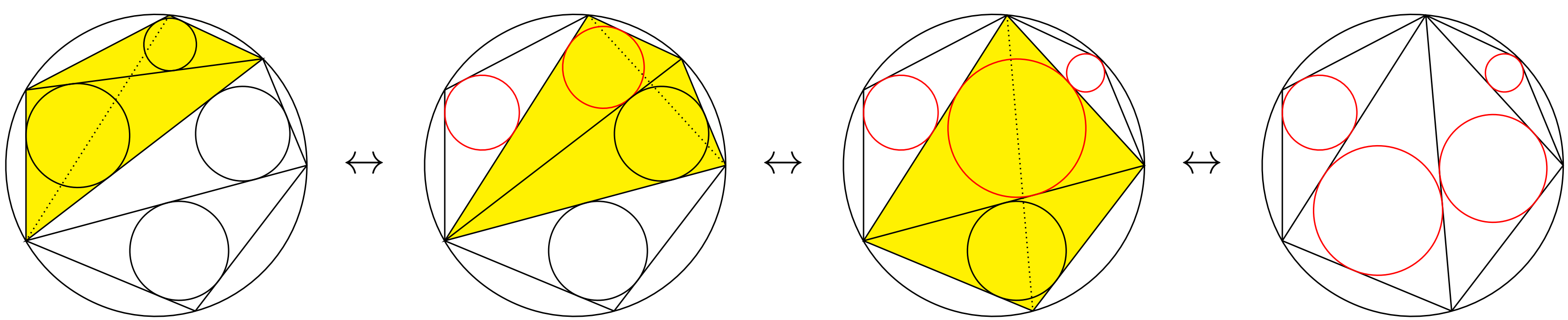
$$r_0 + r_2 = r_1 + r_3$$

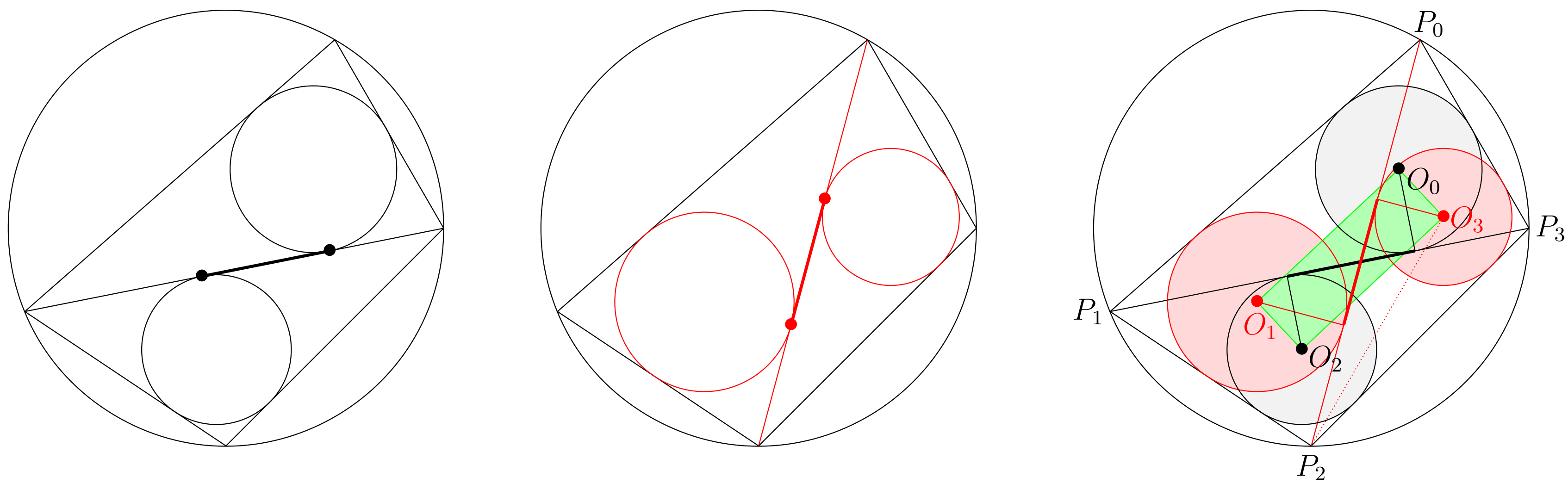
§ Japanese Theorem

Theorem. No matter how one triangulates a cyclic polygon, the sum of the radii of all the incircles of triangles is constant.



Theorem [Ryokan Maruyama(Sangaku, 1800)]. For quadrilaterals.

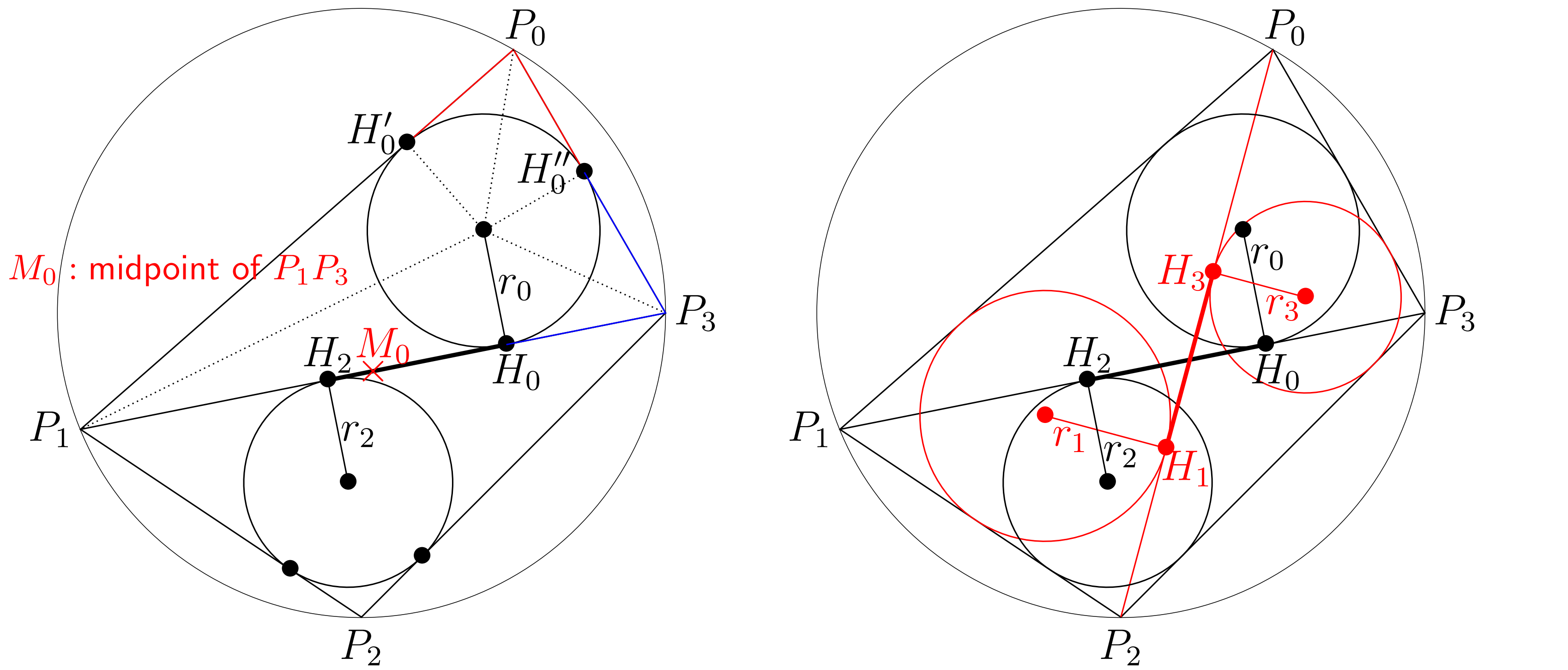




Several proofs : 解説は Uegaki(2002, 1〜5), Oshima(2022)

1. Equal distances between points of contact along diagonals
2. Carnot's Theorem: lengths of perpendiculars dropped from circumcenter of a triangle to each side and radii of circum/incircles
3. center of the incircle ⇒ vertex of rectangle+Chapple's Theorem
4. Difference between the radii incircles
5. Use formula of trigonometric functions (sum formula etc.)
6. Thébault's theorem (⇐Extended Ptolemy's Theorem)
7. Use complex plane $\mathbb{C}$

1. points: P_0, P_1, P_2, P_3, incircles: O_0, O_1, O_2, O_3, radii: r_0, r_1, r_2, r_3



$$P_0H_0' = P_0H_0'', \text{ etc. } \Rightarrow$$

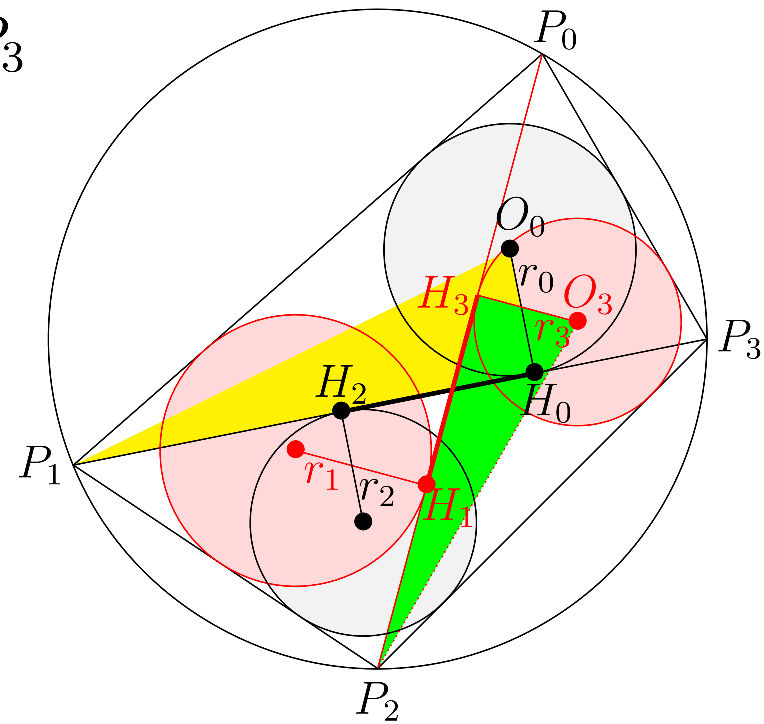
$$\begin{aligned} P_0P_1 - P_3P_0 &= P_0H_0' + P_1H_0' - (P_0H_0'' + P_3H_0'') \\ &= P_1H_0 - P_3H_0 = \pm 2M_0H_0, \text{ etc. } \Rightarrow \end{aligned}$$

$$P_0P_1 - P_3P_0 + P_2P_3 - P_1P_2 = P_1H_0 - P_3H_0 + P_3H_2 - P_1H_2 = \pm 2(H_0H_2)$$

$$P_0P_1 - P_1P_2 + P_2P_3 - P_3P_0 = P_0H_1 - P_2H_1 + P_2H_3 - P_0H_3 = \pm 2(H_1H_3)$$

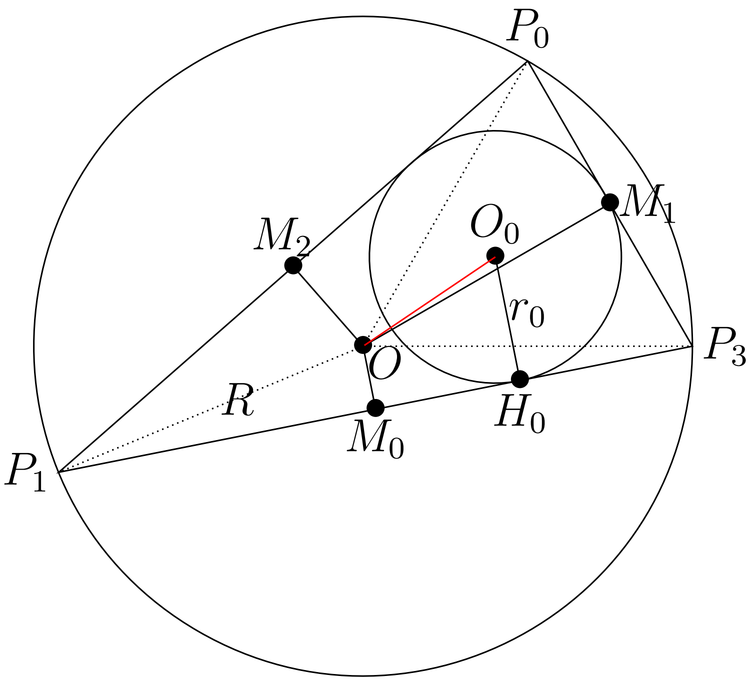
$$H_0H_2 = H_1H_3$$

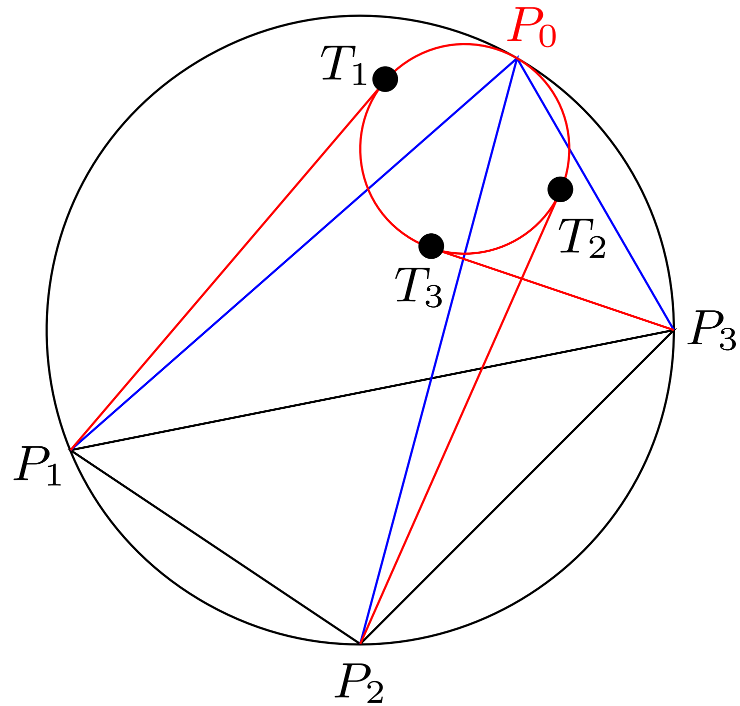
$$\angle O_0P_1H_0 = \tfrac{1}{2}\angle P_0P_1P_3 = \tfrac{1}{2}\angle P_0P_2P_3$$
$$= \angle O_3P_2H_3$$
$$\Rightarrow \ \triangle O_0P_1H_0 \sim \triangle O_3P_2H_3$$
$$\Rightarrow \frac{r_0}{r_3} = \frac{O_0H_0}{O_3H_3} = \frac{P_1H_0}{P_2H_3}$$
$$\Rightarrow \begin{cases} r_0P_2H_3 = r_3P_1H_0 \\ r_1P_3H_0 = r_0P_2H_1 \\ r_2P_0H_1 = r_1P_3H_2 \\ r_3P_1H_2 = r_2P_0H_3 \end{cases}$$



$$r_0(P_2H_3 - P_2H_1) + r_2(P_0H_1 - P_0H_3)$$
$$= r_1(P_3H_2 - P_3H_0) + r_3(P_1H_0 - P_1H_2),$$
$$\Rightarrow (r_0 + r_2)H_1H_3 = (r_1 + r_3)H_0H_2 \ \Rightarrow \ r_0 + r_2 = r_1 + r_3$$

The above proof is give by T. Yoshida (1819–1892) in『壁算法附録解』





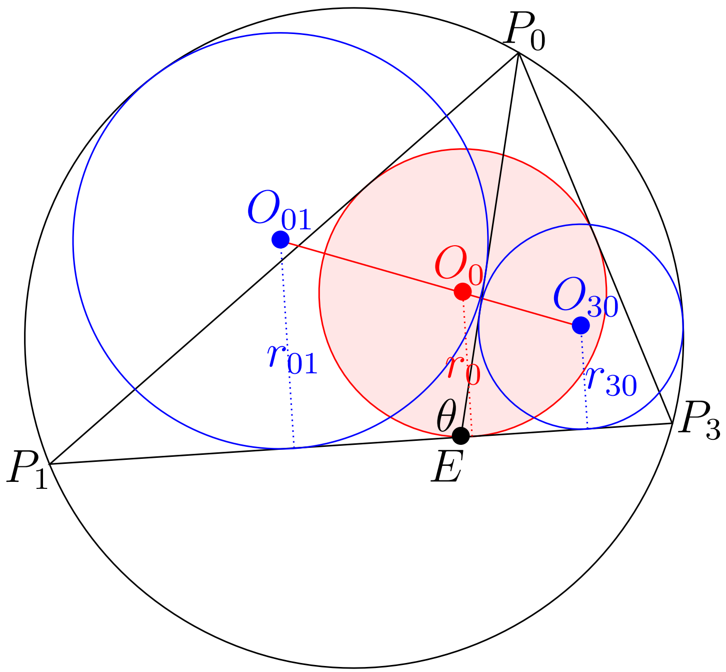
Circmcircle O: center, R: radius($=\ \ P_1$), $h_i := OM_i$

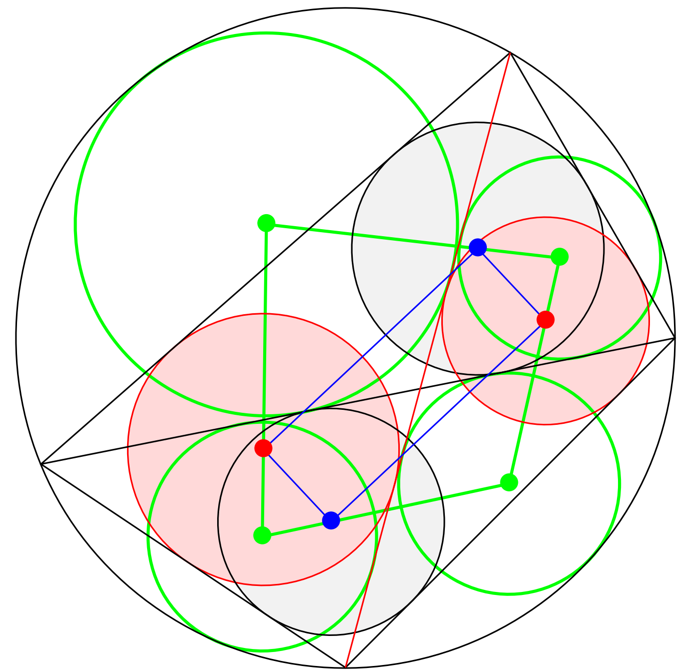
Chapple's Th. : $OO_o^2 = R^2 - 2Rr_0$

Carnot's Th. : $R + r_0 = \begin{cases} h_0 + h_1 + h_2 - 2h_i & (\overline{OM_i}\text{:outside}) \\ h_0 + h_1 + h_2 & (O\text{:inside}) \end{cases}$

Ptolemy's Th. : $P_0P_1 \cdot P_2P_3 + P_1P_2 \cdot P_3P_0 = P_0P_2 \cdot P_1P_3$

Extended Ptolemy's Th. : $T_1P_1 \cdot P_2P_3 + P_1P_2 \cdot P_3T_3 = T_2P_2 \cdot P_1P_3$





Sawayama–Thébault's Th. : O_0 is on the line segment $O_{01}O_{30}$,

$$r_0 = r_{01}\cos^2\frac{\theta}{2} + r_{30}\sin^2\frac{\theta}{2}$$

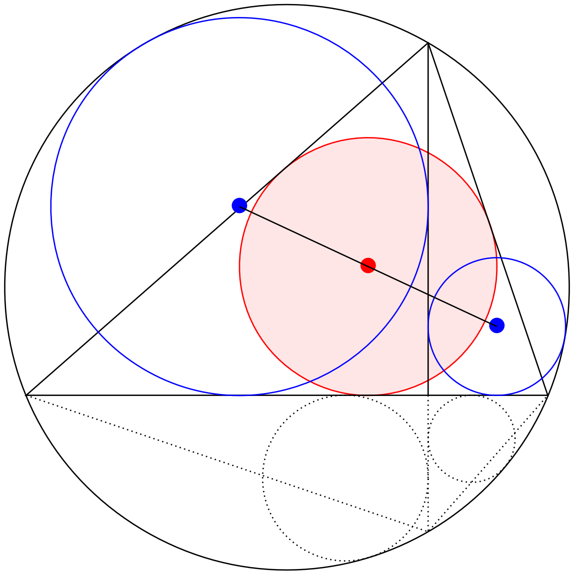
Thébault 1938 :

Streefkerk 1973 (Duch), Veldkamp 1989, Taylor 1983 (proof requires 24pages)

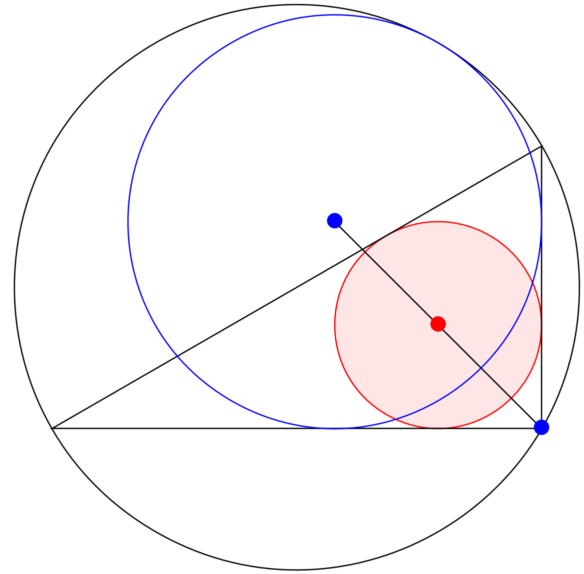
Sawayama 1905, Shail 2001, Ayme 2003, Čelin 2011, Doai– 2016 etc.

Gueron 2002 (Ext. Ptolemy⇒Thébault),

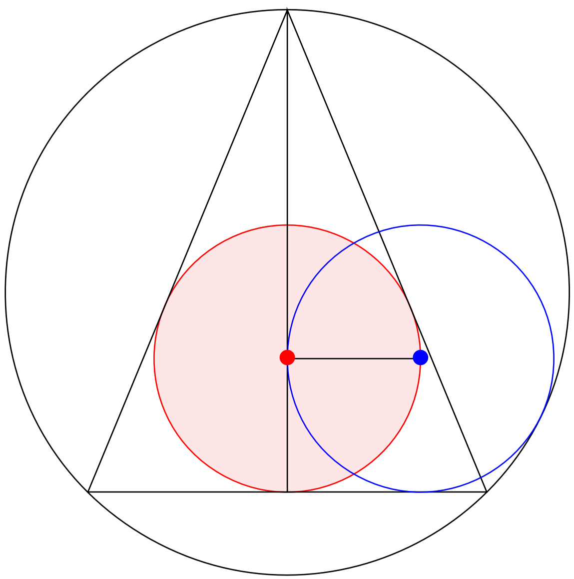
Reyes 2002 (⇒Japanese Theorem)



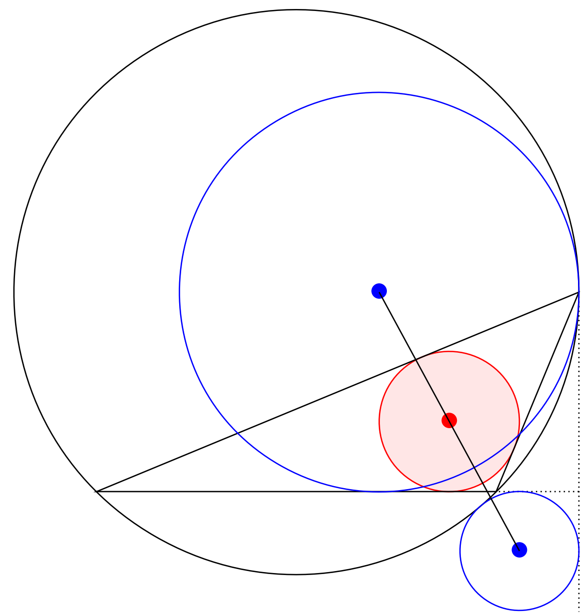
Fukushima, Shirine 田村大元神社 (1901) 5-th in 2



Oufunato, Shrine 根城八幡宮 (1941)



福島県田村大元神社 (1901) 1 枚目 7 問目



Fukushima Prefecture, Tamura vilage, Shrine 田村大元神社

Dedicated to the shrine 1901

from Kodera Yu, Wasan's house http://www.wasan.jp/

Fukushima Prefecture, Tamura vilage, Shrine 田村大元神社

Dedicated to the shrine 1901

from Kodera Yu, Wasan's house http://www.wasan.jp/

Complex plane

A triangle with side lengths a, b, c, with arrea S, radius of the incircle r

$$S = \frac{1}{4}\sqrt{(b+c-a)(c+a-b)(a+b-c)(a+b+c)} \quad \text{(Heron's formula)}$$

$$= \frac{1}{2}(a+b+c)r$$

$$\Rightarrow \qquad \boxed{r = \frac{1}{2}\sqrt{\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}}}$$

Vertices of a cyclic quadrilateral : $P_j = z_j^2 \quad (j = 0, 1, 2, 3)$

$$|z_j| = 1, \quad 0 \leq \operatorname{Arg} z_0 < \operatorname{Arg} z_1 < \operatorname{Arg} z_2 < \operatorname{Arg} z_3 < \pi$$

$$P_iP_j = |z_i^2 - z_j^2| = \sqrt{(z_i^2 - z_j^2)(\tfrac{1}{z_i^2} - \tfrac{1}{z_j^2})} = \frac{z_i^2 - z_j^2}{\sqrt{-1}z_i z_j} \quad (0 \leq i < j \leq 3)$$

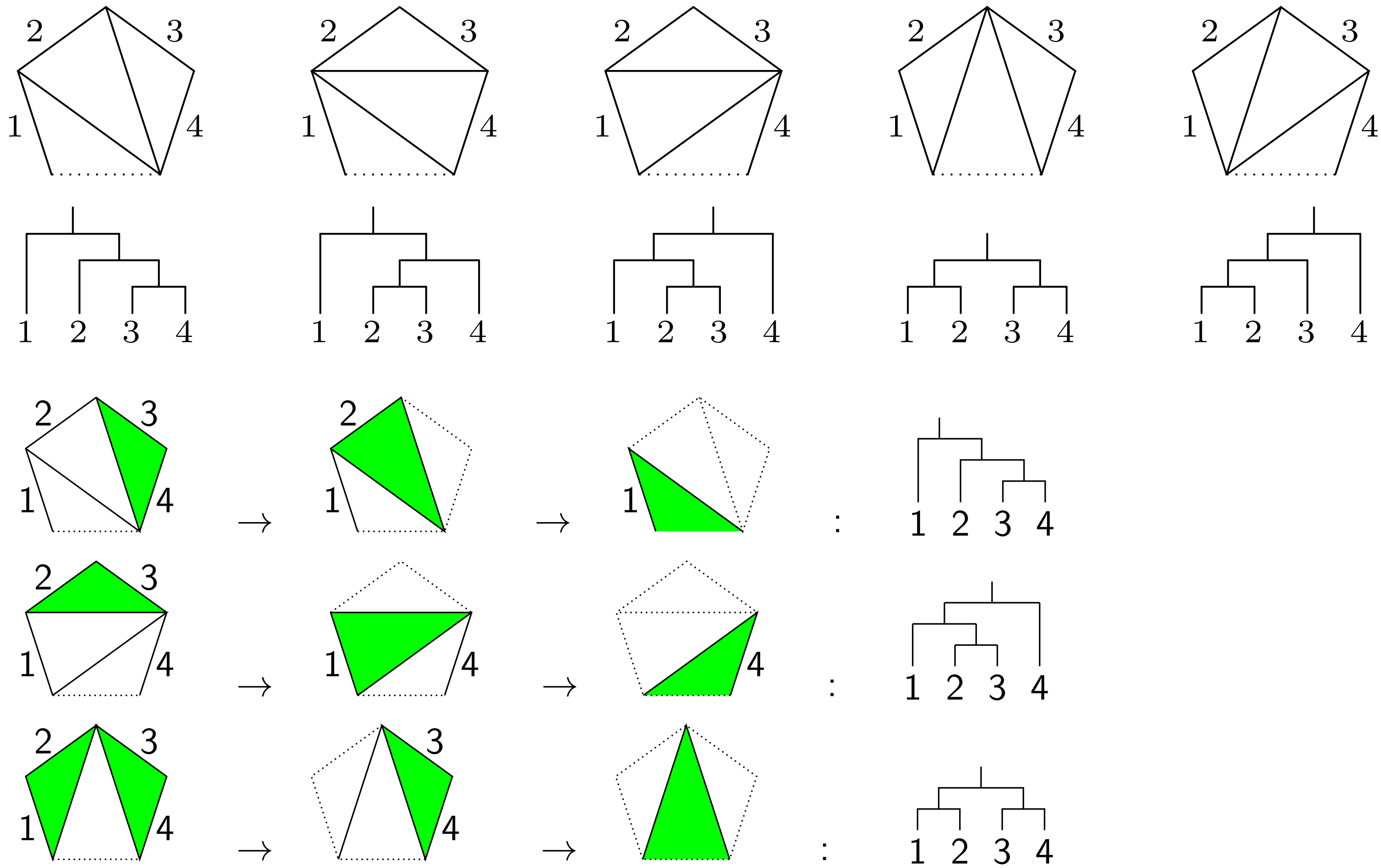
$r(z_i, z_j, z_k)$: radi of incircle of the triangle with vertices z_i^2, z_j^2, z_k^2

$$r(z_0, z_1, z_2) = \frac{(z_2 + z_0)(z_1 - z_0)(z_2 - z_1)}{-2z_0 z_1 z_2}$$

$$r(z_0, z_1, z_3) - r(z_0, z_1, z_2) = \frac{(z_1 - z_0)(z_3 - z_2)(z_0 z_1 + z_2 z_3)}{-2z_0 z_1 z_2 z_3}$$

$$= r(z_0, z_2, z_3) - r(z_1, z_2, z_3)$$

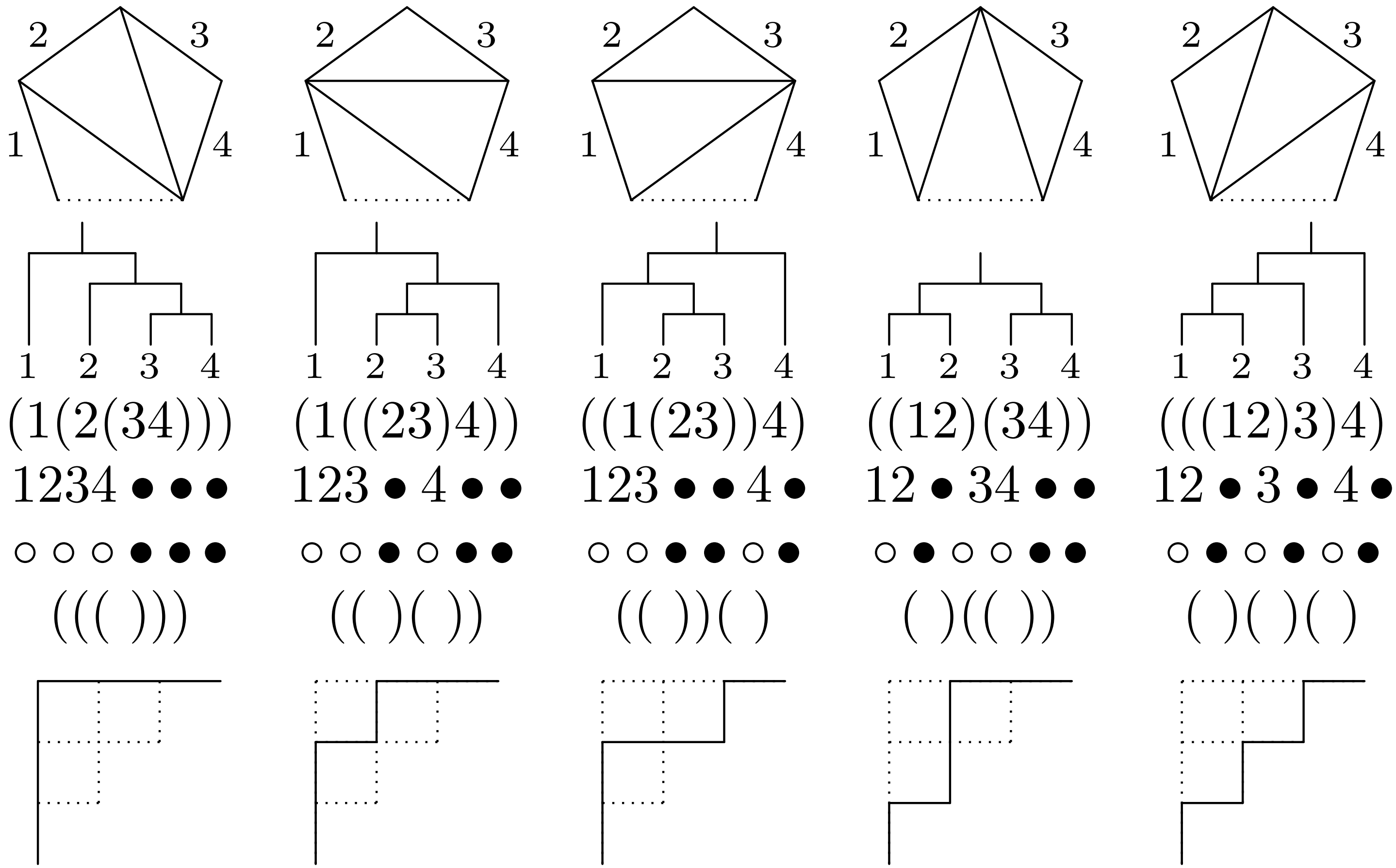
§ Triangulations of polygon and tournament



C_n Catalan number:#triangulations of $n+2$-gon $(n = 3 \Rightarrow C_3 = 5)$

#figures of tournaments of $n+1$ teams

§ Triangulation of convex $n+2$-gon & Catalan number



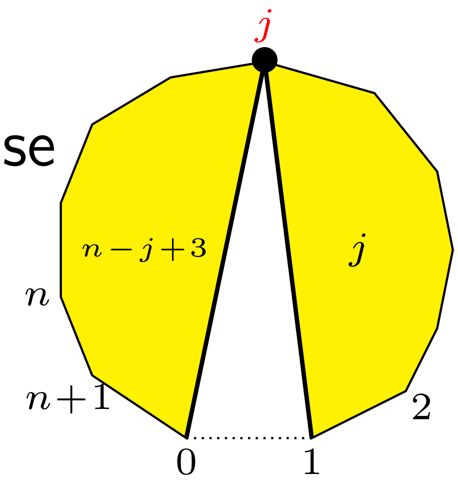
- single-elimination tournaments of $n+1$ teams
- order of calculation of a product of $n+1$ matrices $A_1 \cdots A_n$
- combinations of n sets of "(" and ")"
- restricted lattice paths from $(0, -n)$ to $(n, 0)$

Vertices of $n{+}2$-gon : $\{0, 1, \ldots, n+1\}$ $(\circlearrowleft)$

j : vertex of a triangle with 0, 1 (minimal) as the base

⇒ classify triangulation of j-gon and $n{-}j{+}3$ -gon with vertices$\{1, \ldots, j\}$ and $\{0, j, \ldots, n+1\}$

⇒ $C_{j-2} \times C_{n-j+1}$ cases$(2 \leq j \leq n+1)$



$j \;=\; 2 \qquad 3 \qquad n+1$

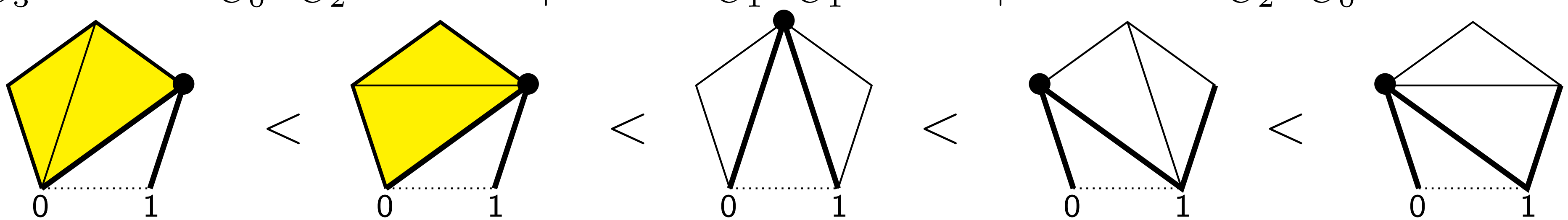
$$\boxed{C_n = C_0 \cdot C_{n-1} + C_1 \cdot C_{n-2} + \cdots + C_{n-1} \cdot C_0} \qquad (C_0 = 1)$$

C_{n-2} triangulations of n-gon can be arranged in order

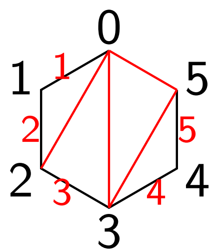
- triangulations of $n{+}2$-gon are arranged in the first part of those of $(n+3)$-gon with putting a vertices between 0 and 1

Triangulation : m-th one is realized in n-gon$(1 \leq m \leq C_{n-2})$

$C_3 = \quad C_0 \cdot C_2 \quad + \quad C_1 \cdot C_1 \quad + \quad C_2 \cdot C_0$



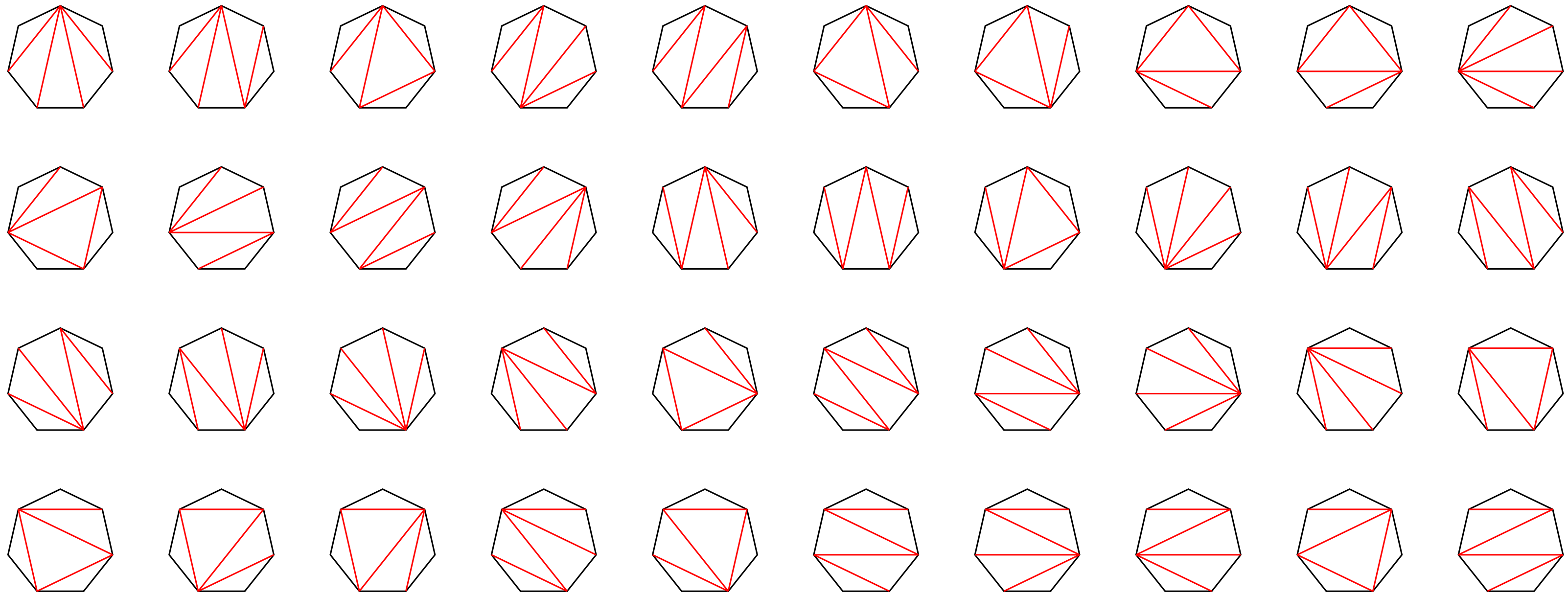
- A triangulation: $n-3$ pairs of diagonals expressed by lists of two vertices in $\{0, \ldots, n-1\}$. (unique minimal representative)

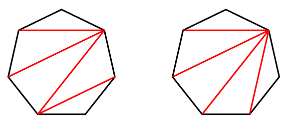


: [[0,2],[0,3],[3,5]]   [2,0,1,2,0,1] (cf. `os_mulidif.rr`)

- List of numbers of diagonals through each vertex
- Tournament : $(((**)*)(**))$ $(((12)3)(45))$
- Admissible 01 sequence of degree $m = n-2$: sequence of 0 and 1 with a certain condition : $\underbrace{01010}_{\#0 \geq \#1}011$ $\circ \bullet \circ \bullet \circ \circ \bullet \bullet$

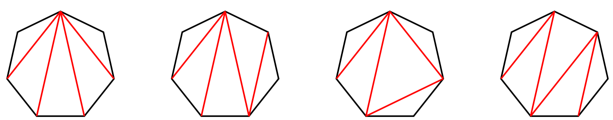
1. Interconversion between these expressions
2. Get k-th triangulation of n-gon
3. Operation : rotation, reflection, flip, enlargement, reduction
4. Table of triangulations (modulo rotation, reflection)
5. Readable source of Ti*k*Z containing triangulations of polygons

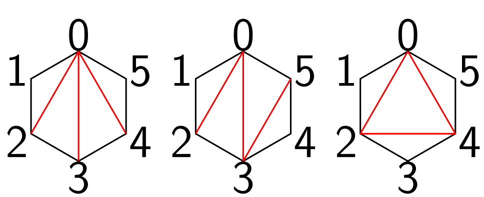


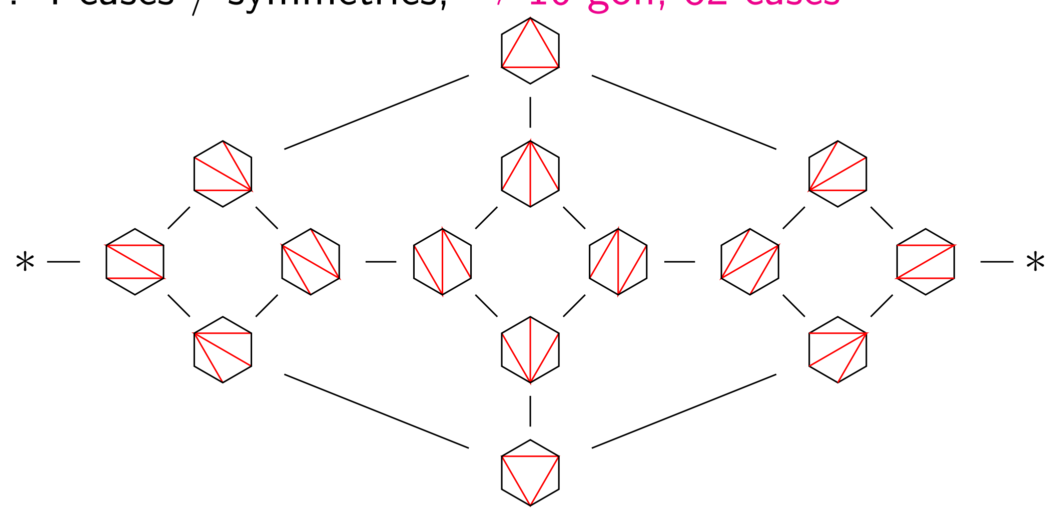


: 7-gon, 42 cases (the north vertex: 0)

: 6-gon : first 14 cases by excluding the north west vertex 1

⇒  : 4 cases / symmetries, → 10-gon, 82 cases





$$u(x) := C_0 + C_1 x + \cdots + C_n x^n + \cdots \qquad (\text{Generating function})$$

$$u^2 = (C_0 + C_1 x + \cdots + C_n x^n + \cdots)(C_0 + C_1 x + \cdots + C_n x^n + \cdots)$$

$$= C_0 C_0 + (C_0 C_1 + C_1 C_0)x + \cdots + (C_0 C_n + \cdots + C_n C_0)x^n + \cdots$$

$$= C_1 + C_2 x + \cdots + C_{n+1} x^n + \cdots$$

$$u - xu^2 = C_0 = 1 \;\Rightarrow\; xu^2 - u + 1 = 0 \;\Rightarrow\; \boxed{u(x) = \frac{1-\sqrt{1-4x}}{2x}}$$

$${}_a\mathrm{C}_k := \frac{a(a-1)\cdots(a-k+1)}{k!} \qquad (k = 0, 1, \ldots)$$

$$(1-4x)^{\frac{1}{2}} = 1 + {}_{\frac{1}{2}}\mathrm{C}_1(-4x) + \cdots + {}_{\frac{1}{2}}\mathrm{C}_{n+1}(-4x)^{n+1} + \cdots$$

$$= 1 + \frac{\frac{1}{2}}{1!}(-4x) + \cdots + \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n)}{(n+1)!}(-4x)^{n+1} + \cdots$$

$$= 1 - \cdots - \frac{1\cdot 3\cdots(2n-1)}{(n+1)!}2^{n+1}x^{n+1} - \cdots$$

$$= 1 - \cdots - \frac{2(2n)!}{(n+1)!n!}x^{n+1} - \cdots$$

$$\boxed{C_n = \frac{(2n)!}{(n+1)!n!} = \frac{{}_{2n}\mathrm{C}_n}{n+1}} \qquad \frac{C_n}{C_{n-1}} = \frac{(2n)(2n-1)}{(n+1)n} = \frac{4n-2}{n+1} \xrightarrow{n\to\infty} 4$$

n	0	1	2	3	4	5	6	7	8	9	10	11
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786

§ Random walk

Starting from the origin, one moves 1 unit in the positive or negative direction on the number line every second

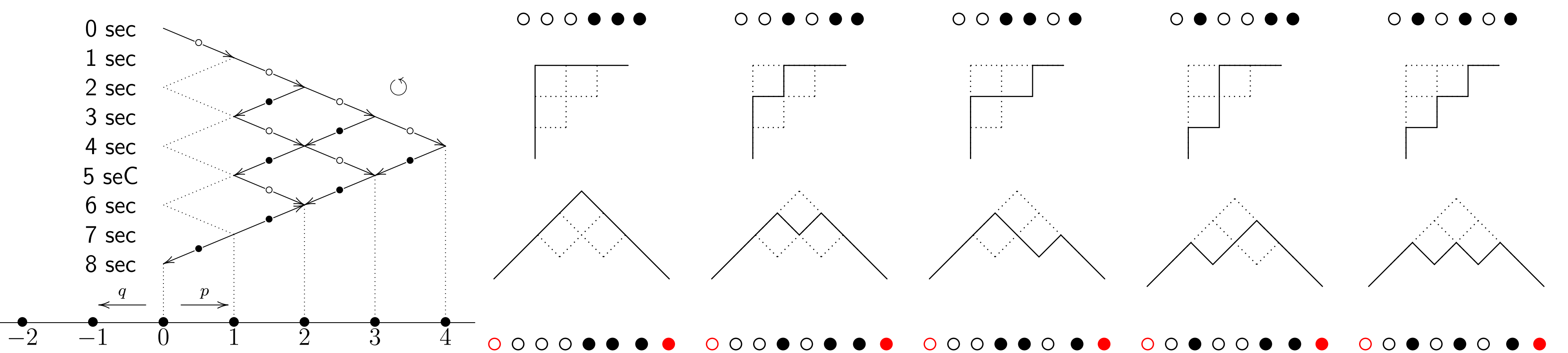
After 2 second : $(+1{:}\circ,\ -1{:}\bullet)$: ○○, ○●, ●○, ●● : 4 cases

positions : $2, 0, 0, -2$ (○● $\Rightarrow\ 0 \xrightarrow{+1} 1 \xrightarrow{-1} 0$)

In 2 cases out of 4 cases, it returns to the origin (50%)

Cases that it returns to the origin after 8 second for the first time

Moreover Cases whose first step is $+1$



$2 \times C_3 = 10$ cases out of $2^8 = 256$ cases ($\sim$ 4%)

It returns after $2n$ sec for the first time : $2C_{n-1}$ cases out of 2^{2n} cases

Assume probability p for $+1$ and q for -1 at every second ($p+q$=1)

Probability returning to the origin just after $2n$ sec for the first time

$$= 2C_{n-1}p^n q^n$$

Probability returning to the origin up to $2N$ sec later at least once

$$P(N) = \sum_{n=1}^{N} 2C_{n-1}(pq)^n$$

$$\lim_{N\to\infty} P(N) = \sum_{n=1}^{\infty} 2C_{n-1}(pq)^n = 2pq\sum_{m=0}^{\infty} C_m(pq)^m$$

$$= 2pq\frac{1-\sqrt{1-4pq}}{2pq} = 1-\sqrt{(p+q)^2-4pq} = 1-|p-q|$$

Probability $P(N)$

$p\backslash 2N$	2	4	6	8	10	20	100	1000	10000	∞
$\frac{1}{2}$	0.5	0.625	0.687	0.726	0.753	0.823	0.920	0.974	0.992	1
$\frac{1}{3}$	0.444	0.543	0.587	0.611	0.626	0.655	0.666	0.666	0.666	$\frac{2}{3}$
$\frac{1}{4}$	0.375	0.445	0.471	0.484	0.490	0.498	0.499	0.499	0.499	0.5

Thank you for your attention !

- https://www.ms.u-tokyo.ac.jp/~oshima

  os_muldif.rr : a library of computer algebra Risa/Asir

  Educational materials: Japanese Theorem (in Japanese)
- Flip of triangulations of incircles:
  https://s-takato.github.io/ketcindysample/misc/offline/jpntheoremv3jsoffL.html
- Table of triangulations of regular polygons
- Single-elimination tournaments
- http://www.wasan.jp/english/index.html Japanese Temple Geometric Problem