Middle convolution of KZ-type equations
(Knizhnik–Zamolodchikov type equations)
and
Single-elimination tournaments

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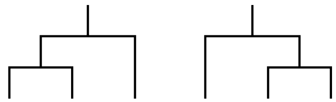
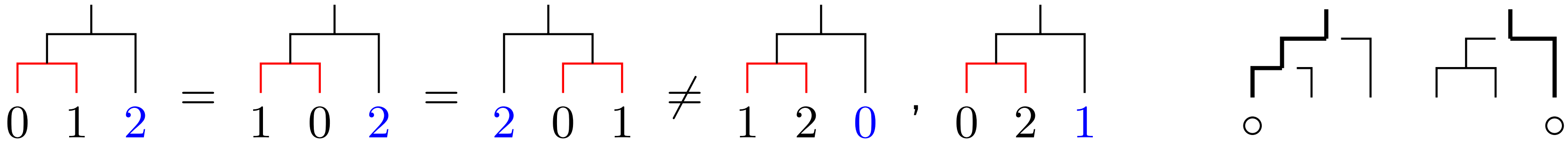
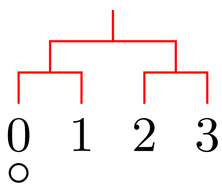
3 teams

2 patterns : 



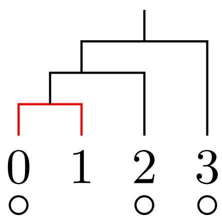
1 type, 2 patterns, 3 (cases of) tournaments, 2 win types

4 teams



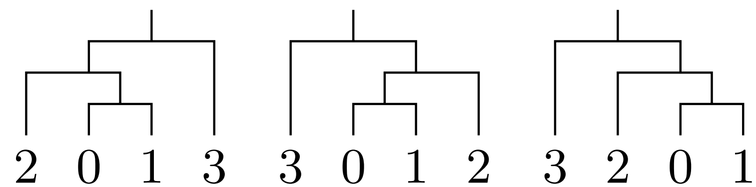
: 1 pattern

: $\frac{4!}{2^3} = 3$ cases



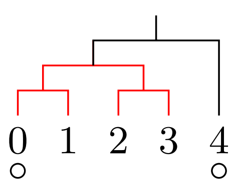
: 4 patterns

: $\frac{4!}{2} = 12$ cases



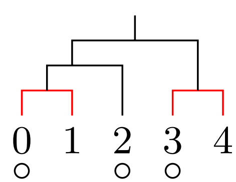
2 types, 5 patterns, 15 tournaments, 4 win types

5 teams



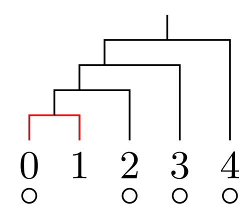
: 2 patterns

: $\frac{5!}{2^3} = 15$ cases,



: 4 patterns

: $\frac{5!}{2^2} = 30$ cases,



: 8 patterns

: $\frac{5!}{2} = 60$ cases

3 types, 14 patterns, 105 tournaments, 9 win types

Numbers of single-elimination tournaments

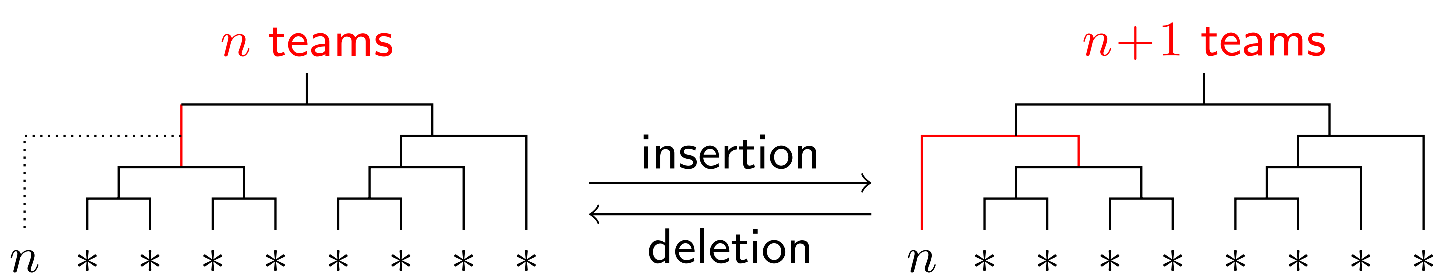
teams	2	3	4	5	6	7	8	9	10	n
patterns	1	2	5	14	42	132	429	1430	4862	$T_n = \frac{(2n-2)!}{n!(n-1)!}$
win types	1	2	4	9	20	46	106	248	582	W_n
types	1	1	2	3	6	11	23	46	98	U_n
tournaments	1	3	15	105	945	10395	135135	2027025	34459425	$K_n = (2n-3)!!$

$$T_n = \sum_{k=1}^{n-1} T_k \cdot T_{n-k}, \quad T_1 = 1, \qquad \text{(patterns)}$$

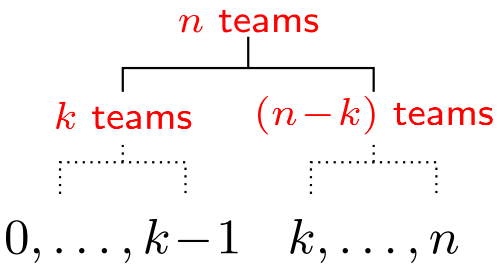
$$W_n = \sum_{1 \le k \le n-1} W_k \cdot U_{n-k}, \quad W_1 = 1, \qquad \text{(win types)}$$

$$U_n = \frac{1}{2}\Bigg(\sum_{1 \le k \le n-1} U_k \cdot U_{n-k} \ \left(+ U_{\frac{n}{2}} \text{ if } n \text{ is even}\right) \Bigg), \quad U_1 = 1, \qquad \text{(types)}$$

$$K_n = \frac{1}{2} \sum_{k=1}^{n-1} {}_nC_k \cdot K_k \cdot K_{n-k}, \quad K_1 = 1. \qquad \text{(tournaments)}$$



$2n-1$ vertical line segments $\Rightarrow K_{n+1} = (2n-1)K_n \Rightarrow K_n = (2n-3)!!$.



$$1 \le k < n \ \Rightarrow \begin{cases} T_n \leftarrow T_k \cdot T_{n-k} \\ W_k \Leftarrow W_k \cdot U_{n-k} \end{cases}$$

$$k < n-k \ \Rightarrow \begin{cases} U_n \leftarrow U_k \cdot U_{n-k} \\ K_n \leftarrow {}_nC_k \cdot K_k \cdot K_{n-k} \end{cases}$$

$$k = n-k \ \Rightarrow \begin{cases} U_n \leftarrow \frac{1}{2}U_k(U_k - 1) + U_k, \\ K_n \leftarrow {}_nC_k\left(\frac{1}{2}K_k(K_k-1)\right) + \left(\frac{1}{2}{}_nC_k\right)K_k \end{cases}$$

Identities

$$K_n = (2n-3)!! = \frac{1}{2}\sum_{k=1}^{n-1} {}_nC_k \cdot (2k-1)!! \cdot (2n-2k-1)!!$$

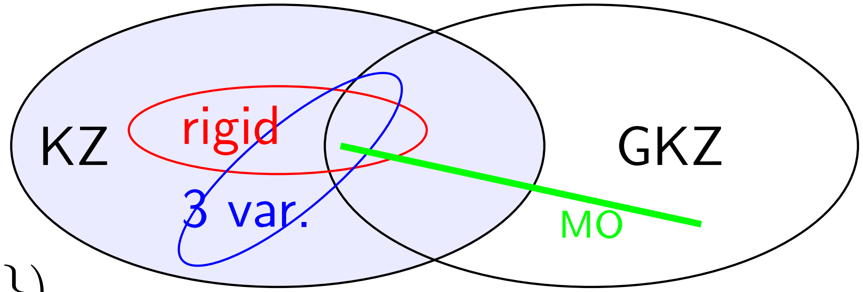
$$T_n = C_{n-1} = \frac{(2n-2)!}{(n-1)!n!} = \sum_{k=1}^{n-k} \frac{(2k-2)!}{(k-1)!k!} \frac{(2(n-k)-2)!}{(n-k-1)!(n-k)!}$$

$$1 = \left(1 - \sum_{k=1}^{\infty} U_n x^k\right)\left(\sum_{\ell=0}^{\infty} W_{\ell+1} x^\ell\right),$$

C_n : Catalan number

U_n : Wedderburn-Etherington number

KZ type system



$$\mathcal{M}: \frac{\partial u}{\partial x_i} = \sum_{\substack{0\le\nu\le n-1\\ \nu\ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u \qquad (i \in L_n := \{0,\dots,n-1\})$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \ A_{ij} = A_{ji} \in M(N,\mathbb{C}), \ A_{ii} = 0$$

with integrability condition $[A_{ij}, A_{k\ell}] = [A_{ij}, A_{ik} + A_{jk}] = 0$

KZ system $\hookleftarrow$ rigid Fuchsian ODE (Haraoka) $\frac{du}{dx} = \frac{A_{01}}{x-x_1}u + \cdots + \frac{A_{0n-1}}{x-x_{n-1}}u$

Fuchian ODE with 3 singular points on $\mathbb{P}^1$

middle convolutions and additions $\to$ accessory parameters and local structues

KZ system : $\mathrm{Sp}\,\mathcal{M} \leftarrow$ ODE: spectral type / (generalized) Riemann scheme

$$A_{i\infty} := -(A_{i0} + A_{i1} + \cdots + A_{in-1}) \qquad (i \in L_n)$$

$$\widetilde{L}_n := L_n \cup \{\infty\}, \quad A_i = A_\emptyset = 0$$

$$A_{i_1\cdots i_k} := \sum_{1\le p<q\le k} A_{i_p i_q} \quad (\{i_1,\dots,i_k\} \subset \widetilde{L}_n)$$ (generalized) residue matrices

$[A_I, A_J] = 0 \quad (I \subset J$ or $I \supset J$ or $I \cap J = \emptyset,\ I, J \subset \widetilde{L}_n)$

$[A_I, A_{L_n}] = 0 \ \Rightarrow\ A_{L_n} = \kappa$: scalar if $\mathcal{M}$ is irreducible ($\kappa = 0$: homogeneous)

- $[A,B]=0 \;\Rightarrow\; \exists$ Simultaneous (generalized) eigenvalue class :

$$A=\begin{pmatrix}0&&&\\&0&*&\\&&0&\\&&&4\end{pmatrix},\qquad B=\begin{pmatrix}1&&&\\&2&&\\&&2&\\&&&3\end{pmatrix}$$

$$\Rightarrow\;[A:B]=\{[0:1]_1,[0:2]_2,[4:3]_1\}=\{[0:1],[0:1]_2,[4:3]\}$$

- $[A_i,A_j]=0\;\;(i,j=1,..,r)\;\Rightarrow\;[A_1:\cdots:A_r]=\{[\lambda_{1,1}:\cdots:\lambda_{r,1}]_{m_1},\cdots\}$

Def. $\mathcal{I}=\{I_\nu \mid \nu=1,\dots,r\}$ is a commuting family of subsets of a set L

$\overset{\text{def.}}{\Longleftrightarrow}\;\; I_\nu\subset L,\;\; |I_\nu|>1$ and $(I_\nu\subset I_{\nu'}$ or $I_\nu\supset I_{\nu'}$ or $I_\nu\cap I_{\nu'}=\emptyset)$

$\mathcal{I}$ is maximal $\overset{\text{def.}}{\Longleftrightarrow}$ $(\mathcal{I},\mathcal{I}'$ are these families with $\mathcal{I}\subset\mathcal{I}'\;\Rightarrow\mathcal{I}=\mathcal{I}')$

Def. $\mathcal{L}_n :=$ {maximal commuting families of subsets of L_n}

$$\mathrm{Sp}\,\mathcal{M}\;:=\Big\{[A_{I_1}:\cdots:A_{I_{n-1}}]\;|\;\mathcal{I}=\{I_1,\dots,I_{n-1}\}\Big\}_{\mathcal{I}\in\mathcal{L}_n}$$

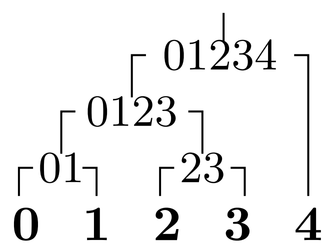
$$\mathrm{Sp}'\,\mathcal{M}\;:=\Big\{[A_{I_1}:\cdots:A_{I_{n-2}}]\;|\;\mathcal{I}=\{I_1,\dots,I_{n-2},L_n\}\Big\}_{\mathcal{I}\in\mathcal{L}_n}$$

Fact. $\mathcal{L}_n\simeq$ {tournaments of teams labeled by L_n}

$|\mathcal{L}_n|=(2n-3)!!\qquad |\mathcal{I}|=n-1\;\;(\mathcal{I}\in\mathcal{L}_n)$

Example. $L=\{0,1,2,3,4\}$

$\mathcal{I}=\big\{\{0,1\},\{0,1,2,3\},\{2,3\},\{0,1,2,3,4\}\big\}$



$\big\{[A_{ij}:A_{k\ell}:A_{ijk\ell}],\,[A_{ij}:A_{ijk}:A_{\ell m}],\,[A_{ij}:A_{ijk}:A_{ijk\ell}]\;|\;\{i,j,k,\ell,m\}=L_5\big\}$

Middle convolution with respect to x_0

The convolution $\widetilde{\mathrm{mc}}_{x_0,\mu}\mathcal{M}$ of $\mathcal{M}$ with $\mu \in \mathbb{C}$ is given by

$$\tilde{\mathcal{M}} : \frac{\partial \tilde{u}}{\partial x_i} = \sum_{0\le\nu<n} \frac{\tilde{A}_{i\nu}}{x_i - x_\nu}\tilde{u} \quad (0 \le i < n)$$

$$\tilde{A}_{0k} = k \overset{k}{\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{01} & \cdots & A_{0k}+\mu & \cdots & A_{0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}} \in M((n-1)N, \mathbb{C})$$

(Dettwiler-Reiter ⇐ Katz),

$$\tilde{A}_{ij} = \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} \overset{\qquad i \qquad\qquad j}{\begin{pmatrix} A_{ij} & & & & & & \\ & \ddots & & & & & \\ & & A_{ij}+A_{0j} & & -A_{0j} & & \\ & & & \ddots & & & \\ & & -A_{0i} & & A_{ij}+A_{0i} & & \\ & & & & & \ddots & \\ & & & & & & A_{ij} \end{pmatrix}} \in M((n-1)N, \mathbb{C})$$

(Haraoka)

$$\iota_j(v) := (v)_j := j\begin{pmatrix}0\\ \vdots\\ v\\ \vdots\\ 0\end{pmatrix} \in \mathbb{C}^{(n-1)N} \quad (v\in\mathbb{C}^N), \quad \iota_j : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N}$$

$$\iota_I := \sum_{i\in I}\iota_i : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N}, \ (v)_I := \iota_I(v) \in \mathbb{C}^{(n-1)N} \quad (v\in\mathbb{C}^N,\ I\subset L_n^0)$$

$$\tilde{\mathcal{K}}_j := \iota_j(\mathrm{Ker}\,A_{0j}) = j\begin{pmatrix}0\\ \vdots\\ \mathrm{Ker}\,A_{0j}\\ \vdots\\ 0\end{pmatrix} \subset \mathbb{C}^{(n-1)N}, \quad L_n^0 := \{1,\dots,n-1\}$$

$$\tilde{\mathcal{K}}_\infty := \ker \tilde{A}_{0,\infty} \overset{\mu\neq 0}{=} \iota_{L_n}\big(\mathrm{Ker}\,(A_{0\infty}-\mu)\big) = \left\{\begin{pmatrix}v\\ \vdots\\ v\end{pmatrix} \mid A_{0\infty}v = \mu v\right\} \subset \mathbb{C}^{(n-1)N}$$

$$\tilde{\mathcal{K}} := \tilde{\mathcal{K}}_\infty + \bigoplus_{j=1}^{n}\tilde{\mathcal{K}}_j \quad (\text{direct sum} \ \Leftarrow\ \mu\neq 0) \ \Rightarrow\ \tilde{A}_I\text{-invariant} \quad (I\subset L_n)$$

$$\bar{A}_I := \tilde{A}_I|_{\mathbb{C}^{(n-1)N}/\tilde{\mathcal{K}}} \in M\big((n-1)N - \dim\tilde{\mathcal{K}}, \mathbb{C}\big)$$

middle convolution $\overline{\mathcal{M}} = \mathrm{mc}_{x_0,\mu}\,\mathcal{M} : \dfrac{\partial \bar{u}}{\partial x_i} = \displaystyle\sum_{0\le\nu<n}\frac{\bar{A}_{i\nu}}{x_i - x_\nu}\bar{u}$

$$\mathrm{mc}_{x_0,-\mu}\circ\mathrm{mc}_{x_0,\mu} = \mathrm{id} \quad (\Leftarrow\ \partial^{\mu}\circ\partial^{-\mu})$$

Fact. $A_{L_n} = \kappa,\ I\subset L_n \Rightarrow\ \bar{A}_I = \bar{A}_{\tilde{L}_n\setminus I} + \kappa + \mu \quad (\kappa = 0)$

Def. For $i \in L_n$ and $I,\, J \subset L_n$

$$\mathrm{md}_{i,J}(I) := \begin{cases} I \cup \{i\} & (I \supset J) \\ I \setminus \{i\} & (I \not\supset J) \end{cases} \qquad \mathrm{me}_{i,J}(I) := \begin{cases} 1 & (i \in I \supset J) \\ 0 & (i \notin I \text{ or } I \not\subset J) \end{cases}$$

$$i \in I \supset J \text{ or } i \notin I \not\supset J \;\Rightarrow \mathrm{md}_{i,J}(I) = I$$

Fact. $\mathrm{md}_{i,J} : \{\text{commuting families}\} \to \{\text{commuting families}\}$

Theorem. i) For $\mathcal{I} = \{I^{(1)}, \ldots, I^{(n-1)}\} \in \mathcal{L}_n$ and $I \in \mathcal{I}$

$$[\tilde{A}_I] = [A_{I\cup\{0\}} + \mu]_{|I|-1} \cup [A_{I\setminus\{0\}}]_{n-|I|}$$

$$[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}] = \bigcup_{J\in\mathcal{I}} [A^J_{I^{(1)}} : \cdots : A^J_{I^{(n-1)}}]$$

$$[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_j} = [A^{\{j\}}_{I^{(1)}} : \cdots : A^{\{j\}}_{I^{(n-1)}}]|_{\ker A_{0j}} \quad (j \in L_n^0),$$

$$[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_\infty} = [A^{L_n}_{I^{(1)}} : \cdots : A^{L_n}_{I^{(n-1)}}]|_{\ker(A_{0\infty}-\mu)} \quad (A_{0\infty} = A_{L_n^0} - A_{L_n})$$

$$A^K_I := A_{\mathrm{md}_{0,K}(I)} + \mathrm{me}_{0,K}(I)\cdot\mu$$

ii) $U^{-1}_{b^0(\mathcal{I})} A_{I^{(i)}} U_{b^0(\mathcal{I})}$ are block upper traiangular matrices

We may assume that $\mathcal{I} = \{I^{(1)}, \ldots, I^{(n-1)}\}$ satisfies

$\mathcal{I}_0 := \{I \mid 0 \in I \in \mathcal{I}\} = \{I^{(k_1)}, \ldots, I^{(k_m)}\}$

1. $k_p < k_q \quad \Leftrightarrow \quad I^{(k_p)} \subsetneq I^{(k_q)}$

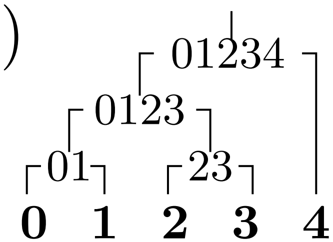
2. $k_p < i \leq j < k_{p+1} \quad \Leftarrow \quad I^{(k_p)} \setminus I^{(k_{p-1})} \supset I^{(i)} \supset I^{(j)} \quad (I^{(k_0)} := \emptyset)$

Def. For $i \in L_n$ and $I \in \mathcal{I}$

$b^i(I) \in \widetilde{\mathcal{I}} := \mathcal{I} \cup \bigcup_{\nu \in L_n} \{\{\nu\}\}$ so that $i \notin b^i(I) \subsetneq I$ and $I \setminus b^i(I) \in \widetilde{\mathcal{I}}$

$$GL\big((n-1)N, \mathbb{C}\big) \ni \big(U_{b^0(\mathcal{I})}\big)_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq n-1}} := \begin{cases} \mathbf{1}_N & \big(i \in b^0(I^{(j)})\big) \\ 0 & \big(i \notin b^0(I^{(j)})\big) \end{cases}$$

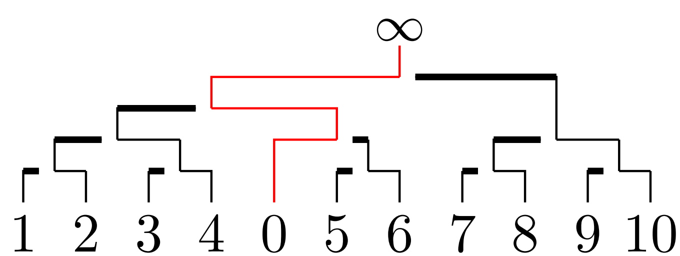
例. **Example**. $L = \{0, 1, 2, 3, 4\}$



$\mathcal{I} = \big\{\{0, 1\}, \{0, 1, 2, 3\}, \{2, 3\}, \{0, 1, 2, 3, 4\}\big\}$

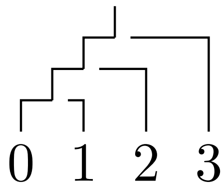
$b^0(\{0, 1\}) = \{1\}$, $b^0(\{0, 1, 2, 3\}) = \{2, 3\}$, $b^0(\{2, 3\}) = \{2\}$, $b^0(L_5) = \{4\}$

$L = \{0, 1, \ldots, 10\}$:



$\mathcal{I}$: $\{0, 5, 6\}, \{5, 6\}, \{0, \ldots, 6\}, \{1, 2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{0, \ldots, 10\}, \{7, 8, 9, 10\}, \{7.8\}, \{9, 10\}$

b^0 : $\{5.6\}, \quad \{5\}, \{1, 2, 3, 4\}, \quad \{1, 2\}, \quad \{1\}, \quad \{3\}, \{7, 8, 9, 10\}, \quad \{7, 8\}, \quad \{7\}, \quad \{9\}$

1. 

$$\mathcal{I} = \{\{0,1\},\{0,1,2\},\{0,1,2,3\}\} \xrightarrow{b^0} \{\{1\},\{2\},\{3\}\}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{01} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{012} = \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix}$$

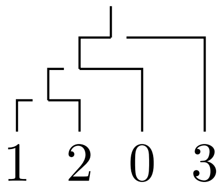
$$[\tilde{A}_{01} : \tilde{A}_{012}] = \{[A_{01}+\mu : A_{012}+\mu], [0 : A_{012}+\mu], [0 : A_{12}]\}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_1} = [A_{01}+\mu : A_{012}+\mu]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_2} = [0 : A_{012}+\mu]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_3} = [0 : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_\infty} = [0 : A_{12}]|_{\ker (A_{0\infty}-\mu)}$$

2. 

$$\mathcal{I} = \{\{0,1,2\},\{1,2\},\{0,1,2,3\}\} \xrightarrow{b^0} \{\{1,2\},\{1\},\{3\}\}$$

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{012} = \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & 0 \\ 0 & 0 & A_{12} \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} A_{012}-A_{01} & -A_{02} & 0 \\ -A_{01} & A_{012}-A_{02} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{12} & -A_{01} & 0 \\ 0 & A_{012} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}$$

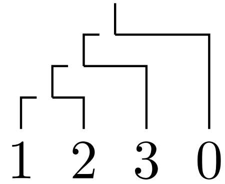
$$[\tilde{A}_{012} : \tilde{A}_{12}] = \{[A_{012}+\mu : A_{12}], [A_{012}+\mu : A_{012}], [A_{12} : A_{12}]\}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_1} = [A_{012}+\mu : A_{012}]|_{\ker A_{01}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_2} = [A_{012}+\mu : A_{012}]|_{\ker A_{02}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_3} = [A_{12} : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_\infty} = [A_{12} : A_{12}]|_{\ker (A_{0\infty}-\mu)}$$

3. 

$$\mathcal{I} = \{\{0,1,2,3\},\{1,2,3\},\{1,2\}\} \xrightarrow{b^0} \{\{1,2,3\},\{1,2\},\{1\}\}$$

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_* U$$

$$\tilde{A}_{123} = \begin{pmatrix} -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \end{pmatrix} \to \begin{pmatrix} A_{123} & -A_{01}-A_{02} & -A_{01} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} A_{012}-A_{01} & -A_{02} & 0 \\ -A_{01} & A_{012}-A_{02} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{12} & 0 & 0 \\ 0 & A_{12} & -A_{01} \\ 0 & 0 & A_{012} \end{pmatrix}$$

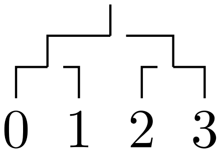
$$[\tilde{A}_{123} : \tilde{A}_{12}] = \{[A_{123} : A_{12}], [0 : A_{12}], [0 : A_{012}]\}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_1} = [0 : A_{012}]|_{\ker A_{01}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_2} = [0 : A_{012}]|_{\ker A_{02}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_3} = [0 : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_\infty} = [A_{123} : A_{12}]|_{\ker (A_{0\infty}-\mu)}$$

4. 

$$\mathcal{I} = \big\{\{0,1\},\{0,1,2,3\},\{2,3\}\big\} \xrightarrow{b^0} \big\{\{1\},\{2,3\},\{2\}\big\}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_* U$$

$$\tilde{A}_{01} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} A_{012}+\mu & A_{02}+A_{03} & A_{02} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{023}-A_{02} & -A_{03} \\ 0 & -A_{02} & A_{023}-A_{03} \end{pmatrix} \to \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23} & -A_{02} \\ 0 & 0 & A_{023} \end{pmatrix}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}] = \{[A_{01}+\mu : A_{23}], [0 : A_{23}], [0 : A_{023}]\}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_1} = [A_{01} + \mu : A_{23}]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_2} = [0 : A_{023}]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_3} = [0 : A_{023}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_\infty} = [0 : A_{23}]|_{\ker (A_{0\infty}-\mu)}$$

Case 1

$J\backslash\widetilde{A}$	$\widetilde{01}$	$\widetilde{012}$
01	01+μ	012+μ
012	0	012+μ
0123	0	12
1	01+μ	012+μ
2	0	012+μ
3	0	12
∞	0	12

Case 2

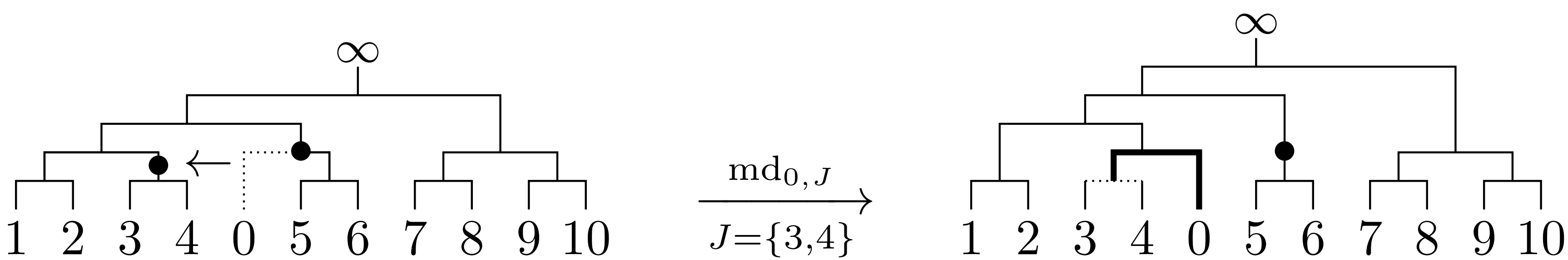
$J\backslash\widetilde{A}$	$\widetilde{012}$	$\widetilde{12}$
012	012+μ	12
12	012+μ	012
0123	12	12
1	012+μ	012
2	012+μ	012
3	12	12
∞	12	12

Case 3

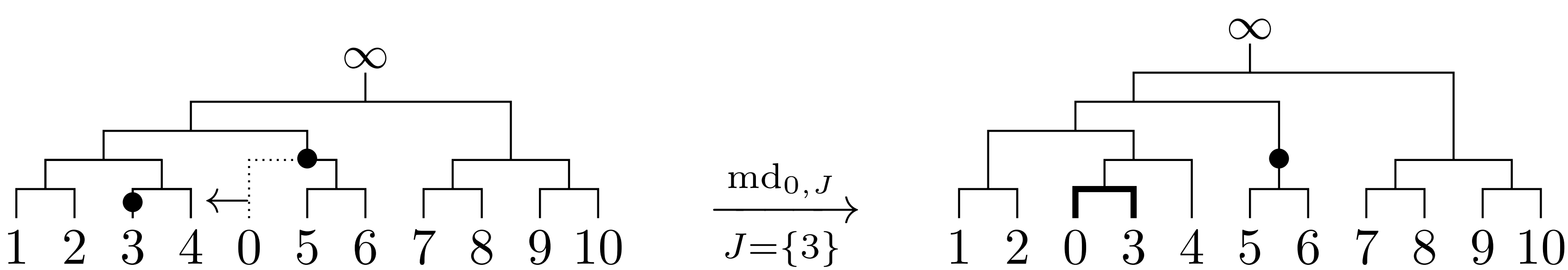
$J\backslash\widetilde{A}$	$\widetilde{123}$	$\widetilde{12}$
0123	123	12
123	0	12
12	0	012
1	0	012
2	0	012
3	0	12
∞	123	12

Case 4

$J\backslash\widetilde{A}$	$\widetilde{01}$	$\widetilde{23}$
01	01+μ	23
0123	0	23
23	0	023
1	01+μ	23
2	0	023
3	0	023
∞	0	23



$$J = \{3,4\} : \{3,4\} \to \{0,3,4\},\ \{1,2,3,4\} \to \{0,1,2,3,4\},\ \{0,5,6\} \to \{5,6\}$$



$$j = 3 : \{3,4\} \to \{0,3,4\},\ \{1,2,3,4\} \to \{0,1,2,3,4\},\ \{0,5,6\} \to \{5,6\},\ +\{0,3\}$$

$$\begin{pmatrix} A_{12}+A_{02} & -A_{02} & 0 \\ -A_{01} & A_{12}+A_{01} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} + \begin{pmatrix} A_{13}+A_{03} & 0 & -A_{03} \\ 0 & A_{13} & 0 \\ -A_{01} & 0 & A_{13}+A_{01} \end{pmatrix} + \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23}+A_{03} & -A_{03} \\ 0 & -A_{02} & A_{23}+A_{02} \end{pmatrix}$$

$$= \begin{pmatrix} A_{0123}-A_{01} & -A_{02} & -A_{03} \\ -A_{01} & A_{0123}-A_{02} & -A_{03} \\ -A_{01} & -A_{02} & A_{0123}-A_{03} \end{pmatrix} \qquad (\Leftarrow A_{12} + A_{13} + A_{14} = A_{123})$$

$$\tilde{A}_{1\cdots k} = \begin{pmatrix} A_{0\cdots k}-A_{01} & \cdots & -A_{0k} & & & \\ \vdots & \cdots & \vdots & & & \\ -A_{01} & \cdots & A_{0\cdots k}-A_{0k} & & & \\ & & & A_{1\cdots k} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots k} \end{pmatrix}$$

$$I \subset L_n \ (= \{1, \ldots, n-1\}), \quad i \in I, \quad j \in L_n \setminus I$$

$$\tilde{A}_I(v)_J = (A_I v)_J \ \text{ in } \ V_J := \iota_J(\mathbb{C}^N) \qquad (I \subset \forall J \subset L_n)$$

$$\tilde{A}_I(v)_i = (A_{0I} v)_i - (A_{0i} v)_I \equiv (A_{0I} v)_i \mod V_I \qquad (i \in I)$$

$$\tilde{A}_I(v)_j = (A_I v)_j \qquad (j \notin I)$$

$$A_I(v)_{L_n} = (A_I v)_{L_n}$$

$$[\tilde{A}_I] = [A_I]_{n-|I|} \cup [A_{0I}]_{|I|-1}$$

$$\tilde{A}_I(v)_i = (A_{0I} v)_i \qquad (v \in \ker A_{0i})$$

$$\tilde{A}_{0\cdots k-1} = \begin{pmatrix} A_{01\cdots k-1}+\mu & & & A_{0k} & \cdots & A_{0n-1} \\ & \ddots & & \vdots & \cdots & \vdots \\ & & A_{01\cdots k-1}+\mu & A_{0k} & \cdots & A_{0n-1} \\ & & & A_{1\cdots k} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots k} \end{pmatrix}$$

$$I \subset L_n \quad (I = \{1, \ldots k-1\} \uparrow), \quad i \in I, \quad j \in L_n \setminus I$$

$$\tilde{A}_{0I}(v)_i = ((A_{0I}+\mu)v)_i$$

$$\tilde{A}_{0I}(v)_j = (A_{0j}v)_I + (A_I v)_j \equiv (A_I v)_j \mod V_I$$

$$(A_{0I}+\mu)v + \sum_{\nu \in L_n \setminus I} A_{0\nu} v = A_I v + \Big(\underbrace{\sum_{\nu=1}^{n-1} A_{0\nu}}_{: -A_{0\infty}} + \mu\Big) v$$

$$\equiv A_I v \mod \ker(A_{0\infty} - \mu)$$

$$[\tilde{A}_{0I}] = [A_I]_{n-1-|I|} + [A_{0I}+\mu]_{|I|}$$

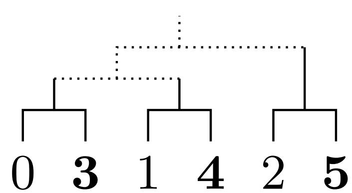
KZ-type equations vs single-elimination Tournaments

KZ-type equation with $n$variables	Tournament of n teams
Family of maximal commuting residue matrices	Tournament
Spectra of KZ-type equation	Set of all tournaments
Variable of middle convolution	Winner of tournament
Base of upper triangulation of the family	Result of all games
Middle convolution	Deletion and Insertion of the winner
Kernels to define middle convolution	Basic/Top insertion of the winner
With other m fixed singular points	Divide n teams into m groups

Example : variables x_0, x_1, x_2, fixed singular points y_3, y_4, y_5 $\Rightarrow$ 105 families

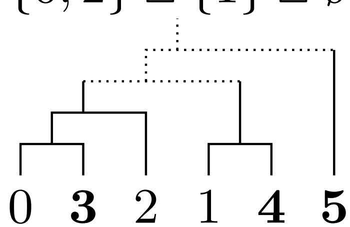
$$\frac{\partial u}{\partial x_i} = \sum_{\substack{0 \le \nu \le 2 \\ \nu \neq i}} \frac{A_{i\nu}}{x_i - x_\nu} u + \sum_{\nu=3}^{5} \frac{A_{i\nu}}{x_i - y_\nu} u \quad (i = 0, 1, 2)$$

$\overset{3}{\{0\}} \sqcup \overset{4}{\{1\}} \sqcup \overset{5}{\{2\}}$



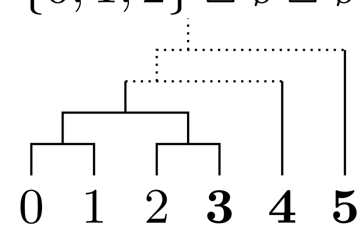
$\{\{0,3\},\{1,4\},\{2,5\}\}$

$\{0,\overset{3}{2}\} \sqcup \overset{4}{\{1\}} \sqcup \overset{5}{\emptyset}$



$\{\{0,3\},\{1,4\},\{0,2,3\}\}$

$\{0,\overset{3}{1},2\} \sqcup \overset{4}{\emptyset} \sqcup \overset{5}{\emptyset}$



$\{\{0,1\},\{2,3\},\{0,1,2,3\}\}$

Thank you for your attention!

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