MIDDLE CONVOLUTIONS OF KZ-TYPE EQUATIONS AND SINGLE-ELIMINATION TOURNAMENTS

TOSHIO OSHIMA

Abstract. We introduce an extension of generalized Riemann scheme of Fuchsian ordinary differential equations in the case of KZ-type equations, which describes the local structure of the equations obtained by resolution of singularities of KZ-type equations. We give the transformation of the extension under middle convolutions. Then we get the corresponding transformation of eigenvalues and their multiplicities of the residue matrices of KZ-type equations under middle convolutions. The result is interpreted by combinatorics of single-eliminated tournaments.

1. Introduction

A Fuchsian differential equation

$$\mathcal{N} : \frac{du}{dx} = \sum_{\nu=1}^{n-1} \frac{A_{0\nu}}{x - x_\nu} u,$$

where u is a column vector of N unknown functions, has singularities at n points $x_1, \dots, x_{n-1}, \infty \in \mathbb{C} \cup \{\infty\}$. The local behavior of a solution of $\mathcal{N}$ in a complex neighborhood of a singular point x_ν is characterized by the conjugacy class of the residue matrix $A_{0\nu} \in M(N, \mathbb{C})$, namely, by the eigenvalues of $A_{0\nu}$ and their multiplicities. The residue matrix at infinity is given by $A_{0\infty} = -(A_{01} + \cdots + A_{0,n-1})$. The collection of the eigenvalues and their multiplicities of the residue matrices is called the (generalized) *Riemann scheme* of $\mathcal{N}$. The equation $\mathcal{N}$ is said to be *rigid* if it is irreducible and completely determined by its Riemann scheme; otherwise, $\mathcal{N}$ depends on a finite number of *accessory parameters*.

Katz [4] showed that any rigid Fuchsian differential equation $\mathcal{N}$ can be reduced to and constructed from the trivial equation $u' = 0$ by a sequence of invertible operations called *additions* and *middle convolutions*. Haraoka [2] further proved that if an ordinary differential equation $\mathcal{N}$ is rigid, it can be extended to a *Knizhnik–Zamolodchikov type (KZ-type) equation*

$$\mathcal{M} : \frac{\partial u}{\partial x_i} = \sum_{\substack{0 \le \nu < n \\ \nu \ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u \qquad (i = 0, \dots, n-1)$$

by setting $x_0 = x$ and treating the singular points $x_1, \dots, x_{n-1}$ as new variables. This extension is obtained by generalizing the middle convolution to KZ-type equations. Dettweiler and Reiter [1] gave an explicit formula for the middle convolution in terms of the residue matrices $A_{0\nu}$, following Katz's definition [4]; Haraoka's result provides an extension of this construction.

These transformations, additions and middle convolutions together with permutations of the variables $x_0, \dots, x_{n-1}$, preserve the KZ-type form but not necessarily the rigidity. Indeed, even if the original KZ-type equation has a rigid variable, the transformed equation may have none.

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Many known and new multivariable hypergeometric functions appear as solutions of KZ-type equations obtained from the trivial equation through such transformations (see, e.g., [13, 5]). It is therefore useful to study the middle convolution for analyzing these equations. In the case of Fuchsian ordinary differential equations, most results in [7] are obtained through such analyses. Since the middle convolution acts as a microlocal transformation, one may expect a corresponding transformation of the fundamental data describing the singularities—such as the eigenvalues and their multiplicities of the residue matrices. For ordinary differential equations, these relations were derived in [1] and [7]. For KZ-type equations, however, the singularity structure is more complicated. This problem was solved in the case $n = 4$ by [10]. In this paper, we study the general case for arbitrary n (cf. [10, Remark 3]).

The local solution of $\mathcal{M}$ around a singularity $x_i = x_j$ is characterized by the eigenvalues of the residue matrix A_{ij} and their multiplicities. The local solution around a singular locus of the form $\{x_0 = x_1,\ x_2 = x_3\}$ is characterized by the conjugacy class of the pair (A_{01}, A_{23}). Here A_{01} commutes with A_{23} due to the integrability condition of $\mathcal{M}$. Hence, the simultaneous eigenvalues of A_{01} and A_{23}, together with their multiplicities, determine this conjugacy class.

At the singularity $\{x_0 = x_1 = x_2\}$, the eigenvalues and multiplicities of the matrix $A_{012} = A_{01} + A_{02} + A_{12}$ provide important data describing the local behavior of the solution. In general, for a subset $I \subset L_n := \{0, 1, \dots, n-1\}$ with $|I| > 1$, we define the *generalized residue matrix* A_I as the sum of all A_{ij} satisfying $\{i, j\} \subset I$. A *maximal commuting family* $\mathcal{I}$ of residue matrices is defined as a maximal subset of $\{A_I \mid I \subset L_n,\ |I| > 1\}$ whose elements commute with one another.

We denote by $\operatorname{Sp}\mathcal{M}$ the data consisting of the simultaneous eigenvalues and their multiplicities of the maximal commuting families of residue matrices of $\mathcal{M}$ (see Definition 3.8), and refer to $\operatorname{Sp}\mathcal{M}$ as the *spectra* of $\mathcal{M}$. There are $(2n-3)!!$ maximal commuting families of residue matrices (Proposition 3.9), and each family contains $n-1$ commuting matrices. These families correspond to the residue matrices associated with hypersurfaces defining normal crossing singularities in a suitable blow-up of a singular point (cf. §4).

The main result of this paper is that additions and middle convolutions of $\mathcal{M}$ induce explicit transformations of the spectra $\operatorname{Sp}\mathcal{M}$ (Theorem 5.1). For instance, in order to determine the eigenvalues and multiplicities of the residue matrix A_{12} for a equation obtained by repeatedly applying additions and middle convolutions to a given KZ-type equation $\mathcal{M}$, one needs to compute the transformation of spectra $\operatorname{Sp}\mathcal{M}$ (see Remark 5.12).

When the singularities of the equation satisfied by Appell's hypergeometric function F_1 are blown up, fifteen normal crossing singularities appear. The simplest KZ-type equation $\mathcal{M}$ is the rank 3 case with $n = 4$, corresponding to the equation satisfied by F_1. In this case, $\operatorname{Sp}\mathcal{M}$ consists of $(2n-3)!! = 15$ decompositions into simultaneous eigenspaces of the residue matrices, whose simultaneous eigenvalues (with free multiplicities) coincide with the characteristic exponents at the fifteen singular points. Thus, $\operatorname{Sp}\mathcal{M}$ can be regarded as an *extension of the Riemann scheme* of $\mathcal{N}$ to the KZ-type equation $\mathcal{M}$ (cf. Example 7.5).

Fuchsian ordinary differential equations $\mathcal{N}$ with three singular points naturally correspond to KZ-type equations $\mathcal{M}$ with $n = 3$ (cf. §7.3). Moreover, many hypergeometric functions of $n-2$ variables appear as solutions of KZ-type equations $\mathcal{M}$ with n variables (see [5], [10]). When $n = 4$, a residue matrix A_I with $|I| = 3$ can be expressed in terms of another residue matrix A_J with $|J| = 2$ (cf. Example 3.10 (i)), and $\operatorname{Sp}\mathcal{M}$ corresponds to the simultaneous eigenspace decompositions of pairs of

residue matrices associated with normal crossing singular points. The transformation of $\operatorname{Sp}\mathcal{M}$ under the middle convolution for $n = 4$ was described in [10]. For $n > 4$, the structure of $\operatorname{Sp}\mathcal{M}$ becomes more intricate, but the combinatorics of single-elimination tournaments provides a useful framework for understanding its organization (cf. § 7.1).

In §2, we review several known combinatorial results related to single-elimination tournaments, such as the enumeration of possible tournament scenarios, with emphasis on their application to the structure of the residue matrices of KZ-type equations. In particular, we introduce *maximal families of commuting subsets* of a finite set, which correspond to maximal commuting families of residue matrices. The numerical sequences depending on n that appear in this section can be found in the database [15].

In §3, after recalling the integrability condition for KZ-type equations, we introduce the notion of the spectra $\operatorname{Sp}\mathcal{M}$.

In §4, we construct a local coordinate system and define a resolution of the singular points of a KZ-type equation. We show that the maximal commuting family of residue matrices coincides with the set of residue matrices corresponding to the normal crossing divisors defining a singular point in the blow-up.

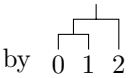
In §5, we study the transformation of residue matrices under the middle convolution for KZ-type equations, and we state our main result, Theorem 5.1. To formulate the theorem, we define transformations of maximal families of commuting subsets of a finite set, corresponding to transformations of tournament structures.

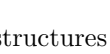
Theorem 5.1 can be applied to the analysis of multivariable hypergeometric functions. Examples for the case $n = 4$ are given in [10, §8], [13, §7], and [5, §5]. A solution of a rigid Fuchsian differential equation $\mathcal{N}$ with more than three singular points can be studied in depth via the KZ-type equation obtained from $\mathcal{N}$ by extending its variables. For example, the differential equation satisfied by Jordan–Pochhammer's hypergeometric function can be extended to a KZ-type equation satisfied by Appell's F_1 or Lauricella's F_D. Furthermore, the theorem can also be applied to KZ-type equations with irregular singularities through their versal unfoldings (cf. [12]). In §6, Theorem 5.1 is illustrated by examples for $n = 4$.

By applying a fractional linear transformation, we may assume that the KZ-type equation $\mathcal{M}$ has no singularity at infinity. In §7.2, we examine the middle convolution for this case. The accessory parameters and rigidity of the KZ-type equation are discussed in §7.3.

In §7.4, we consider KZ-type equations with fixed singular points. In this situation, the spectra $\operatorname{Sp}\mathcal{M}$ correspond to single-elimination tournaments with certain restrictions. When the equation involves only one variable, our results reduce to those of [1]. Finally, in §7.5, we study a residue matrix related to semilocal monodromy, as introduced in [11], for KZ-type equations

2. Single-elimination tournaments

We consider single-elimination tournaments involving n teams. In this note, the term tournament always refers to a single-elimination tournament. A tournament consisting of three teams labeled by 0, 1 and 2 is represented by 

. This diagram indicates that team 0 and team 1 play in the first round, and the winner of that game plays against team 2 in the final. If the teams are indistinguishable, there are two possible structures and  . For tournaments with n teams, each pattern corresponds to a binary rooted tree with $n-1$ internal nodes and n leaves embedded in the plane. The total number of such tournaments is given

by the Catalan number C_{n-1}. In this case, there are $n-1$ games and each game determines a final winner of some teams by a sub-tournament.

A tournament of three teams is determined by the pair of teams playing the first round. There are three possible tournaments, described by

$$\big\{\{0,1\},\{0,1,2\}\big\},\ \big\{\{1,2\},\{0,1,2\}\big\},\ \big\{\{0,2\},\{0,1,2\}\big\} \tag{2.1}$$

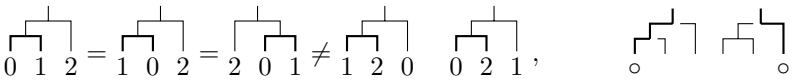
Each expression represents the set of games in a tournament, where each game is labeled by the set of participating teams, and the winner of the game determines the champion among them. Since the final game corresponds to the set of all teams, we may describe a tournament by listing all games except the final one. We call this representation the reduced form of the tournament. Namely the tournaments of three teams are

$$\big\{\{0,1\}\big\},\ \big\{\{1,2\}\big\},\ \big\{\{0,2\}\big\} \tag{2.2}$$

in reduced expression.

Two tournaments that can be transformed into each other by a permutation of teams are considered isomorphic. Each isomorphism class is called a *type* of tournament. The number of isomorphic classes of tournaments of n teams is the n-th Wedderburn-Etherington number and a tournament can be expressed by $n-1$ subsets or $n-2$ proper subsets of the set of teams.

Furthermore, we consider types of tournaments distinguished by specifying the final winner, which we refer to as *win-types*. There is one type of the tournaments of 3 teams and there are two win-types of them.

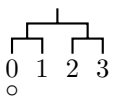


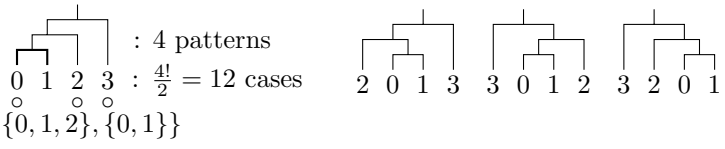
The above left hand side shows the expression of the tournament and the thick segment indicates that the two tournaments selecting two teams playing the game are isomorphic by a suitable permutation of teams. The right hand side gives the tournaments distinguished only by the final winner indicated by $\circ$.

For tournaments involving three teams, there is one type, two structural patterns, three distinct tournaments, and two win-types. (In summary: 1 type, 2 patterns, 3 tournaments, and 2 win-types.)

We present examples of the tournaments of 4 and 5 teams in reduced form:

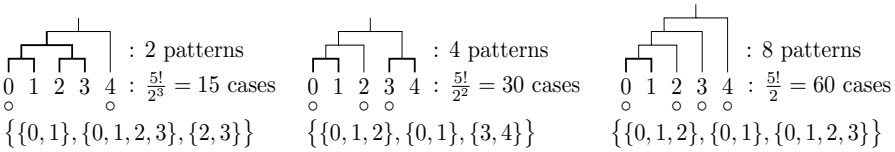
4 teams





2 types, 5 (=1+4) patterns, 15 (=3+12) tournaments, 4 (=1+3) win-types

5 teams



3 types, 14 patterns, 105 (=15+30+60) tournaments, 9 (=2+3+4) win-types

We summarize these numbers in the following table.

Numbers of single-elimination tournaments

teams	2	3	4	5	6	7	8	9	n
patterns	1	2	5	14	42	132	429	1430	$T_n=\frac{(2n-2)!}{n!(n-1)!}$
win-types	1	2	4	9	20	46	106	248	W_n
types	1	1	2	3	6	11	23	46	U_n
tournaments	1	3	15	105	945	10395	135135	2027025	$K_n=(2n-3)!!$

We denote by K_n, U_n, W_n and T_n the numbers of tournaments, types, win-types, patterns, respectively, for the tournaments involving n teams. These quantities satisfy the following recurrence relations:

$$\text{(patterns)} \qquad T_n = \sum_{k=1}^{n-1} T_k \cdot T_{n-k}, \quad T_1 = 1,$$

$$\text{(win-types)} \qquad W_n = \sum_{k=1}^{n-1} W_k \cdot U_{n-k}, \quad W_1 = 1,$$

$$\text{(types)} \qquad U_n = \frac{1}{2}\Big(\sum_{k=1}^{n-1} U_k \cdot U_{n-k} \ \big(+U_{\frac{n}{2}} \text{ if } n \text{ is even}\big)\Big), \quad U_1 = 1,$$

$$\text{(tournaments)} \qquad K_n = \frac{1}{2}\sum_{k=1}^{n-1} {}_nC_k \cdot K_k \cdot K_{n-k}, \quad K_1 = 1.$$

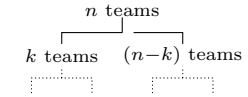
Consider a tournament involving n teams. The teams playing the final game are the winners of two subtournaments: one consisting of k teams and the other of $n-k$ teams with $k = 1, \ldots, n-1$. Each of these subtournaments is itself a tournament of k and $n-k$ teams, respectively. By considering the structure of the entire tournament under a suitable relabeling of the teams, we obtain the corresponding numbers of cases as follows.

$$T_n \leftarrow T_k \cdot T_{n-k} \qquad (1 \le k < n),$$

$$W_n \leftarrow W_k \cdot U_{n-k} \qquad (1 \le k < n),$$

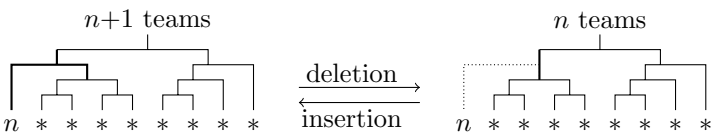
$$U_n \leftarrow \begin{cases} U_k \cdot U_{n-k} & (1 \le k < n-k), \\ \frac{1}{2}U_k(U_k-1) + U_k & (k = n-k), \end{cases}$$

$$K_n \leftarrow \begin{cases} {}_nC_k \cdot K_k \cdot K_{n-k} & (1 \le k < n-k), \\ {}_nC_k\big(\frac{1}{2}K_k(K_k-1)\big) + \big(\frac{1}{2}{}_nC_k\big)K_k & (k = n-k). \end{cases}$$



Thus, the recurrence relations for T_n and W_n are straightforward, since the first k teams and the remaining $n-k$ teams are distinguished. For U_n and K_n, we may assume $k \le n-k$. Taking into account the symmetry when $k = n-k$, we obtain the recurrences above.

Now consider a tournament of $n+1$ teams labeled by $0, 1, \ldots, n$. If team n does not participate, we remove the first game that team n would have played, yielding naturally have a tournament of n teams, as illustrated below:



Conversely, given a tournament of n teams labeled $0, 1, \ldots, n-1$, we may insert the first game of the team n in any of the $2n-1$ vertical line segments of the tournament diagram. That is, the new game can be either the first game of one of n teams or a game whose opponent is the winner of one of the previous games.

Each pair consisting of a tournament of n teams and one of its vertical line segments corresponds uniquely to a tournament of $n+1$ teams under these insertion and deletion operations. Hence we obtain

$$K_{n+1} = (2n-1)K_n. \tag{2.3}$$

Moreover, we have the following equalities obtained from recurrence relations:

$$K_n = (2n-3)!! = \frac{1}{2}\sum_{k=1}^{n-1} {}_nC_k \cdot (2k-1)!! \cdot (2n-2k-1)!!,$$

$$T_n = \frac{(2n-2)!}{(n-1)!n!} = \sum_{k=1}^{n-k} \frac{(2k-2)!}{(k-1)!k!}\frac{(2(n-k)-2)!}{(n-k-1)!(n-k)!},$$

$$1 = \Big(1 - \sum_{k=1}^{\infty} U_n x^k\Big)\Big(\sum_{k=0}^{\infty} W_{k+1}x^k\Big).$$

Remark 2.1. (i) We call the deletion of a team *basic* when the first game of the team coincides with the first game of its opponent. Similarly, the insertion of a team into a tournament is called *basic* if the inserted game is also the first game for the other team. Note that the inverse of a basic deletion is a basic insertion, and that at least one basic deletion is always possible for any tournament. Hence, every tournament with n teams can be constructed by successive basic insertions starting from the tournament with two teams. Consequently, for given two tournaments of n teams, a suitable sequence of $n-1$ pairs of deletions and basic insertions transforms one tournament into the other. In this process, it is sufficient to keep track only of the inserted games.

(ii) We call an insertion that introduces a game between the new team and the winner of the original tournament a top insertion. If there is a champion among the teams and the challenger to the champion is determined by a tournament among the remaining teams, this corresponds to a tournament obtained by a top insertion.

Applying successive top insertions to the game between two teams labeled 0 and 1, we obtain a tournament of n teams labeled by $0, \dots, n-1$, where the n-th game is played by the team labeled $n-1$ and the team selected through the preceding games.

(iii) The four diagrams shown just before Remark 5.12 illustrate the transformations of a tournament by a pair consisting of a deletion and an insertion. The last two diagrams correspond to a basic insertion and a top insertion, respectively.

Let L be a finite set with $|L| > 1$, where $|L|$ denotes the cardinality of L.

Definition 2.2. A collection $\mathcal{I} = \{I_\nu \mid \nu = 1, \dots, r\}$ is called *a commuting family* of L if $|I_\nu| > 1$ and $I_\nu \subset L$ for $\nu = 1, \dots, r$ and

$$I_\nu \cap I_{\nu'} = \emptyset \text{ or } I_\nu \subsetneqq I_{\nu'} \text{ or } I_\nu \supsetneqq I_{\nu'} \quad \text{for} \quad 1 \le \nu < \nu' \le r. \tag{2.4}$$

Moreover, if there is no commuting family $\mathcal{I}'$ of L satisfying $\mathcal{I} \subsetneqq \mathcal{I}'$, then $\mathcal{I}$ is called a *maximal commuting family* of L.

Let $\mathcal{I}$ be a maximal commuting family of L with $|L| > 2$. If there is an element I_0 of $\mathcal{I}$ with $|I_0| = |L| - 1$, then $\mathcal{I} \setminus \{L\}$ is a maximal commuting family of I_0. If such I_0 does not exist in $\mathcal{I}$, there exist I_1 and I_2 in $\mathcal{I}$ which satisfy $I_1 \cap I_2 = \emptyset$ and

$$\mathcal{I} \setminus \{L\} = \mathcal{I}_1 \cup \mathcal{I}_2,\ \mathcal{I}_i := \{I \in \mathcal{I} \mid I \subset I_i\} \quad (i = 1, 2). \tag{2.5}$$

Then each $\mathcal{I}_i$ is a maximal family of I_i for $i = 1$ and 2. Thus, we have the following theorem.

Theorem 2.1. *There is a natural bijection between the set of maximal commuting families of L and the set of single-elimination tournaments whose teams are labelled by the elements of L.*

The maximal commuting family of L correspond to the single-elimination tournaments of teams labeled by the elements of L through the expression of the games in terms of teams labels as in (2.1). In particular

$$|\mathcal{I}| = |L| - 1.$$

Definition 2.3. For a maximal commuting family $\mathcal{I}$ of L, we define

$$\widetilde{\mathcal{I}} := \mathcal{I} \cup \bigcup_{\nu \in L} \{\{\nu\}\}$$

and introduce maps b, $\bar{b}$ and $\bar{b}'$

$$b(I) = \{\bar{b}(I), \bar{b}'(I)\} \qquad (I \in \mathcal{I},\ \bar{b}(I),\ \bar{b}'(I) \in \widetilde{\mathcal{I}}) \tag{2.6}$$

such that $I = \bar{b}(I) \sqcup \bar{b}'(I)$, i.e.,

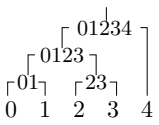
$$\bar{b}(I) \cap \bar{b}'(I) = \emptyset, \quad \bar{b}(I) \cup \bar{b}'(I) = I.$$

We interpret $\bar{b}$ as specifying the *losing side* for each game in the tournament. If $i \in L$ is the final winner, then $i \notin \bar{b}(I)$ for all $I \in \mathcal{I}$. A map $\bar{b}$ satisfying this condition is denoted by b^i. Moreover, we denote by $I_{\bar{b}}$ the labels of the two teams played the game corresponding to $I \in \mathcal{I}$. Then

$$I_{\bar{b}} := I \setminus \bigcup_{I \supsetneq J \in \mathcal{I}} \bar{b}(J). \tag{2.7}$$

Example 2.4. As an example, let $L = \{0, 1, 2, 3, 4\}$ and

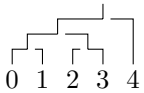
$$\mathcal{I} = \big\{\{0,1\}, \{0,1,2,3\}, \{2,3\}, \{0,1,2,3,4\}\big\}$$

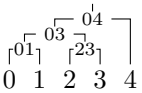


and

$$\begin{aligned}
b(\{0,1\}) &= \big\{\{0\}, \{1\}\big\}, & b^0(\{0,1\}) &= \{1\},\\
b(\{2,3\}) &= \big\{\{2\}, \{3\}\big\}, & b^0(\{2,3\}) &= \{2\},\\
b(\{0,1,2,3\}) &= \big\{\{2,3\}, \{0,1\}\big\}, & b^0(\{0,1,2,3\}) &= \{2,3\},\\
b(\{0,1,2,3,4\}) &= \big\{\{4\}, \{0,1,2,3\}\big\}, & b^0(\{0,1,2,3,4\}) &= \{4\}.
\end{aligned}$$

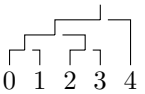
Indicating the losing side by a gap of segment in the tournament diagram, we obtain the following left diagram. Then the teams played each game are determined by $\bar{b}$ as shown in the right diagram.





$$\begin{cases}
\{0,1\}_{\bar{b}} = \{0,1\},\\
\{2,3\}_{\bar{b}} = \{2,3\},\\
\{0,1,2,3\}_{\bar{b}} = \{0,3\},\\
\{0,1,2,3,4\}_{\bar{b}} = \{0,4\},
\end{cases}$$

There is another diagram such that the team 0 is the final winner of the tournament determined by $\mathcal{I}$. In this case, $b^0(\{2,3\}) = \{3\}$ and $\{0,1,2,3\}_{\bar{b}} = \{0,2\}$.



3. Spectra of KZ-type equations

Consider the system of differential equations

$$\mathcal{M}: \frac{\partial u}{\partial x_i} = \sum_{\substack{0\le\nu\le n-1\\ \nu\neq i}} \frac{A_{i\nu}}{x_i - x_\nu} u \qquad (i = 0, \dots, n-1), \tag{3.1}$$

where $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ is a column vector and $A_{ij} = A_{ji} \in M(N, \mathbb{C})$ are matrices. This system is called a Knizhnik-Zamolodchikov type (KZ-type) equation of rank N (cf. [KZ]). Here $M(N, \mathbb{C})$ denote the space of $N \times N$ complex matrices, and each A_{ij} is referred to as the residue matrix along $x_i = x_j$. They satisfy the following integrability condition

$$[A_{ij}, A_{k\ell}] = 0 \qquad \big(\forall\{i, j, k, \ell\} \subset L_n\big), \tag{3.2}$$

$$[A_{ij}, A_{ik} + A_{jk}] = 0 \qquad \big(\forall\{i, j, k\} \subset L_n\big), \tag{3.3}$$

where all indices i, j, k, ℓ are distinct. Hereafter in this note, we use the notation

$$L_n := \{0, 1, \dots, n-1\}, \quad \tilde{L}_n := L_n \cup \{\infty\}, \quad L_n^i := L_n \setminus \{i\}. \tag{3.4}$$

Definition 3.1. Define the *general residue matrices* by

$$\begin{aligned} A_{i\infty} &= -(A_{i0} + A_{i1} + \cdots + A_{i,n-1}) \qquad (i \in L_n),\\ A_{ii} &= A_\emptyset = A_i = 0,\\ A_{i_1 i_2 \cdots i_k} &:= \sum_{1\le p<q\le k} A_{i_p i_q} \qquad \big(\{i_1, \dots, i_k\} \subset \tilde{L}_n\big). \end{aligned}$$

The matrix $A_{i\infty}$ is called the residue matrix of $\mathcal{M}$ along $x_i = \infty$.

This definition, together with the integrability condition, yields the following lemma.

Lemma 3.2 ([10, §2])**.** (i) $$\sum_{i=0}^{n-1} A_{i\infty} = -2A_{L_n},$$

(ii) *For any subsets I and J of $\tilde{L}_n$, we have*

$$\begin{aligned} A_I - A_{\tilde{L}_n \setminus I} &= A_{L_n} && \big(I \subset L_n\big),\\ [A_I, A_J] &= 0 && \big(I \cap J = \emptyset \text{ or } I \subset J \text{ or } I \supset J\big),\\ [A_{L_n}, A_I] &= 0. \end{aligned}$$

Since $[A_{L_n}, A_{ij}] = 0$ for $0 \le i < j < n$, we obtain the following.

Corollary 3.3. *Suppose the system $\mathcal{M}$ is irreducible, i.e., there exists no nontrivial proper subspace $V \subset \mathbb{C}^N$ such that $A_{ij}V \subset V$ for $1 \le i < j < n$. Then*

$$A_{L_n} = \kappa I_N \tag{3.5}$$

for some $\kappa \in \mathbb{C}$.

Definition 3.4. Let $\mathrm{Ad}\big((x_p - x_q)^\lambda\big)$ denote the transformation of $\mathcal{M}$ induced by $u \mapsto (x_p - x_q)^\lambda u$. It changes the residue matrix A_{pq} to $A_{pq} + \lambda$ and leaves all other $A_{i\nu}$ in (3.1) unchanged. This transformation is called an *addition*.

If $\mathcal{M}$ is irreducible, the transform $\mathrm{Ad}\big((x_p - x_q)^{-\kappa}\big)$ sends $\mathcal{M}$ to an equivalent system with $A_{L_n} = 0$.

Definition 3.5. A KZ-type equation $\mathcal{M}$ is called *homogeneous* if $A_{L_n} = 0$. In this case,

$$\begin{aligned} A_I &= A_{L_n \setminus I} \qquad (I \subset L_n), \\ A_{i\infty} &= A_{L_n \setminus \{i\}} \qquad (i \in L_n). \end{aligned} \tag{3.6}$$

Remark 3.6. Let $\mathcal{I}$ be a commuting family of $\tilde{L}_n$. Then

$$\{I \in \mathcal{I} \mid \infty \notin I\} \cup \{\tilde{L}_n \setminus I \mid \infty \in I \in \mathcal{I}\}$$

is also a commuting family of of $\tilde{L}_n$.

Remark 3.7. Since the residue matrices are invariant by the coordinate transformation $x_0 \mapsto ax_0 + b$, we may specialize $(x_1, x_2) = (0, 1)$ without loss of generality. This reduces the number of variable to $n-2$. Appell's functions F_1 and F_2, among others, can be are realized as solutions of certain KZ-type equations with $n = 4$ (cf. [5]).

Definition 3.8. The set $\{A_I \mid I \in \mathcal{I}\}$ defined by the maximal commuting family of $\mathcal{I}$ of L_n is called a *maximal commuting family of residue matrices* of $\mathcal{M}$.

The result in the former section implies the following.

Proposition 3.9. (i) *Let $\mathcal{I}$ be a maximal commuting family of a finite set L. Then $|\mathcal{I}| = |L| - 1$.*

(ii) *$\mathcal{M}$ possesses $(2n-3)!!$ maximal commuting families of residue matrices.*

Example 3.10. We give examples of maximal commuting families of residue matrices, where i, j, k, ℓ, m, p, q are distinct indices. Since every maximal family contains A_{L_n}, we list the families excluding A_{L_n}

(i) When $n = 4$, the families are

$$\{A_{ij}, A_{k\ell}\},\ \{A_{ij}, A_{ijk}\} \qquad \big(\{i, j, k, \ell\} = \{0, 1, 2, 3\}\big).$$

If $\mathcal{M}$ is homogeneous, then $A_{0\infty} = A_{123}$ etc. and these $W_4 = 15$ families cf. [10]) are

$$\{A_{ij}, A_{k\ell}\} \qquad \big(\{i, j, k, \ell\} \subset \{0, 1, 2, 3, \infty\}\big).$$

(ii) When $n = 5$, there are $W_5 = 105$ maximal commuting families of residue matrices (cf. [8, p.94]) :

$$\{A_{ij}, A_{k\ell}, A_{ijk\ell}\},\ \{A_{ij}, A_{ijk}, A_{\ell m}\},\ \{A_{ij}, A_{ijk}, A_{ijk\ell}\},$$

where $\big(\{i, j, k, \ell, m\} = \{0, 1, 2, 3, 4\}\big)$. If $\mathcal{M}$ is homogeneous, these become

$$\{A_{ij}, A_{k\ell}, A_{pq}\},\ \{A_{ij}, A_{ijk}, A_{pq}\} \qquad (\{i, j, k, \ell, p, q\} = \{0, 1, 2, 3, 4, 5, \infty\}).$$

We next recall the notation introduced in [10]. Let the (generalized) eigenvalues and their multiplicities of a matrix $A \in M(N, \mathbb{C})$ be written as

$$[A] = \{[\lambda_1]_{m_1}, \dots, [\lambda_r]_{m_r}\},$$

where $m_1 + \cdots + m_r = N$ and $\prod_{i=1}^{r}(A - \lambda_i) = 0$.

We may assume $m_i \geq m_j$ if $\lambda_i = \lambda_j$ and moreover

$$\operatorname{rank} \prod_{\nu=1}^{k}(A - \lambda_\nu) = N - (m_1 + \cdots + m_k) \qquad (k = 1, \dots, r),$$

or $A \in M(N, \mathbb{C})$ is a limit of matrices satisfying this condition; however we do not assume this in this paper.

When two matrices $A,\ B \in M(N,\mathbb{C})$ commute, we can consider their *simultaneous eigenspace decomposition* and we introduce the notation for their simultaneous eigenvalues and their multiplicities. For example, if

$$A = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix},\ B = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 3 \end{pmatrix},$$

then

$$[A] = \{[0]_3, [-1]_1\} = \{[0]_3, -1\},\ [B] = \{[1]_1, [2]_2, [3]_1\} = \{1, [2]_2, 3\}$$

and the simultaneous eigenvalues and their multiplicities are written by

$$[A:B] = \big\{[0:1]_1, [0:2]_2, [-1:3]_1\big\} = \big\{[0:1], [0:1]_2, [-1:3]\big\}.$$

In general, when $[B_i, B_j] = 0$ for all $i, j = 1, \dots, k$, we write

$$[B_1 : \cdots : B_k] = \big\{[\lambda_{1,1} : \cdots : \lambda_{k,1}]_{m_1}, \cdots, [\lambda_{1,r} : \cdots : \lambda_{k,r}]_{m_r}\big\}$$

and for a positive integer p,

$$[B_1 : \cdots : B_k]_p = \big\{[\lambda_{1,1} : \cdots : \lambda_{k,1}]_{pm_1}, \cdots, [\lambda_{1,r} : \cdots : \lambda_{k,r}]_{pm_r}\big\}.$$

Matrices that commute can always be simultaneously transformed to upper triangular matrices in a suitable base.

Definition 3.11. Let $\mathcal{L}_n$ be the set of maximal commuting families of L_n. The *spectra* of $\mathcal{M}$ is defined by

$$\operatorname{Sp}\mathcal{M} := \big\{[A_{I_1} : \cdots : A_{I_{n-1}}] \mid \mathcal{I} = \{I_1, \dots, I_{n-1}\}\big\}_{\mathcal{I}\in\mathcal{L}_n},$$
$$\operatorname{Sp}'\mathcal{M} := \big\{[A_{I_1} : \cdots : A_{I_{n-2}}] \mid \mathcal{I} = \{I_1, \dots, I_{n-2}, L_n\}\big\}_{\mathcal{I}\in\mathcal{L}_n}.$$

In most cases, A_{L_n} is either a scalar matrix or 0, and we use $\operatorname{Sp}'\mathcal{M}$ instead of $\operatorname{Sp}\mathcal{M}$. For instance, the transformation considered in [5, §5] preserve homogeneity, so we may assume $A_{L_n} = 0$.

Remark 3.12. Let $\{I_1, \dots, I_k\}$ be a maximal commuting family of of L_n. Then the spectrum $[\sum_i c_i A_{I_i}]$ ($c_i \in \mathbb{C}$) can be obtained from $\operatorname{Sp}\mathcal{M}$. For example, since $A_{01} + \cdots + A_{0k} = A_{01\cdots k} - A_{1\cdots k}$, the spectrum $[A_{01} + \cdots + A_{0k}]$ follows from $\operatorname{Sp}\mathcal{M}$ (cf. [11], § 7.5).

4. Blowing up of singular points

A KZ-type equation is written in Pfaffian form as

$$du = \Omega u, \quad \Omega = \sum_{0\le i<j<n} A_{ij}\frac{d(x_i - x_j)}{x_i - x_j}. \tag{4.1}$$

For the index set $L_n = \{0, 1, \dots, n-1\}$, let

$$\mathcal{I} = \{I_1, \dots, I_{n-2}, I_{n-1}\} \in \mathcal{L} \ \text{ with } \ I_{n-1} = L_n \tag{4.2}$$

be a maximal commuting family of L_n. Using the notation given in Definition 2.3 and (2.7), we set

$$b(L_n) = \{J,\ J'\}, \quad J = \{j_0, \dots, j_k\}, \quad J' = \{j'_0, \dots, j'_{k'}\}, \tag{4.3}$$
$$(I_i)_{\bar b} = \{n_i, n'_i\}, \quad n_i,\ n'_i \in L_n. \tag{4.4}$$

Then $k + k' = n - 2$ with $k, k' \ge 0$, and J and J' correspond to the semi-final games of the tournament $\mathcal{I}$. Note that n_i and n'_i correspond to the teams played the game I_i, which are determined by the outcomes of the previous games in the tournament.

The pair $b(L_n)$ corresponds to the singular point

$$x_{j_0} = \cdots = x_{j_k}, \qquad x_{j'_0} = \cdots = x_{j'_{k'}} \tag{4.5}$$

of the KZ-type equation. A local coordinate system $(x_{j_1}, \dots, x_{j_k}, x_{j'_1}, \dots, x_{j'_{k'}})$, with $x_{j_0} = 0$ and $x_{j'_0} = 1$, is valid in a neighborhood of the point. We now construct a local coordinate system that gives a resolution of the singularity.

Definition 4.1. Define local coordinates $(X_1, \dots, X_{n-2})$ by

$$x_{n_i} - x_{n'_i} = \prod_{I_i \subset I_\ell \neq L_n} X_\ell. \tag{4.6}$$

Sometimes, we denote X_ℓ by X_{I_ℓ} for $1 \leq \ell \leq n-2$.

Remark 4.2. Under the notation in Definition 2.3, $I_i = \bar{b}(I_i) \sqcup \bar{b}'(I_i)$ and we may assume $n_i \in \bar{b}(I_i)$ and $n'_i \in \bar{b}'(I_i)$. Then the condition

$$\begin{aligned} |x_{n_i} - x_\nu| \leq \epsilon |x_{n_i} - x_{n'_i}|,\ |x_{n'_i} - x_{\nu'}| \leq \epsilon |x_{n_i} - x_{n'_i}| \\ (\nu \in \bar{b}(I_i),\ \nu' \in \bar{b}'(I_i),\ i \in L_n) \end{aligned} \tag{4.7}$$

with $0 < \epsilon \ll 1$ corresponds to a neighborhood of the origin in the local coordinate $(X_1, \dots, X_{n-1})$. For $\mathcal{I} \in \mathcal{L}$ satisfying (4.3), the following theorem provides a resolution of the singular point (4.5).

Theorem 4.1. *The 1-form*

$$\Omega - \sum_{\ell=1}^{n-2} A_{I_\ell} \frac{dX_\ell}{X_\ell}$$

is non-singular in a neighborhood of the origin.

Proof. Since the condition $I^o_{i,j} \subset I$ for $I \in \mathcal{I}$ is equivalent to $\{i,j\} \subset I$ under the notation in Lemma 4.3 below, the lemma and the Definition 4.1 implies that

$$x_i - x_j = f_{i,j}(X) \cdot \prod_{\{i,j\} \subset I_\ell \in \mathcal{I} \setminus \{L_n\}} X_\ell, \tag{4.8}$$

where $f_{i,j}$ is a polynomial in the variables X_ℓ with $1 \leq \ell \leq n-2$ and the constant term of $f_{i,j}$ equals 1 or -1. Hence

$$\frac{d(x_i - x_j)}{x_i - x_j} - \sum_{\{i,j\} \subset I_\ell \in \mathcal{I} \setminus \{L_n\}} \frac{dX_\ell}{X_\ell}$$

is regular in a neighborhood of the origin, which proves the theorem. □

Lemma 4.3. *Let $\mathcal{I}$ be a maximal commuting family of $L = \{0, \dots, n-1\}$, and let $\bar{b}$ be the map defined in Definition 2.3. Putting*

$$x_I := x_{n_I} - x_{n'_I} \quad \textit{with} \quad I_{\bar{b}} = \{n_I, n'_I\} \subset L \tag{4.9}$$

for $I \in \mathcal{I}$. Then for $i, j \in L$ with $i \neq j$, we define $I^o_{i,j}$ the minimal subset in $\mathcal{I}$ containing both i and j. Then x_I $(I \in \mathcal{I})$ are linearly independent over $\mathbb{C}$, and

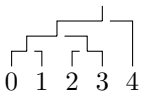
$$x_i - x_j = \sum_{I \in \mathcal{I}} \epsilon_I x_I \quad \textit{with} \quad \begin{cases} \epsilon_I = 0 & (I \supsetneqq I^o_{i,j} \textit{ or } I \cap I^o_{i,j} = \emptyset), \\ \epsilon_I \in \{1, -1\} & (I = I^o_{i,j}), \\ \epsilon_I \in \mathbb{Z} & (I \subsetneqq I^o_{i,j}). \end{cases} \tag{4.10}$$

Proof. Let $(I^o_{i,j})_{\bar{b}} = \{i_0, j_0\}$. The statement is evident when $|I^o_{i,j}| = 2$. We prove (4.10) by induction on the cardinality of $I^o_{i,j}$. We may assume $i \in \bar{b}(I^o_{i,j})$ and $j \in \bar{b}'(I^o_{i,j})$ by exchanging i and j if necessary. We may further assume $i_0 \in \bar{b}(I^o_{i,j})$ and $j_0 \in \bar{b}'(I^o_{i,j})$. Then $i = i_0$ or $I^o_{i,i_0} \subsetneqq I^o_{i,j}$. Moreover $j = j_0$ or $I^o_{j,j_0} \subsetneqq I^o_{i,j}$. Since

$$x_i - x_j = (x_{i_0} - x_{j_0}) + (x_i - x_{i_0}) - (x_j - x_{j_0}),$$

the hypothesis of the induction establishes (4.10). Because the dimension of the space $\sum_{i,j \in L} \mathbb{C}(x_i - x_j)$ is $n-1$ and $|\mathcal{I}| = n-1$, the elements x_I $(I \in \mathcal{I})$ are linearly independent. □

Remark 4.4. To aid understanding the above proof, let us illustrate concretely, following the above discussion, an example of (4.8) – which plays a key role for the proof of Theorem 4.1 – in the case corresponding to Example 2.4 and the final example in this section. Then,



$$
\begin{aligned}
\mathcal{I} &= \big\{\{0,1\},\{2,3\},\{0,1,2,3\},\{0,1,2,3,4\}\big\}\\
x_3 - x_0 &= x_{0123} = X_{0123},\\
x_1 - x_0 &= x_{01} = X_{01}X_{0123},\\
x_2 - x_3 &= x_{23} = X_{23}X_{0123},\\
I^o_{02} &= I^o_{12} = \{0,1,2,3\},\\
(X,Y,Z) &:= (X_{01}, X_{23}, X_{0123}),\\
x_0 - x_2 &= (x_0 - x_3) - (x_2 - x_3)\\
&= -x_{0123} - x_{23}\\
&= -X_{0123} - X_{23}X_{0123} = -(1+Y)Z,\\
x_1 - x_2 &= (x_1 - x_0) - (x_2 - x_3) + (x_0 - x_3)\\
&= x_{01} - x_{23} - x_{0123}\\
&= X_{01}X_{0123} - X_{23}X_{0123} - X_{0123} = -(1 - X + Y)Z.
\end{aligned}
$$

Here, $\frac{d(x_0-x_2)}{x_0-x_2} - \frac{dZ}{Z}$ and $\frac{d(x_1-x_2)}{x_1-x_2} - \frac{dZ}{Z}$ are non-singular at $(X,Y,Z) = 0$.

Examples

Let

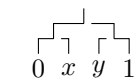
$$\Omega' := \sum_{\ell=1}^{n-2} A_{I_\ell} \frac{dX_\ell}{X_\ell} \tag{4.11}$$

be the singular part of Ω stated in Theorem 4.1. According to the theorem, we present the resolution of the singular point corresponding to each type of maximal commuting family of residue matrices in the cases $n = 4$ and $n = 5$.

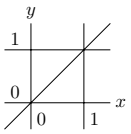
$\mathbf{n = 4}$

$$\Omega = A_{x0}\frac{dx}{x} + A_{y0}\frac{dy}{y} + A_{x1}\frac{d(x-1)}{x-1} + A_{y1}\frac{d(y-1)}{y-1} + A_{xy}\frac{d(x-y)}{x-y}.$$

$(x,y) = (0,1)$:



$$\begin{cases} x_1 - x_0 = x = X, \\ x_2 - x_3 = y - 1 = Y, \end{cases}$$



$$\begin{cases} X = x, \\ Y = y - 1 \end{cases} \qquad \begin{cases} \frac{dx}{x} = \frac{dX}{X}, \\ \frac{dy}{y} = \frac{dY}{Y} \end{cases}$$

$$\Omega' = A_{x0}\frac{dX}{X} + A_{y1}\frac{dY}{Y}$$

$(x,y) = (0,0)$:

$$\begin{cases} x_2 - x_0 = y = Y, \\ x_1 - x_0 = x = XY, \\ x_2 - x_1 = y - x = (1-X)Y, \end{cases}$$

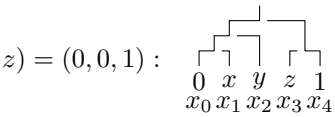
$$\begin{cases} X = \frac{x}{y}, \\ Y = y \end{cases} \qquad \begin{cases} \frac{dx}{x} = \frac{dX}{X} + \frac{dY}{Y}, \ \frac{dy}{y} = \frac{dY}{Y} \\ \frac{d(x-y)}{x-y} = \frac{dY}{Y} + \frac{d(X-1)}{X-1}, \end{cases}$$

$$\Omega' = A_{x0}\frac{dX}{X} + A_{xy0}\frac{dY}{Y}, \qquad A_{xy0} := A_{x0} + A_{y0} + A_{xy} \ \ (|x| \ll |y| \ll 1).$$

Remark 4.5. In [13, 5], the local coordinates $(X, Y) = (\frac{y}{x}, y)$ are used to desingularize the origin, where (X, Y) lies in a neighborhood of $(\infty, 0)$. This coordinate transformation preserves the KZ-type equations, and the point $(\infty, 0)$ becomes a normal crossing singularity of the equations.

$$\mathbf{n = 5}$$

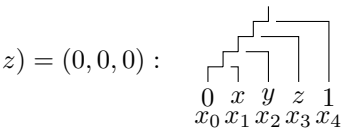
$$\Omega = A_{x0}\frac{dx}{x} + A_{y0}\frac{dy}{y} + A_{z0}\frac{dz}{z} + A_{x1}\frac{d(x-1)}{x-1} + A_{y1}\frac{d(y-1)}{y-1} + A_{z1}\frac{d(z-1)}{z-1} + A_{xy}\frac{d(x-y)}{x-y} + A_{yz}\frac{d(y-z)}{y-z} + A_{xz}\frac{d(x-z)}{x-z}.$$

$(x, y, z) = (0, 0, 1)$: 

$$\begin{cases} x_2 - x_0 = y = Y, \\ x_1 - x_0 = x = XY, \\ x_3 - x_4 = z - 1 = Z, \end{cases}$$

$$\begin{cases} X = \frac{x}{y} \\ Y = y, \\ Z = z - 1, \end{cases} \qquad \begin{cases} x - y = (X-1)Y, \\ z = Z + 1, \end{cases}$$

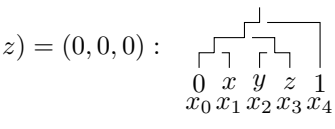
$$\Omega' = A_{x0}\frac{dX}{X} + A_{xy0}\frac{dY}{Y} + A_{z1}\frac{dZ}{Z} \qquad (|x| \ll |y| \ll 1,\ |z-1| \ll 1)$$

$(x, y, z) = (0, 0, 0)$: 

$$\begin{cases} x_3 - x_0 = z = Z, \\ x_2 - x_0 = y = YZ, \\ x_1 - x_0 = x = XYZ, \end{cases}$$

$$\begin{cases} X = \frac{x}{y}, \\ Y = \frac{y}{z}, \\ Z = z, \end{cases} \qquad \begin{cases} y - z = (Y-1)Z, \\ x - y = (X-1)YZ, \\ x - z = (XY-1)Z, \end{cases}$$

$$\Omega' = A_{x0}\frac{dX}{X} + A_{xy0}\frac{dY}{Y} + A_{xyz0}\frac{dZ}{Z} \qquad (|x| \ll |y| \ll |z| \ll 1)$$

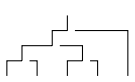
$(x, y, z) = (0, 0, 0)$: 

$$\begin{cases} x_3 - x_0 = z = Z, \\ x_1 - x_0 = x = XZ, \\ x_2 - x_3 = y - z = YZ, \end{cases}$$

$$\begin{cases} X = \frac{x}{z}, \\ Y = \frac{y-z}{z}, \\ Z = z, \end{cases} \qquad \begin{cases} y = (1+Y)Z, \\ x - y = (X - Y - 1)Z, \\ x - z = (X-1)Z, \end{cases}$$

$$\Omega' = A_{x0}\frac{dX}{X} + A_{yz}\frac{dY}{Y} + A_{xyz0}\frac{dZ}{Z} \qquad (|x|,\ |y-z| \ll |z| \ll 1).$$

Here, we may use the following local coordinates (cf. Example 2.4).



$$\begin{cases} X = \frac{x}{y}, \\ Y = \frac{z-y}{y}, \\ Z = z, \end{cases} \qquad (|x| \ll 1,\ |z-y| \ll |y| \ll 1).$$

Then Ω' changes, although their difference is non-singular at the origin.

5. Middle convolutions of KZ-type equations

The *convolution* $\widetilde{\mathrm{mc}}_{x_0,\mu}\,\mathcal{M}$ of the system $\mathcal{M}$ (3.1) with $\mu \in \mathbb{C}$ is defined by

$$\widetilde{\mathcal{M}}\ :\ \frac{\partial \tilde{u}}{\partial x_i} = \sum_{0\le\nu<n} \frac{\tilde{A}_{i\nu}}{x_i - x_\nu}\tilde{u} \qquad (0 \le i < n),$$

$$\tilde{A}_{0k} = \begin{array}{c} \\ k \end{array} \begin{pmatrix} & & \overset{k}{\downarrow} & & \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{01} & \cdots & A_{0k}+\mu & \cdots & A_{0n-1} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ & & \underset{i}{\uparrow} & & \underset{j}{\uparrow} \end{pmatrix} \in M\big((n-1)N,\mathbb{C}\big) \quad \text{(Dettweiler-Reiter [1])},$$

$$\tilde{A}_{ij} = \begin{array}{c} \\ i \\ \\ \\ j \\ \\ \\ \end{array} \begin{pmatrix} A_{ij} & & & & & & \\ & \ddots & & & & & \\ & & A_{ij}+A_{0j} & & -A_{0j} & & \\ & & & \ddots & & & \\ & & -A_{0i} & & A_{ij}+A_{0i} & & \\ & & & & & \ddots & \\ & & & & & & A_{ij} \end{pmatrix} \in M\big((n-1)N,\mathbb{C}\big) \quad \text{(Haraoka [2])}.$$

In particular, the compatibility condition of $\mathcal{M}$ guarantees that of $\widetilde{\mathcal{M}}$.

Furthermore, we set $\tilde{A}_{ii} = 0$, $\tilde{A}_{i\infty} = -\tilde{A}_{i0} - \cdots - \tilde{A}_{i,n-1}$ and $\tilde{A}_{i_1\dots i_k} = \sum_{1\le p<q\le k} \tilde{A}_{i_p i_q}$ for $\{i_1,\dots,i_k\} \subset \tilde{L}_n$.

We now introduce notation to define and analyze the middle convolution.

Definition 5.1. For an integer $n > 1$ and a positive integer N, define

$$L_n^0 = \{1,2,\dots,n-1\}, \quad L_n = \{0,1,2,\dots,n-1\},$$

$$\iota_j(v) := (v)_j := {}_j\begin{pmatrix} 0 \\ \vdots \\ v \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{(n-1)N}, \quad \iota_j : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N} \quad (v \in \mathbb{C}^N,\ j \in L_n^0),$$

$$\iota_I := \sum_{i\in I} \iota_i : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N},\ (v)_I := \iota_I(v) \in \mathbb{C}^{(n-1)N} \quad (v \in \mathbb{C}^N,\ I \subset L_n^0),$$

$$V_I := \iota_I(\mathbb{C}^N) \simeq \mathbb{C}^N.$$

For the system $\mathcal{M}$ given by (3.1), define the following subspaces of $\mathbb{C}^{(n-1)N}$:

$$\mathcal{K}_i := \iota_i(\ker\, A_{0i}) = {}_i\begin{pmatrix} 0 \\ \vdots \\ \ker A_{0i} \\ \vdots \\ 0 \end{pmatrix} \subset \mathbb{C}^{(n-1)N},$$

$$\mathcal{K}_\infty := \ker \tilde{A}_{0\infty} \overset{\mu\neq 0}{=} \iota_{L_n^0}\big(\ker(A_{0\infty}-\mu)\big) = \left\{ \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \mid A_{0\infty}v = \mu v \right\} \subset \mathbb{C}^{(n-1)N},$$

$$\mathcal{K} := \mathcal{K}_\infty + \bigoplus_{i=1}^{n} \mathcal{K}_i.$$

Note that if $\mu \neq 0$, then $\ker \tilde{A}_{0\infty}$ equals $\iota_{L_n^0}\big(\ker(A_{0\infty}-\mu)\big)$ and $\mathcal{K}$ is the direct sum of $\mathcal{K}_1,\dots,\mathcal{K}_{n-1}$ and $\mathcal{K}_\infty$.

Since $\tilde{A}_I\mathcal{K} \subset \mathcal{K}$, each $\tilde{A}_I$ induces a linear transformation on the quotient space $\mathbb{C}^{(n-1)N}/\mathcal{K}$. These linear transformation are presented by matrices

$$\bar{A}_I \in M\big((n-1)N - \dim \mathcal{K}, \mathbb{C}\big) \tag{5.1}$$

with respect to a basis of the quotient space.

Definition 5.2 ([1], [2])**.** The *middle convolution* $\overline{\mathcal{M}} = \mathrm{mc}_{x_0,\mu}\,\mathcal{M}$ of $\mathcal{M}$ is defined by

$$\overline{\mathcal{M}} : \frac{\partial \bar{u}}{\partial x_i} = \sum_{\nu \in L_n \setminus \{i\}} \frac{\bar{A}_{i\nu}}{x_i - x_\nu}\bar{u} \quad (i \in L_n).$$

As is proved by [1], if the ordinary differential equation in the variable x_0 defined by $\mathcal{M}$ is irreducible and μ is generic, $\mathrm{mc}_{x_0,\mu}\,\mathcal{M}$ is also irreducible, and

$$\mathrm{mc}_{x_0,\mu'} \circ \mathrm{mc}_{x_0,\mu} = \mathrm{mc}_{x_0,\mu+\mu'}, \qquad \mathrm{mc}_{x_0,0} = \mathrm{id}. \tag{5.2}$$

In most cases, the irreducibility of this ODE coincides with that of $\mathcal{M}$ (cf. [9]).

When $n = 4$, the convolution of $A_{123} = A_{12} + A_{13} + A_{23}$ is

$$\begin{aligned}\tilde{A}_{123} &= \begin{pmatrix} A_{12}+A_{02} & -A_{02} & 0 \\ -A_{01} & A_{12}+A_{01} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} + \begin{pmatrix} A_{13}+A_{03} & 0 & -A_{03} \\ 0 & A_{13} & 0 \\ -A_{01} & 0 & A_{13}+A_{01} \end{pmatrix} \\ &\quad + \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23}+A_{03} & -A_{03} \\ 0 & -A_{02} & A_{23}+A_{02} \end{pmatrix} \\ &= \begin{pmatrix} A_{0123}-A_{01} & -A_{02} & -A_{03} \\ -A_{01} & A_{0123}-A_{02} & -A_{03} \\ -A_{01} & -A_{02} & A_{0123}-A_{03} \end{pmatrix} \in M(3N, \mathbb{C}).\end{aligned}$$

Let m be a positive integer smaller than n. Then

$$\tilde{A}_{1\cdots m} = \begin{pmatrix} A_{0\cdots m}-A_{01} & \cdots & -A_{0m} & & & \\ \vdots & \ddots & \vdots & & & \\ -A_{01} & \cdots & A_{0\cdots m}-A_{0m} & & & \\ & & & A_{1\cdots m} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots m} \end{pmatrix} \in M((n-1)N).$$

Lemma 5.3. *Retain the notation in Definition 5.1. For $I \subset L_n$ with $|I| > 1$, $j, k \in L_n^0$, and $v \in \mathbb{C}^N$, we have*

$$\begin{aligned} \tilde{A}_I(v)_J &= (A_I v)_J \in V_J && (I \subset J \subset L_n^0), \\ \tilde{A}_I(v)_j &= (A_{0I}v)_j - (A_{0i}v)_I && (I \ni j), \\ \tilde{A}_I(v)_k &= (A_I v)_k \in V_{\{j\}} && (I \not\ni k), \\ [\tilde{A}_I] &= [A_I]_{n-|I|} \cup [A_{0I}]_{|I|-1}, \\ \tilde{A}_I(v)_j &= (A_{0I}v)_j \in \mathcal{K}_j && (v \in \ker A_{0j},\ I \ni j), \\ \tilde{A}_I(v)_k &= (A_I v)_k \in \mathcal{K}_k && (v \in \ker A_{0k},\ I \not\ni k), \\ \tilde{A}_I(v)_{L_n^0} &= (A_I v)_{L_n^0} \in \mathcal{K}_\infty && (v \in \ker(A_{0\infty} - \mu)). \end{aligned}$$

Proof. By the symmetry among the coordinates $(x_1, \ldots, x_n)$, it suffices to prove the lemma in the case $I = \{1, \ldots, m\}$. The lemma then follows directly from the

explicit expression of A_I given above. Note that the last three equalities in the above follow from the commutation relations

$$[A_{0I}, A_{0j}] = [A_I, A_{0k}] = [A_I, A_{0\infty}] = 0 \qquad (j \in I,\ k \notin I).$$

□

Since $\tilde{A}_{0\cdots m} = \tilde{A}_{1\cdots m} + \tilde{A}_{01} + \cdots + \tilde{A}_{0m}$, we obtain

$$\tilde{A}_{0\cdots m} = \begin{pmatrix} A_{01\cdots m}+\mu & & & A_{0,m+1} & \cdots & A_{0,n-1} \\ & \ddots & & \vdots & \cdots & \vdots \\ & & A_{01\cdots m}+\mu & A_{0,m+1} & \cdots & A_{0,n-1} \\ & & & A_{1\cdots m} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots m} \end{pmatrix}.$$

Then $\widetilde{A}_{0\ldots n-1}$ is a block diagonal matrix whose diagonal entries are $A_{0\cdots n-1}+\mu$ and we have the following lemma.

Lemma 5.4. *For $I \subset L_n^0$ and $j,\ k \in L_n^0$ and $v \in \mathbb{C}^N$, we have*

$$\begin{aligned}
\tilde{A}_{0I}(v)_j &= ((A_{0I}+\mu)v)_j \in V_{\{j\}} && (I \ni j),\\
\tilde{A}_{0I}(v)_k &= (A_I v)_k + (A_{0k}v)_I && (I \not\ni k),\\
[\tilde{A}_{0I}] &= [A_{0I}+\mu]_{n-1-|I|} \cup [A_I]_{|I|},\\
\tilde{A}_{0I}(v)_j &= ((A_{0I}+\mu)v)_j \in \mathcal{K}_j && (v \in \ker A_{0j},\ I \ni j),\\
\tilde{A}_{0I}(v)_k &= (A_I v)_k \in \mathcal{K}_k && (v \in \ker A_{0k},\ I \not\ni k),\\
\tilde{A}_{0I}(v)_{L_n^0} &= (A_I v)_{L_n^0} \in \mathcal{K}_\infty && (v \in \ker(A_{0\infty}-\mu)).
\end{aligned}$$

The last equality in this lemma follows from the relation

$$(A_{0I}+\mu)v + \sum_{\nu \in L_n^0 \setminus I} A_{0\nu}v = A_I v + \Big(\sum_{\nu=1}^{n-1} A_{0\nu} + \mu\Big)v = A_I v - (A_{0\infty}-\mu)v.$$

Definition 5.5. Let $\mathcal{I}$ be a maximal commuting family of $L_n = \{0,\ldots,n-1\}$. We decompose $\mathcal{I}$ into its subsets $\mathcal{I}_k$, and fix labels $I_{k,\nu}$ and an order on the elements of $\mathcal{I}$ so that they satisfy the following relations.

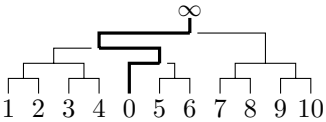
$$\begin{aligned}
&\mathcal{I}_0 := \{I \in \mathcal{I} \mid 0 \in I\} = \{I_{k,0} \mid 1 \le k \le m\}\\
&\qquad \text{where } I_{1,0} \subset I_{2,0} \subset \cdots \subset I_{m,0},\\
&\mathcal{I}_k := \{I \in \mathcal{I} \mid I \subset I_{k,0} \setminus I_{k-1,0}\} = \{I_{k,\nu} \mid \nu = 1,\ldots,m_k\} \quad (1 \le k \le m)\\
&\qquad \text{where } I_{0,0} = \{0\} \text{ and if } \nu < \nu', \text{ then } I_{k,\nu} \supset I_{k,\nu'} \text{ or } I_{k,\nu} \cap I_{k,\nu'} = \emptyset,\\
&I_{k,\nu} \le I_{k',\nu'} \overset{\text{def.}}{\iff} k < k' \text{ or } (k = k' \text{ and } \nu \le \nu'),\\
&I^{(\ell)} := I_{k,\nu} \text{ with } \ell = |\{I \in \mathcal{I} \mid I \le I_{k,\nu}\}| \qquad (1 \le \ell < n).
\end{aligned}$$

Here the numbers ℓ indicate the order of the elements of $\mathcal{I}$. Namely,

$$\mathcal{I} = \{I^{(\ell)} \mid \ell = 1,\ldots,n-1\}, \quad I^{(1)} < I^{(2)} < \cdots < I^{(n-1)}. \tag{5.3}$$

Thus, the maximal commuting family $\mathcal{I}$ of L_n becomes a totally ordered set. Note, however, that these labels and the order are not necessarily uniquely determined as is indicated in the following example.

The order can be easily visualized when $\mathcal{I}$ is expressed by a tournament diagram of the teams labeled by the elements of L_n, where the team labeled 0 is the final winner.

Example 5.6. Consider the tournament diagram 

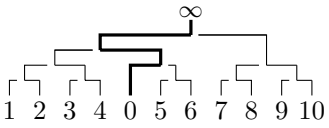
,

$$\mathcal{I}_0 = \{I_{1,0} = \{0,5,6\},\ I_{2,0} = \{0,1,2,3,4,5,6\},\ I_{3,0} = \{0,1,2,\dots,9,10\}\},$$
$$\mathcal{I}_1 = \{I_{1,1} = \{5,6\}\},$$
$$\mathcal{I}_2 = \{I_{2,1} = \{1,2,3,4\},\ I_{2,2} = \{1,2\},\ I_{2,3} = \{3,4\}\},$$
$$\mathcal{I}_3 = \{I_{3,1} = \{7,8,9,10\},\ I_{3,2} = \{7,8\},\ I_{3,3} = \{9,10\}\},$$

$I_{1,0} < I_{1,1} < I_{2,0} < I_{2,1} < I_{2,2} < I_{2,3} < I_{3,0} < I_{3,1} < I_{3,2} < I_{3,3}$.

For example, we may change the above by $I_{2,2} = \{3,4\}$ and $I_{2,3} = \{1,2\}$.

We now recall the map b^0 introduced in Definition 2.3. The set of teams $b^0(I)$ is not uniquely determined by a given tournament. However, if we identify $b^0(I)$ with the teams that lose the game labeled by I, then the definition of b^0 corresponds precisely to the tournament diagram where the winner of each game is indicated. An example of such diagram is shown below:

(5.4) 

Here $n = 11$ and the corresponding $b^0(I^{(1)}), b^0(I^{(2)}), \dots, b^0(I^{(n-1)})$ are

$$\underset{0,5,6}{\overset{6}{\{5,6\}}},\ \underset{5,6}{\overset{5}{\{5\}}},\ \underset{0,1,2,3,4,5,6}{\overset{4}{\{1,2,3,4\}}},\ \underset{1,2,3,4}{\overset{2}{\{1,2\}}},\ \underset{1,2}{\overset{1}{\{1\}}},\ \underset{3,4}{\overset{3}{\{3\}}},\ \underset{0,1,\dots,9,10}{\overset{10}{\{7,8,9,10\}}},\ \underset{7,8,9,10}{\overset{8}{\{7,8\}}},\ \underset{7,8}{\overset{7}{\{7\}}},\ \underset{9,10}{\overset{9}{\{9\}}}.$$

The numbers written below the set $b^0(I^{(\ell)})$ of teams indicate $I^{(\ell)}$, the labels of the teams participating the game $I^{(\ell)}$. The number written above the set indicates that the game corresponding to $b^0(I^{(\ell)})$ is the final game won by the team labeled by that number. Here, for example, $\underset{3,4}{\overset{3}{\{3\}}}$ means that team labeled by 3 lost in the first game and that the teams labeled by 3 and 4 participated the game.

Definition 5.7. We define subspaces of $\mathbb{C}^{(n-1)N}$ as follows (cf. Definitions 2.3 and 5.1).

$$W^{(\ell)} := W_I := \sum_{\nu=1}^{\ell} V_{b^0(I^{(\nu)})} \qquad (I = I^{(\ell)} \in \mathcal{I}),$$
$$U_{b^0(\mathcal{I})} := \big(U_{ij}\big)_{\substack{1\le i<n-1\\ 1\le j\le n-1}} \in GL((n-1)N, \mathbb{C}), \quad \text{where}$$
$$U_{ij} = \begin{cases} E_N & (j \in b^0(I^{(i)})), \\ O_N & (j \notin b^0(I^{(i)})). \end{cases}$$

Here E_N denotes the identity matrix and O_N the zero matrix in $M(N, \mathbb{C})$.

In the case Example 5.6, we have

$$W^{(1)} = W_{\{0,5,6\}} = V_{\{5,6\}}, \qquad W^{(2)} = W_{\{5,6\}} = V_{\{5\}} + W^{(1)},$$
$$W^{(3)} = W_{\{0,1,2,3,4\}} = V_{\{1,2,3,4\}} + W^{(2)}, \qquad W^{(4)} = W_{\{1,2,3,4\}} = V_{\{1,2\}} + W^{(3)},$$
$$W^{(5)} = W_{\{1,2\}} = V_{\{1\}} + W^{(4)}, \qquad W^{(6)} = W_{\{3,4\}} = V_{\{3\}} + W^{(5)},$$
$$W^{(7)} = W_{\{0,\dots,10\}} = V_{\{7,8,9,10\}} + W^{(6)}, \qquad W^{(8)} = W_{\{7,8,9,10\}} = V_{\{7,8\}} + W^{(7)},$$
$$W^{(9)} = W_{\{7,8\}} = V_{\{7\}} + W^{(8)}, \qquad W^{(10)} = W_{\{9,10\}} = V_{\{9\}} + W^{(9)}.$$

Lemma 5.8. (i) *For $0 \notin I \in \mathcal{I}$, we have*

$$V_I \subset W_I. \tag{5.5}$$

$$V_{\bar{b}(I)},\, V_{\bar{b}'(I)} \subset W_I. \tag{5.6}$$

(ii) $\dim W^{(\ell)} = \ell N \quad (\ell = 1, \dots, n-1)$

(iii) *Let I, $K \in \mathcal{I}$ and $v \in \mathbb{C}^N$. Put $J = b^0(K)$.*

If $0 \in I$, then

$$\tilde{A}_I(v)_J = \begin{cases} \big((A_I + \mu)v\big)_J & (I \supset K), \\ (A_{I\setminus\{0\}}v)_J + \big(\sum_{\nu\in J} A_{0\nu}v\big)_{I\setminus\{0\}} & (I \not\supset K). \end{cases}$$

If $0 \notin I$, then

$$\tilde{A}_I(v)_J = \begin{cases} (A_I v)_J & (I \not\supset K), \\ (A_{0I}v)_J - \big(\sum_{\nu\in J} A_{0\nu}v\big)_I & (I \supset K). \end{cases}$$

(iv) $\tilde{A}_I W^{(\ell)} \subset W^{(\ell)} \quad (I \in \mathcal{I},\ 1 \le \ell < n)$.

Proof. Let $I = I_{k,i}$. We first prove (5.5) and (5.6) by induction on i. Since $I = \bar{b}(I) \sqcup \bar{b}'(I)$, we remark that (5.6) follows from (5.5). When $i = 1$, (5.5) is obvious because $b^0(I_{k,0}) = I_{k,1}$. Assume $i > 1$. Then there exists $J \in \mathcal{I}$ such that $I \subset J \in \mathcal{I}_k$ and $I \in b(J)$. Then $V_I \subset W_J \subset W_I$ by the induction hypothesis for (5.6) with replacing I by J.

Hence,

$$\begin{aligned} \sum_{k=1}^{m}\sum_{\nu=0}^{m_k} V_{b^0(I_{k,\nu})} = \sum_{k=1}^{m}\sum_{\nu=0}^{m_k} W_{I_{k,\nu}} &= \sum_{k=1}^{m}\Big(V_{b^0(I_{k,0})} + \sum_{\nu=1}^{m_k}(V_{\bar{b}(I_{k,\nu})} + V_{\bar{b}'(I_{k,\nu})})\Big) \\ &\supset \sum_{j=1}^{n-1} V_{\{j\}} \simeq \mathbb{C}^{(n-1)N}. \end{aligned}$$

Since $\sum_{k=1}^{m}(m_k + 1) = n - 1$, the sum $\sum_{k=1}^{m}\sum_{\nu=0}^{m_k} V_{b^0(I_{k,\nu})}$ is a direct sum decomposition of $\mathbb{C}^{(n-1)N}$, which implies (ii).

Let $I, K \in \mathcal{I}$. Putting $J = b^0(K)$, we have

$$\begin{aligned} &0 \in I \supset K \Rightarrow I \setminus \{0\} \supset J, \\ &0 \in I \not\supset K \Rightarrow I \setminus \{0\} \cap J = \emptyset, \\ &0 \notin I \supset K \Rightarrow I \supsetneqq J, \\ &0 \notin I \not\supset K \Rightarrow I \cap J = \emptyset \ \text{ or } \ I \subset J. \end{aligned}$$

Hence Lemma 5.3 and Lemma 5.4 show (iii). In particular, we have (iv). □

Definition 5.9. Let L be a finite set. For $i \in L$ and nonempty subsets I, $K \subset L$, define

$$\mathrm{md}_{i,K}(I) := \begin{cases} I \cup \{i\} & (I \supset K), \\ I \setminus \{i\} & (I \not\supset K), \end{cases} \qquad \mathrm{me}_{i,K}(I) := \begin{cases} 1 & (i \in I \supset K), \\ 0 & (i \notin I \text{ or } I \not\supset K). \end{cases}$$

Remark 5.10. (i) If $i \in I \supset K$ or $i \notin I \not\supset K$, then $\mathrm{md}_{i,K}(I) = I$.

(ii) If the elements of a family $\{I_\nu \mid \nu = 0, 1, \dots, r\}$ of L commute with each other (cf. (2.4)) and $|I_\nu| > 1$ for $\nu = 1, \dots, r$, then so do the elements of $\{I_0 \cup \{i\}\} \cup \{\mathrm{md}_{i,I_0}(I_\nu) \mid (\nu = 1, \dots, r)\}$.

We remark that if $J = W^{(\ell)}$, $\widetilde{A}_I$ induces a linear transformation on the quotient space $W^{(\ell)}/W^{(\ell-1)} \simeq V_{b^0(J)}$. Then, from Lemma 5.3, Lemma 5.4 and Lemma 5.8, we obtain the following theorem.

Theorem 5.1. (i) *Let $\mathcal{I}$ be a maximal commuting family of the set $L_n = \{0, 1, \dots, n-1\}$ and write $\mathcal{I} = \{I^{(1)}, \dots, I^{(n-1)}\}$. For $I \in \mathcal{I}$, under the notation in Definition 5.9, we have*

$$\begin{aligned}
[\tilde{A}_I] &= [A_{I\cup\{0\}} + \mathrm{me}_{0,I}(I)\cdot\mu]_{|I|-1} \sqcup [A_{I\setminus\{0\}}]_{n-|I|},\\
[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}] &= \bigsqcup_{J\in\mathcal{I}} [A^J_{I^{(1)}} : \cdots : A^J_{I^{(n-1)}}],\\
[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_j} &= [A^{\{j\}}_{I^{(1)}} : \cdots : A^{\{j\}}_{I^{(n-1)}}]|_{\ker A_{0j}} \quad (j \in L^0_n),\\
[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_\infty} &= [A^{L_n}_{I^{(1)}} : \cdots : A^{L_n}_{I^{(n-1)}}]|_{\ker(A_{0\infty}-\mu)},
\end{aligned}$$

where we set

$$A^K_I := A_{\mathrm{md}_{0,K}(I)} + \mathrm{me}_{0,K}(I)\cdot\mu. \tag{5.7}$$

(ii) *By conjugation, the matrices $\widetilde{A}_I$ for $I \in \mathcal{I}$ are simultaneously transformed into upper triangular block matrices $U^{-1}_{b^0(\mathcal{I})}\widetilde{A}_I U_{b^0(\mathcal{I})}$.*

Remark 5.11. (i) In the theorem above, we have

$$A^J_{L_n} = A^{\{j\}}_{0\cdots n-1} = \mu + A_{0\cdots n-1} \quad \text{and} \quad A^{L_n}_{L_n} = A_{L^0_n}.$$

Therefore the term $\tilde{A}_{L_n}$ is often omitted from $[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]$.

Moreover, note that

$$A_{0\infty} = A_{L^0_n} - A_{L_n}$$

and Corollary 3.3 and

$$\mathrm{mc}_{x_0,\mu} = \mathrm{Ad}\big((x_p - x_q)^{-\lambda}\big) \circ \mathrm{mc}_{x_0,\mu} \circ \mathrm{Ad}\big((x_p - x_q)^{\lambda}\big) \quad (1 \le p < q \le n-1).$$

(ii) The generalized Riemann scheme

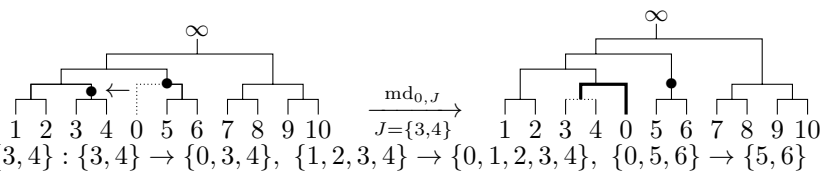
$$\big\{[\bar{A}_{ij}] \mid \{i,j\} \subset \tilde{L}_n\big\}$$

of the residue matrices $\mathrm{mc}_{x_0,\mu}\mathcal{M}$ is obtained by Theorem 5.1 together with $\mathrm{Sp}\,\mathcal{M}$.

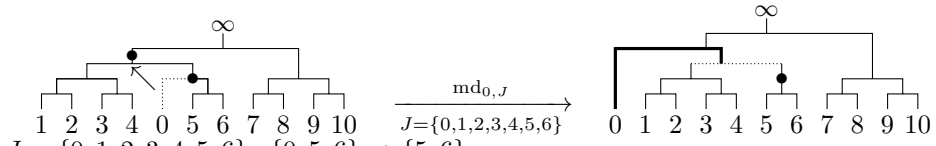
We now examine the relationship between the theorem and tournaments. We represent $\mathcal{I}$ by its corresponding tournament, and denote the team with label i as (i). The theorem describe the transformation of $\mathrm{Sp}\,\mathcal{M}$ in terms of the matrices A^K_I defined in (5.7) for $I \in \mathcal{I}$. The corresponding transformation $\mathrm{md}_{0,I}$ of the tournament $\mathcal{I}$ can be viewed as inserting (0) after deleting it. The deleted game is replaced by the preceding game and the newly inserted game, which becomes the first game involving (0), is determined as follows.

- $(A^J_I)_{I\in\mathcal{I}}$: the new game is played with the winner of the game J and the preceding game of the opponent is replaced by the new game.
- $(A^j_I|_{\ker A_{0j}})_{I\in\mathcal{I}}$: the new game is played with (j) as a basic insertion.
- $(A^\infty_I|_{\ker(A_{0\infty}-\mu)})_{I\in\mathcal{I}}$: the new game is played with the winner of the tournament as a top insertion.

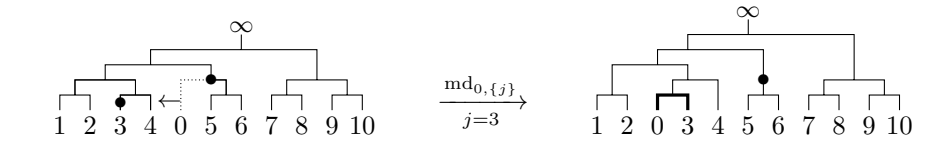
We illustrate the above correspondence by means of diagrams and examples below:



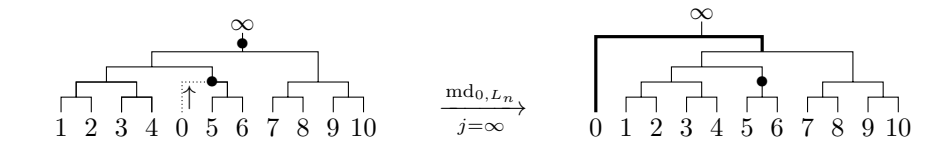
$J = \{3,4\} : \{3,4\} \to \{0,3,4\},\ \{1,2,3,4\} \to \{0,1,2,3,4\},\ \{0,5,6\} \to \{5,6\}$



$J = \{0,1,2,3,4,5,6\} : \{0,5,6\} \to \{5,6\}$



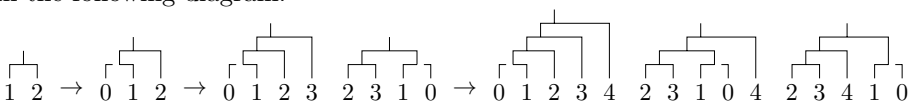
$j = 3 : \{3,4\} \to \{0,3,4\},\ \{1,2,3,4\} \to \{0,1,2,3,4\},\ \{0,5,6\} \to \{5,6\},\ +\{0,3\}$



$j = \infty : \{0,5,6\} \to \{5,6\},\ \{0,1,\dots,6\} \to \{1,\dots,6\},\ \{0,1,\dots,10\} \to \{1,\dots,10\}$

Remark 5.12. Middle convolutions $\mathrm{mc}_{x_0,\mu}$, additions $\mathrm{Ad}\big((x_i - x_j)^\lambda\big)$ and permutations of suffices $\{0,1,\dots,n-1\}$ acts as transformations on the space of KZ-type equations. The induced change of spectra $\mathrm{Sp}\,\mathcal{M}$ is obtained by Theorem 5.1.

We examine the necessary data of $\mathcal{M}$ to get $[A_{12}]$ of an equation which is obtained by a successive application of these transformation to the original KZ-type equation $\mathcal{M}$. Since $[\tilde{A}_{12}|_{\mathcal{K}_1}] = [A_{012}|_{\ker A_{01}}]$, we need $[A_{01} : A_{012}]$ considering additions. Considering more permutations, we need $[A_{12} : A_{123}]$ and $[A_{23} : A_{123}]$ in general. Considering a middle convolution with respect to x_0, we moreover need $[A_{01} : A_{012} : A_{0123}]$ and $[A_{01} : A_{23} : A_{0123}]$. These considerations are represented schematically in the following diagram.



These changes of patterns correspond to basic insertions. Since any tournament is obtained by a successive basic insertions (cf. Remark 2.1), the full spectral data $\mathrm{Sp}\,\mathcal{M}$ are required in general.

On the other hand, convolutions, additions and permutations of suffices define transformation on the space $\{[A_I] \mid I \subset L_n\}$ which does not contain simultaneous eigenspace decompositions. To get the eigenspace decomposition of a residue matrix or a commuting family of residue matrices of the equation obtained by applying middle convolutions and additions to an original KZ-type equation $\mathcal{M}$, the necessary data contained in $\mathrm{Sp}\,\mathcal{M}$ is depend on the procedure of the application, for example, whether $\ker A_{0j}$ is zero or not. In practice, it is useful to identify the necessary data for actual computations of simultaneous eigenspace decompositions of commuting residues of $\mathcal{M}$. For example, we refer to [10, Theorem 4.1] or §7.4 when the middle convolutions are restricted only on some variables.

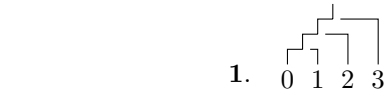
6. Examples

The transformation of $\mathrm{Sp}\,\mathcal{M}$ under the middle convolution of $\mathcal{M}$ is obtained from Theorem 5.1. Since the transformation is symmetric with respect to the indices $\{1,\dots,n-1\}$ of the variables $(x_0,\dots,x_{n-1})$, it suffices to examine the transformation of the maximal commuting families of residue matrices corresponding to the

representatives of win-types. For $n = 3,\ 4,\ 5,\ 6,\ 7, \ldots$, the number of win-types are $W_n = 2,\ 4,\ 9,\ 20,\ 46, \ldots$, respectively.

In this section we fix $n = 4$ and examine the $W_4 = 4$ cases. The results remain valid under any permutation of the indices $\{1, 2, 3\}$. Note that the total number of maximal commuting families of residue matrices is $K_4 = 14$. For simplicity, we assume that $\mathcal{M}$ is homogeneous. Then $A_{0123} = 0,\ \tilde{A}_{01234} = \mu$.

In the following examples, $\mathcal{I} = \{I^{(1)}, I^{(2)}, I^{(3)}\}$ denotes a maximal commuting family whose elements are arranged in the order given in (5.3), and $U = U_{b^0(\mathcal{L})}$ is the block matrix in Theorem 5.1 that transforms each $\mathcal{A}_I$ into a block upper-triangular form, and E_N and O_N in Definition 5.7 are denoted simply by 1 and 0, respectively.

1. 

$$\mathcal{I} = \big\{\{0,1\},\{0,1,2\},\{0,1,2,3\}\big\} \xrightarrow{b^0} \big\{\{1\},\{2\},\{3\}\big\}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{01} = \begin{pmatrix} A_{01}{+}\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} A_{01}{+}\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{012} = \begin{pmatrix} A_{012}{+}\mu & 0 & A_{03} \\ 0 & A_{012}{+}\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{012}{+}\mu & 0 & A_{03} \\ 0 & A_{012}{+}\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}] = \{[A_{01} + \mu : A_{012} + \mu], [0 : A_{012} + \mu], [0 : A_{12}]\}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_1} = [A_{01} + \mu : A_{012} + \mu]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_2} = [0 : A_{012} + \mu]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_3} = [0 : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_\infty} = [0 : A_{12}]|_{\ker (A_{0\infty} - \mu)}$$

2.

$$\mathcal{I} = \big\{\{0,1,2\},\{1,2\},\{0,1,2,3\}\big\} \xrightarrow{b^0} \big\{\{1,2\},\{1\},\{3\}\big\}$$

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{012} = \begin{pmatrix} A_{012}{+}\mu & 0 & A_{03} \\ 0 & A_{012}{+}\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{012}{+}\mu & 0 & A_{03} \\ 0 & A_{012}{+}\mu & 0 \\ 0 & 0 & A_{12} \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} A_{012}{-}A_{01} & -A_{02} & 0 \\ -A_{01} & A_{012}{-}A_{02} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{12} & -A_{01} & 0 \\ 0 & A_{012} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}$$

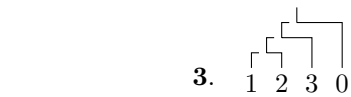
$$[\tilde{A}_{012}:\tilde{A}_{12}] = \{[A_{012}+\mu:A_{12}],[A_{012}+\mu:A_{012}],[A_{12}:A_{12}]\}$$
$$[\tilde{A}_{012}:\tilde{A}_{12}]|_{\mathcal{K}_1} = [A_{012}+\mu:A_{012}]|_{\ker A_{01}}$$
$$[\tilde{A}_{012}:\tilde{A}_{12}]|_{\mathcal{K}_2} = [A_{012}+\mu:A_{012}]|_{\ker A_{02}}$$
$$[\tilde{A}_{012}:\tilde{A}_{12}]|_{\mathcal{K}_3} = [A_{12}:A_{12}]|_{\ker A_{03}}$$
$$[\tilde{A}_{012}:\tilde{A}_{12}]|_{\mathcal{K}_\infty} = [A_{12}:A_{12}]|_{\ker (A_{0\infty}-\mu)}$$

A_I^J (cf. Theorem 5.1)

$J\backslash I$	$\widetilde{012}$	$\widetilde{12}$	$\widetilde{0123}$
012	012+μ	12	μ
12	012+μ	012	μ
0123	12	12	μ

$j\backslash I$	$\widetilde{012}$	$\widetilde{12}$	$\widetilde{0123}$
1	012+μ	012	μ
2	012+μ	012	μ
3	12	12	μ
∞	12	12	μ

Remark 6.1. The simultaneous eigenspace decomposition of $\left(\tilde{A}_I\right)_{I\in\mathcal{I}}$ is obtained by A_I^J in Theorem 5.1. The above left table is the $(n-1)\times(n-1)$ matrix whose (J,I)-element with $I\in\mathcal{I}$ and $J\in\mathcal{I}$ gives the suffix S such that $A_S = A_I^J$. Moreover $012+\mu$ in the table means $A_{012}+\mu$. Then S contains 0 if and only if $I\supset J$ and the term "$+\mu$" appears if and only if $0\in I\supset J$. Here the last column corresponding to $\tilde{A}_{0123}$ is omitted. Similarly, the above right table shows $A_I^{\{j\}}$ and $A_I^{L_n}$ which describe $\left(\tilde{A}_I|_{\mathcal{K}_j}\right)_{I\in\mathcal{I}}$ and $\left(\tilde{A}_I|_{\mathcal{K}_\infty}\right)_{I\in\mathcal{I}}$, respectively.

3. 

$$\mathcal{I} = \{\{0,1,2,3\},\{1,2,3\},\{1,2\}\} \xrightarrow{b^0} \{\{1,2,3\},\{1,2\},\{1\}\}$$
$$U = \begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix},\quad V = U^{-1} = \begin{pmatrix}0&0&1\\0&1&-1\\1&-1&0\end{pmatrix},\quad \tilde{A}_* \to V\tilde{A}_*U$$
$$\tilde{A}_{123} = \begin{pmatrix}-A_{01}&-A_{02}&-A_{03}\\-A_{01}&-A_{02}&-A_{03}\\-A_{01}&-A_{02}&-A_{03}\end{pmatrix} \to \begin{pmatrix}A_{123}&-A_{01}-A_{02}&-A_{01}\\0&0&0\\0&0&0\end{pmatrix}$$
$$\tilde{A}_{12} = \begin{pmatrix}A_{012}-A_{01}&-A_{02}&0\\-A_{01}&A_{012}-A_{02}&0\\0&0&A_{12}\end{pmatrix} \to \begin{pmatrix}A_{12}&0&0\\0&A_{12}&-A_{01}\\0&0&A_{012}\end{pmatrix}$$
$$[\tilde{A}_{123}:\tilde{A}_{12}] = \{[A_{123}:A_{12}],[0:A_{12}],[0:A_{012}]\}$$
$$[\tilde{A}_{123}:\tilde{A}_{12}]|_{\mathcal{K}_1} = [0:A_{012}]|_{\ker A_{01}}$$
$$[\tilde{A}_{123}:\tilde{A}_{12}]|_{\mathcal{K}_2} = [0:A_{012}]|_{\ker A_{02}}$$
$$[\tilde{A}_{123}:\tilde{A}_{12}]|_{\mathcal{K}_3} = [0:A_{12}]|_{\ker A_{03}}$$
$$[\tilde{A}_{123}:\tilde{A}_{12}]|_{\mathcal{K}_\infty} = [A_{123}:A_{12}]|_{\ker (A_{0\infty}-\mu)}$$

4.

$$\mathcal{I} = \{\{0,1\},\{0,1,2,3\},\{2,3\}\} \xrightarrow{b^0} \{\{1\},\{2,3\},\{2\}\}$$
$$U = \begin{pmatrix}1&0&0\\0&1&1\\0&1&0\end{pmatrix},\quad V = U^{-1} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&-1\end{pmatrix},\quad \tilde{A}_* \to V\tilde{A}_*U$$
$$\tilde{A}_{01} = \begin{pmatrix}A_{01}+\mu&A_{02}&A_{03}\\0&0&0\\0&0&0\end{pmatrix} \to \begin{pmatrix}A_{01}+\mu&A_{03}+A_{02}&A_{02}\\0&0&0\\0&0&0\end{pmatrix}$$

$$\tilde{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{023}-A_{02} & -A_{03} \\ 0 & -A_{02} & A_{023}-A_{03} \end{pmatrix} \to \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23} & -A_{02} \\ 0 & 0 & A_{023} \end{pmatrix}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}] = \{[A_{01}+\mu : A_{23}], [0 : A_{23}], [0 : A_{023}]\}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_1} = [A_{01}+\mu : A_{23}]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_2} = [0 : A_{023}]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_3} = [0 : A_{023}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_\infty} = [0 : A_{23}]|_{\ker (A_{0\infty}-\mu)}$$

Case **1**

$\widetilde{A}$	$\widetilde{01}$	$\widetilde{012}$
01	01+μ	012+μ
012	0	012+μ
0123	0	12
1	01+μ	012+μ
2	0	012+μ
3	0	12
∞	0	12

Case **3**

$\widetilde{A}$	$\widetilde{123}$	$\widetilde{12}$
0123	123	12
123	0	12
12	0	012
1	0	012
2	0	012
3	0	12
∞	123	12

Case **4**

$\widetilde{A}$	$\widetilde{01}$	$\widetilde{23}$
01	01+μ	23
0123	0	23
23	0	023
1	01+μ	23
2	0	023
3	0	023
∞	0	23

We now present a computer program in [6] that reproduces the results in this section using computer algebra system `Risa/Asir`. The resulting data, including the tournament figures, are displayed in a PDF file generated via TEX. The examples for $n \leq 6$ obtained by this program are in [14].

```
/*  Generate TeX source of Examples 1-4 in this section  */
 N=4;    /* N-2=2 variables HG, N=5 : 3 variables HG */
 T=os_md.symtournament(N|to="T");       /* T: types  */
 R=os_md.xytournament(T,0|verb=21,winner="all");
                                        /* R: win-types */
 for(S="";R!=[];R=cdr(R)){
   C=car(R);
   S=S+"\\raisebox{-2mm}{"              /* S: source of TeX */
    +os_md.xytournament(C[0],0|teams=C[1],winner=0) /* fig. */
    +"}\\qquad"+rtostr(C[4])+"\\ $\\to$ \\ "+rtostr(C[3])
    +os_md.midKZ(C[2],C[3]);            /* spectra */
 }
 os_md.dviout(S);                       /* display */
```

In the above program

- In the 2-nd line, the number of variables $n = 4$ is set by `N=4`.
- In the 3-rd line, all types `T` for `N` teams are computed.
- In the 4-th line, all win-types $\mathcal{I}$ are obtained and stored in `R`.
- From the 6-th line on wards, $\mathrm{Sp}(\mathrm{mc}_{x_0,\mu}\mathcal{M})$ and the results shown in this section are converted into a TEX source string `S`, and finally displayed as a PDF file.

Remark 6.2. The top insertion embeds the tournaments of $n-1$ teams into those of n teams. The image of this embedding consists of the tournaments of n teams satisfying $b^0(L_n) = \{n-1\}$. This corresponds to the KZ-type equation $\mathcal{M}$ with $A_{i,n-1} = 0$ for $0 \leq i \leq n-2$. Hence, the results for KZ-type equations with $n-1$ variables follow from those with n variables.

In the first two cases in this section correspond to this embedding, and the results for $n = 3$ are obtained by omitting $b^0(\{0, 1, 2, 3\}) = \{3\}$. That is, they are derived from leading 2×2 blocks of the matrices in these examples. Moreover, we omit the term $\mathcal{K}_3$ and the final terms in the simultaneous eigenspace decompositions, while the term A_{012} may remain.

7. Further considerations

7.1. **Single-elimination tournaments.** We summarize the relation between the KZ-type equations and single-elimination tournaments discussed in this paper :

KZ-type equation with n variables	Tournament of n teams
Family of maximal commuting residue matrices	Tournament
Spectra of KZ-type equation	Set of all tournaments
Singular points	Semi-final games
Local coordinates for desingularization	Result of tournaments prior to semi-final
Variable of middle convolution	Winner of tournament
Base of upper triangulation of the family	Result of all games
Middle convolution	Deletion and insertion of the winner
Kernels to define middle convolution	Basic/Top insertion of the winner
With other m fixed singular points (§7.4)	Division of n teams into m groups

In the above table, for example, "Result of tournaments prior to semi-final" corresponds to the pairs (n_i, n_i') for $I_i \neq L_n$ in Definition 4.1.

7.2. **Infinite points.** The KZ-type equation $\mathcal{M}$ in §3 is defined on the configuration space of $n+1$ points of $\mathbb{P}^1$. By applying a linear fractional transformation that maps the point at infinity to a finite point, we obtain a KZ-type equation in $n+1$ variables that has no singularity at infinity. Then all singular points become finite, which may make it easier to study the symmetry among them. If the original equation has $n-1$ variables, the transformed KZ-type equation has n variables and is characterized by the condition

$$A_{i\infty} := -\sum_{\nu=0}^{n-1} A_{i\nu} = 0 \qquad (0 \leq i < n) \tag{7.1}$$

on the residue matrices A_{ij}. Hence we assume the following.

Definition 7.1. We call ∞ a *pseudo-singular point* if there exist $\mu_i \in \mathbb{C}$ such that

$$A_{i\infty} = \mu_i \qquad (0 \leq i < n). \tag{7.2}$$

Here, each μ_i mean a scalar matrix.

If we apply $\mathrm{Ad}\big((x_0-x_1)^{\lambda}(x_0-x_2)^{\lambda}(x_1-x_2)^{-\lambda}\big)$ to $\mathcal{M}$, $A_{0\infty}$ changed into $A_{0\infty}-\lambda$ and $A_{i\infty}$ for $i \neq 0$ are unchanged. Hence, a KZ-tye equation with a pseudo-singular point at infinity can be transformed into one with a non-singular point at infinity by suitable additions, namely, into an equation satisfying (7.1). We examine the middle convolution of $\tilde{A}_{i\infty}$. Note that

$$\tilde{A}_{0\infty} = -\sum_{\nu=1}^{n-1} A_{0\nu} = \begin{pmatrix} -A_{01}-\mu & -A_{02} & \cdots & -A_{0,n-1} \\ -A_{01} & -A_{02}-\mu & \cdots & -A_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{01} & -A_{02} & \cdots & -A_{0,n-1}-\mu \end{pmatrix},$$

$$\tilde{A}_{1\infty} = -\sum_{\nu=0}^{n-1} A_{1\nu} = \begin{pmatrix} A_{0\infty}+A_{1\infty}+A_{01}-\mu & 0 & \cdots & 0 \\ A_{01} & A_{1\infty} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{01} & 0 & \cdots & A_{1\infty} \end{pmatrix}.$$

Hence, for the variable x_0, the middle convolution of the KZ-type equation $\mathcal{M}$ satisfying (7.2) is defined by mc_{x_0,μ_0}. Then

$$\widetilde{A}_{0\infty}(v)_{L_n^0} = \big((A_{0\infty}-\mu)v\big)_{L_n^0}, \qquad \widetilde{A}_{1\infty}(v)_{L_n^0} = \big((A_{01}+A_{1\infty})v\big)_{L_n^0},$$
$$\widetilde{A}_{0\infty} = -\mu \mod V_{L_n^0}, \qquad \widetilde{A}_{i\infty} = \mu_i \mod V_{L_n^0} \quad (1 \le i < n)$$

under the notation in Definition 5.1, and therefore

$$\mathcal{K}_\infty = V_{L_n^0}, \quad \bar{A}_{0\infty} = -\mu, \quad \bar{A}_{i\infty} = \mu_i \quad (0 < i < n) \tag{7.3}$$

and $\mathrm{mc}_{x_0,\mu_0}\mathcal{M}$ also has a pseudo-singular point at infinity.

7.3. **Spectra and accessory parameters.** A basic problem for KZ-type equations is to describe the image and fiber of the map $\mathcal{M} \mapsto \mathrm{Sp}\,\mathcal{M}$. We discuss several aspects of this problem below.

Consider the Fuchsian ordinary differential equation

$$\mathcal{N} : \frac{du}{dx_0} = \frac{A_{01}}{x_0-x_1}u + \frac{A_{02}}{x_0-x_2}u \tag{7.4}$$

with three singular points x_1, x_2, and ∞. Let

$$A_{12} = -A_{01} - A_{02}, \tag{7.5}$$

that is, $A_{12} = A_{0,\infty}$. Then $\mathcal{N}$ extends to the KZ-type equation $\mathcal{M}$ in the variables (x_0, x_1, x_2). Conversely, one may assume (7.5) for an irreducible KZ-type equation with $n = 3$ (cf. Definition 3.5).

In general, $\mathcal{N}$ is not rigid and may involve $2r$ accessory parameters ($r = 0, 1, \ldots$). For example, if A_{01} and A_{02} are generic matrices in $M(3,\mathbb{C})$, then $\mathcal{N}$ has two accessory parameters. Additions and middle convolutions are invertible transformations that preserve the number of accessory parameters. Hence we obtain KZ-type equations in n variables with $2r$ accessory parameters ($n = 3, 4, \ldots$). Conversely, the number of accessory parameters of a KZ-type equation can be determined once it is reduced to a KZ-type equation with three variables.

A KZ-type equation $\mathcal{M}$ is said to be *rigid* if it is completely determined by $\mathrm{Sp}\,\mathcal{M}$ and has no accessory parameters. It is of interest to study whether a given KZ-type equation can be transformed into another KZ-type equation through successive applications of the above transformations. This problem is particularly significant for rigid equations, as well as for identifying irreducible KZ-type equations whose rank cannot be reduced by any such transformation.

A KZ-type equation $\mathcal{M}$ obtained from the trivial equation $u' = 0$ by a successive applications of these transformations is rigid. In this case, the symmetries of $\mathrm{Sp}\,\mathcal{M}$ yield various relations among the solutions, analogous to Kummer's relations for the Gauss hypergeometric functions (cf. [5, Remark 5.17]).

Moreover, in the general holonomic setting, one can describe $\mathrm{Sp}\,\mathcal{M}$ in terms of residue matrices as follows. For a holonomic system $\mathcal{M}$, we blow up its singular locus so that the resulting singularities have normal crossings, and obtain commuting residue matrices associated with the normal crossing divisors (cf. [3]). The *spectra* $\mathrm{Sp}\,\mathcal{M}$ is defined as the set of conjugacy classes of these commuting residue matrices at the normal crossing singular points. When the residue matrices are semisimple, each conjugacy class corresponds to the set of simultaneous eigenvalues together with their multiplicities.

7.4. **Fixed singular points.** We study the KZ-type equation

$$\mathcal{M} : \frac{\partial u}{\partial x_i} = \sum_{\substack{0\le\nu\le n-1\\ \nu\ne i}} \frac{A_{i\nu}}{x_i-x_\nu}u + \sum_{q=1}^{m}\frac{B_{iq}}{x_i-y_q}u \qquad (i = 0,\ldots,n-1) \tag{7.6}$$

which possesses singular loci defined by $x_i = y_q$ for fixed points $y_q \in \mathbb{C}$ $(q = 1, \dots, m)$ in addition to those defined by $x_i = x_j$. Without loss of generality, we may assume that $\mathcal{M}$ has a pseudo-singular point at infinity.

For $\{i_1, \dots, i_p\} \subset \{0, \dots, n-1\}$ and $\{j_1, \dots, j_q\} \subset \{n, \dots, n+m-1\}$, we define

$$A_{i_1,\dots,i_p;j_1,\dots,j_q} := \sum_{1 \le \nu < \nu' \le p} A_{i_\nu i_{\nu'}} + \sum_{\substack{1 \le \nu \le p \\ 1 \le \nu' \le q}} B_{i_\nu j_{\nu'} - n + 1}.$$

We regard $y_j = x_{n-1+j}$ for $j = 1, \dots, m$. Observe that

$$A_{i_1,\dots,i_p;j_1,\dots,j_q} = A_{i_1,\dots,i_p,j_1,\dots,j_q} - A_{j_1,\dots,j_q},$$

where the terms $A_{j_\nu j_{\nu'}}$ on the right-hand side cancel.

The integrability condition for $\mathcal{M}$ is given by

$$\begin{aligned} [A_{ij}, A_{k\ell}] = [A_{i;q}, A_{j;q'}] = [A_{ij}, A_{k;q}] = 0, \\ [A_{ij}, A_{ijk}] = [A_{ij}, A_{ij;q}] = [A_{i;q}, A_{ij;q}] = 0, \end{aligned} \tag{7.7}$$

where i, j, k, $\ell \in L_n$ and q, $q' \in \{n, n+1, \dots, n+m-1\}$ are distinct. Since, for instance,

$$[A_{01}, A_{01\cdots k;q}] = [A_{01}, \sum_{0 \le i < j \le k} A_{i,j} + A_{01;q} + \sum_{i=2}^{k} A_{i;q}] = 0,$$

it follows that

$$\begin{aligned} [A_I, A_J] &= 0 \qquad (I \cap J = \emptyset \text{ or } I \subset J \text{ or } I \supset J), \\ [A_I, A_{J;q}] &= 0 \qquad (I \cap J = \emptyset \text{ or } I \subset J), \\ [A_{I;q}, A_{J;q'}] &= 0 \qquad (I \cap J = \emptyset \text{ and } q \ne q') \end{aligned} \tag{7.8}$$

for I, $J \subset \{0, 1, \dots, n-1\}$ and $\{q, q'\} \subset \{n, n+1, \dots, n+m-1\}$.

Throughout this subsection, we assume condition (7.2) holds.

Definition 7.2. Let (L, L') be a pair of finite disjoint sets. A *maximal commuting family* $\mathcal{I}$ of (L, L') is defined as follows. Write

$$L_n = \bigsqcup_{j \in L'} S_j,$$

and let each $\mathcal{I}^{(j)}$ be a maximal commuting family of $S_j \cup \{j\}$. Then $\mathcal{I} = \{\mathcal{I}^{(j)} \mid S_j \ne \emptyset,\ j \in L'\}$. Here $\mathcal{I}^{(j)} = \emptyset$ if $S_j = \emptyset$.

Moreover, putting $L' = \{r_1, \dots, r_m\}$, we define

$$\widehat{\mathcal{I}} := \mathcal{I} \cup \bigcup_{j=2}^{m} \{\widehat{S}_j\}, \qquad \widehat{S}_j := \bigcup_{\nu=1}^{j} \big(S_\nu \cup \{r_\nu\}\big). \tag{7.9}$$

Here, n and m denote the numbers of the variables and the fixed points, respectively, and we have $n = |S_1| + \cdots + |S_m|$.

The middle convolution can be applied to $\mathcal{M}$ with respect to any variable $x_0, \dots, x_{n-1}$. The resulting changes in the eigenspace decompositions of the residue matrices are determined by Theorem 5.1. We consider the residue matrices in (7.3) associated with $\mathcal{I}$, and take their basis with respect to $\widehat{\mathcal{I}}$. In particular, when $n = 1$, this reduces to the result obtained in [1].

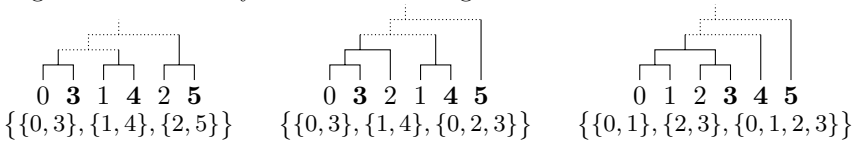
Remark 7.3. The tournament corresponding to $\widehat{\mathcal{I}}$ is characterized as follows: The teams $(n+m-j)$ and $(n+m-1-j)$ face each other in a game $j-1$ rounds prior to the final, provided that both have won their preceding games $(j = 1, \dots, m)$.

Example 7.4. The representatives of the maximal commuting families of $(L, L') = (\{0,1,2\},\{3,4,5\})$ under permutations of the elements of L and those of L' are

$$\begin{aligned}
&1+1+1 : \big\{\{0,3\},\{1,4\},\{2,5\}\big\},\\
&2+1+0 : \big\{\{0,3\},\{1,4\},\{0,2,3\}\big\}, \quad \big\{\{0,1\},\{2,3\},\{0,1,4\}\big\},\\
&3+0+0 : \big\{\{0,1\},\{2,3\},\{0,1,2,3\}\big\}, \quad \big\{\{0,1\},\{0,1,2\},\{0,1,2,3\}\big\},\\
&\qquad\quad \big\{\{0,1\},\{0,1,3\},\{0,1,2,3\}\big\}, \quad \big\{\{0,3\},\{0,1,3\},\{0,1,2,3\}\big\}.
\end{aligned}$$

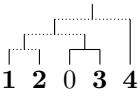
When $S_3 = \{0,1,2\}$ and $S_4 = S_5 = \emptyset$, we have 4 families corresponding to the case $3+0+0$ $(= |S_1| + |S_2| + |S_3|)$ above, since $W_{3+1} = 4$. The numbers of maximal commuting families are $3K_4 = 45$, $3! \cdot 3K_3 = 54$, and 6 in the cases $3+0+0$, $2+1+0$, and $1+1+1$, respectively. The total number of such families is 105.

We present tournament diagrams corresponding to commuting families, where the games indicated by dotted lines are ignored.



Example 7.5 (ordinary differential equation)**.** Suppose $n = 1$. Then there are m maximal commuting families of (L, L'). For example, when $m = 4$, we have

$$\big\{\mathcal{I}\big\} = \big\{\{\{0,1\}\},\{\{0,2\}\},\{\{0,3\}\},\{\{0,4\}\}\big\}, \quad \widehat{\mathcal{I}} = \{\widehat{\{0,3\}}\} \ :$$



and the spectra $\operatorname{Sp}\mathcal{M} = \big\{[A_{0;1}],[A_{0;2}],[A_{0;3}],[A_{0;4}]\big\}$ is the Riemann scheme of the ordinary differential equation

$$\mathcal{M} : \frac{du}{dx} = \frac{A_{0;1}}{x-y_1}u + \frac{A_{0;2}}{x-y_2}u + \frac{A_{0;3}}{x-y_3}u \qquad (A_{0;1}+A_{0;2}+A_{0;3} = -\mu_0 \in \mathbb{C}).$$

Remark 7.6. Suppose $m = 1$. The integrability condition of the KZ-type equation $\mathcal{M}$ equals to that of the extended KZ-type equation with $n+1$ variables which are imposed on residue matrices A_{ij} $(0 \le i < j \le n)$. Hence the KZ-type equation adding the partial derivation with respect to the variable $x_n = y_1$ satisfies the integrability condition. On the other hand, the product of a solution to the extended equation and any function of x_n satisfies the original equation. Therefore, the space of solutions to the original equation with variables $(x_0,\dots,x_n)$ is infinite-dimensional.

7.5. **Semilocal monodromy.** We extends a result on ordinary differential equations given in [11] to KZ-type equations. As an example, [11] computes $[\widetilde{A}_{03} + \widetilde{A}_{04}]$ from $[A_{0i}]$ and $[A_{03} + A_{04}]$ etc. in the case $L = \{0\}$ and $L' = \{1,2,3,4\}$. Here use the notation in the previous subsection, and $\widetilde{A}_{03}$ and $\widetilde{A}_{04}$ are convolutions of A_{03} and A_{04}, respectively.

We focus on the computation of $[\widetilde{A}_{03} + \widetilde{A}_{04}]$ for $L = \{0,1,2\}$ and $L' = \{3,4,5\}$. Formally, we have $A_{03} + A_{04} = A_{034} - A_{34}$, and we proceed formally as though A_{34} and similar terms were defined.

This corresponds to the tournaments in which teams 3 and 4 meet in a semi-final or quarter-final game if they have advanced through earlier rounds. We may further restrict to tournaments in which teams 3 and 4 play a semi-final or quarter final game if they have advanced through earlier rounds.

For $\{0,1,2\} \supset I_1 \supset I_2$, the difference $\widetilde{A}_{I_1 34} - \widetilde{A}_{I_2 34}$ is expressed by residue matrices in (7.6).

To derive the eigenspace decomposition of $\widetilde{A}_{I_1 34} - \widetilde{A}_{I_2 34}$ from the residue matrices A_{ij} of (7.6), we consider tournaments containing the games corresponding to $I_1 34$ and $I_2 34$ and apply Theorem 5.1 to the maximal commuting families of residue matrices associated with these tournaments. The resulting simultaneous eigenspace decompositions involve $A_{I'_1 34}$ and $A_{I'_2 34}$ with $I'_1 \supset I'_2$, but it suffices to compute those corresponding to the differences $A_{I'_1 34} - A_{I'_2 34}$. In this setting, it may occur that $I'_1 = I'_2$ even when $I_1 \supsetneqq I_2$.

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Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro, Tokyo 153-3914, Japan

Email address: `oshima@ms.u-tokyo.ac.jp`