MIDDLE CONVOLUTIONS OF KZ-TYPE EQUATIONS AND SINGLE-ELIMINATION TOURNAMENTS

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Abstract. We introduce an extension of the generalized Riemann scheme for Fuchsian ordinary differential equations in the case of KZ-type equations. This extension describes the local structure of equations obtained by resolving the singularities of KZ-type equations. We present the transformation of this extension under middle convolutions. As a consequence, we derive the corresponding transformation of the eigenvalues and multiplicities of the residue matrices of KZ-type equations under middle convolutions. We interpret the result in terms of the combinatorics of single-elimination tournaments.

1. Introduction

A Fuchsian system

$$\mathcal{N}: \frac{du}{dx} = \sum_{\nu=1}^{n-1} \frac{A_{0\nu}}{x - x_\nu} u$$

for a column vector u of N unknown functions has singularities at n points $x_1, \dots, x_{n-1}, \infty \in \mathbb{C} \cup \{\infty\}$. The local structure of the solution to $\mathcal{N}$ near a singularity x_ν is characterized by the conjugacy class of the residue matrix $A_{0\nu} \in M(N, \mathbb{C})$; that is, by the eigenvalues and their multiplicities. The residue matrix at ∞ is give by $A_{0\infty} = -(A_{01} + \cdots + A_{0\infty})$. The collection of the eigenvalues and their multiplicities of the residue matrices is referred to as the (generalized) **Riemann scheme** of $\mathcal{N}$. The equation $\mathcal{N}$ is called **rigid** if it is irreducible and uniquely determined by its Riemann scheme. If not, $\mathcal{N}$ has a finite number of accessory parameters. For example, there are 188 rigid Fuchsian systems of order at most 8 as is listed in [7, §13.2.3].

Katz [4] showed that any rigid Fuchsian system $\mathcal{N}$ can be reduced to the trivial equation $u' = 0$ by a sequence of invertible transformations called **additions** and **middle convolutions**. Haraoka [2] extended this result to Knizhnik-Zamolodchikov-type (KZ-type) systems, proving that if $\mathcal{N}$ is rigid, it can be extended to a KZ-type equation

$$\mathcal{M}: \frac{\partial u}{\partial x_i} = \sum_{\substack{0 \le \nu < n \\ \nu \ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u \quad (i = 0, \dots, n-1)$$

by setting $x_0 = x$ and treating the singular points $x_1, \dots, x_{n-1}$ as variables. This extension is achieved via a generalization of the middle convolution to KZ-type equations. The middle convolution of $\mathcal{N}$ was explicitly described by Dettweiler-Reiter [1] in terms of the residue matrices $A_{0\nu}$ following Kat'z definition and Haraoka's result extends this to the KZ-type setting. These transformations, including permutations of the variables $x_0, \dots, x_{n-1}$ and middle convolutions with respect to other variables x_i, preserve the KZ-type structure but do not necessarily the rigidity, in fact, even if the original KZ-type equation has a rigid variable, the transformed system may have none.

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Many known or and new multivariate hypergeometric functions arise with several as solutions to KZ-type equations obtained via these transformations from the trivial equation (cf. [5, 13]). Analyzing such equations via middle convolution proves to be fruitful: in the case of Fuchsian ODE's, most of the results in [7] are obtained using this approach. Since the middle convolution is a microlocal transformation, is is expected to induce corresponding of fundamental data describing singularities such as the eigenvalues and their multiplicities of residue matrices: in the case of ODE's, this is explicitly described in [1]. This issue is resolved in [10, Remark 3] for the case $n = 4$, corresponding to hypergeometric functions of two variables. In this paper we study the problem for general n.

The local solution of $\mathcal{M}$ at the singularity $x_i = x_j$ is determined by the eigenvalues and their multiplicities of the residue matrix A_{ij}. At the singularity $\{x_0 = x_1,\ x_2 = x_3\}$, the local behavior is characterized by the conjugacy class of the commuting pair (A_{01}, A_{23}), and thus, by their simultaneous eigenvalues and their multiplicities. At the singularity $\{x_0 = x_1 = x_2\}$, the key data are eigenvalues and multiplicities of the matrix $A_{012} = A_{01} + A_{02} + A_{12}$ are important data characterizing the local solution. In general, for $I \subset L_n := \{0, 1, \dots, n{-}1\}$ with $|I| > 1$, we define a **generalized residue matrix** A_I as the sum of A_{ij} for all $\{i, j\} \subset L_n$. A **maximal commuting family** $\mathcal{I}$ of such matrices residue matrices is a maximal subset of $\{A_I \mid I \subset L_n,\ |I| > 1\}$ in which all elements commute. We denote by $\operatorname{Sp}\mathcal{M}$ the collection of simultaneous eigenvalues and their multiplicities of these maximal commuting families (Definition 3.8) and call it the **spectrum** of of $\mathcal{M}$. There are $(2n{-}3)!!$ such families, each consisting of $n - 1$ commuting matrices corresponding to the residue matrices along hypersurfaces defining a normal crossing singularity after blowing up a singular point point (cf. §4).

The main result of this paper is that additions and middle convolutions applied to $\mathcal{M}$ induce transformations on $\operatorname{Sp}\mathcal{M}$, and we give explicit formulas for these transformations (Theorem 5.1). For example, we need $\operatorname{Sp}\mathcal{M}$ to get the eigenvalues and their multiplicities of the residue matrix A_{12} of the equation obtained by applying additions and middle convolutions several times to an original KZ-type equation $\mathcal{M}$ (Remark 5.12).

By blowing up the singularities of the equation satisfied by Appell's hypergeometric series F_1, we obtain 15 normal crossing singularities. The simplest KZ-type equation $\mathcal{M}$ is the equation of rank 3 system with $n = 4$ satisfied by F_1. In this case $\operatorname{Sp}\mathcal{M}$ consists of $(2n-3)!! = 15$ decompositions into simultaneous eigenspaces of residue matrices, where the simultaneous eigenvalues are multiplicity-free and coincide with the characteristic exponents at the 15 singular points. Thus, $\operatorname{Sp}\mathcal{M}$ serves as a generalization of the Riemann scheme of $\mathcal{N}$ to the KZ-type equation $\mathcal{M}$ (cf. Example 7.5).

A KZ-type equation $\mathcal{M}$ with $n = 3$ naturally corresponds to a Fuchsian system $\mathcal{N}$ with three singular points (cf. §7.3). Moreover, many multivariate hypergeometric functions in $n{-}2$ variables arise as solutions to KZ-type equations $\mathcal{M}$ ([5], [10]). When $n = 4$, each generalized residue matrix A_I with $|I| = 3$ can be expressed in terms of a residue matrix A_J with $|J| = 2$ (cf. Example 3.10 (i)). In this case, $\operatorname{Sp}\mathcal{M}$ can be interpreted as simultaneous eigenspace decompositions of matrix pairs at normal crossing singular points, and the transformation of $\operatorname{Sp}\mathcal{M}$ under middle convolution is described in [10]. For $n > 4$, the structure of $\operatorname{Sp}\mathcal{M}$ is more intricate, but the combinatorics of single-elimination tournaments provides insighats into its organization.

In §2 some known results of combinatorics related to single-elimination tournaments, such as the number of ways in tournament scenarios, are explained with a focus on the application to the structure of residue matrices of KZ-type equations. In particular, we introduce maximal families of commuting subsets of a finite set which correspond maximal commuting residue matrices. The numbers parametrized by n in this section can be referred to the data base [14].

In §3, after reviewing the integrability condition of a KZ-type equation, we introduce $\mathrm{Sp}\,\mathcal{M}$.

In §4, giving a local coordinate system, we define a resolution of the singular point of a KZ-equation and show that the maximal commuting family of residue matrices equals the set of residue matrices corresponding to the normal crossing divisors which define a singular point in the blowing up.

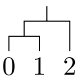
In §5, we examine the transformation of residue matrices under the middle convolution of a KZ-type equation and give our main result Theorem 5.1 in this paper. To state the theorem, we define transformations of maximal families of commuting subsets of a finite set which correspond transformations of tournaments.

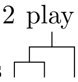
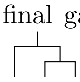
Theorem 5.1 can be applied to analyze hypergeometric functions with several variables. Examples of the application with $n = 4$ are given in [10, §8], [13, §7], [5, §5]. A solution to a rigid Fuchsian system $\mathcal{N}$ with more than 3 singular points can be deeply analyzed through the KZ-equation obtained from $\mathcal{N}$ by the extension of variables. For example, the ordinary differential equation satisfied by Jordan-Pochhammer's hypergeometric function is extended to a KZ-type equation satisfied by Appell's F_1 or Lauricella's F_D. Moreover the theorem can be applied to KZ-type equations with irregular singularities through their versal unfoldings (cf. [12]).

In §6, Theorem 5.1 is explained by examples with $n = 4$.

Under a fractional linear transformation, KZ-type equation $\mathcal{M}$ can be assumed to have no singularity at infinity and in §7.1, we examine the middle convolution of the equation. In §7.2, we study KZ-type equation $\mathcal{M}$ with fixed singular points. In this case, $\mathrm{Sp}\,\mathcal{M}$ corresponds to single-elimination tournaments with a certain restriction. The equation has only one variable, our result corresponds to that in [1]. Accessory parameters and rigidity of a KZ-type equation and $\mathrm{Sp}\,\mathcal{M}$ for a Fuchsian holonomic system $\mathcal{M}$ are explained in §7.3.

2. Single-elimination tournaments

We consider single-elimination tournaments of n teams. In this paper tournaments always mean single-elimination tournaments. A tournament of 3 teams distinguished by the labels 0, 1 and 2 are expressed by 

. This chart means that team 0 and team 1 play first and the winner of this game and team 2 play the final game. If the teams are not distinguished, there are two patterns  and . In the case of the tournaments of n teams, the patterns correspond to binary one-rooted trees with $n-1$ non-leaf nodes and n leaves figured in a plane. The total number of them are given by the Catalan number C_{n-1}. In this case, there are $n-1$ games and each game determines a final winner of some teams by a sub-tournament.

A tournament of three teams is determined by the teams playing the first game and there are 3 cases of the tournaments, which are described by

$$\big\{\{0,1\},\{0,1,2\}\big\},\ \big\{\{1,2\},\{0,1,2\}\big\},\ \big\{\{0,2\},\{0,1,2\}\big\} \tag{2.1}$$

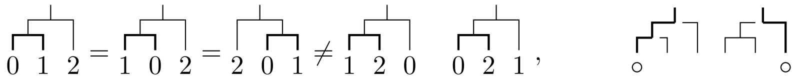
These are the sets of games in the tournaments and each game is labeled by a set of teams such that the game determines the final winner of the teams. Since the final game corresponds to the set of all teams, we may express the tournaments by the games excluding the final games, which we call shortened expression. Namely, the set of tournaments of three teams is

$$\{\{0,1\}\},\ \{\{1,2\}\},\ \{\{0,2\}\} \tag{2.2}$$

in shortened expression.

If a tournament is changed to another tournament by a permutation of teams, we think that these tournaments are isomorphic and call this isomorphic class a type of the tournament. The number of isomorphic classes of tournaments of n teams is the n-th Wedderburn-Etherington number and a tournament can be expressed by $n-1$ subsets or $n-2$ proper subsets of the set of teams.

We moreover consider types of tournaments indicating the final winner and we call them win types. There is one type of the tournaments of 3 teams and there are two win types of them.

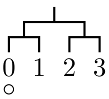


The above left hand side shows the expression of the tournament and the thick segment indicates that the two tournaments selecting two players of the game are isomorphic by a suitable permutation of teams. The right hand side gives the tournaments distinguished only by the final winner indicated by ∘.

In the case of tournaments of three teams there are one type, three cases of tournaments, two patterns and two win types (1 type, 2 patterns, 3 tournaments, 2 win types).

We give examples of the tournaments of 4 and 5 teams in shortened form:

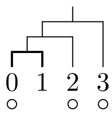
4 teams



: 1 pattern

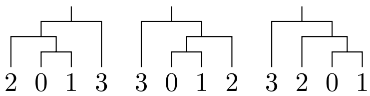
: $\frac{4!}{2^3} = 3$ cases

$\{\{0,1\},\{2,3\}\}$



: 4 patterns

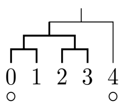
: $\frac{4!}{2} = 12$ cases



$\{\{0,1,2\},\{0,1\}\}$

2 types, 5 (=1+4) patterns, 15 (=3+12) tournaments, 4 (=1+3) win types

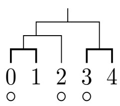
5 teams



: 2 patterns

: $\frac{5!}{2^3} = 15$ cases

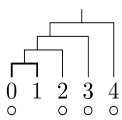
$\{\{0,1\},\{0,1,2,3\},\{2,3\}\}$



: 4 patterns

: $\frac{5!}{2^2} = 30$ cases

$\{\{0,1,2\},\{0,1\},\{3,4\}\}$



: 8 patterns

: $\frac{5!}{2} = 60$ cases

$\{\{0,1,2\},\{0,1\},\{0,1,2,3\}\}$

3 types, 14 patterns, 105 (=15+30+60) tournaments, 9 (=2+3+4) win types

Moreover we have

Numbers of single-elimination tournaments

teams	2	3	4	5	6	7	8	9	n
patterns	1	2	5	14	42	132	429	1430	$T_n = \frac{(2n-2)!}{n!(n-1)!}$
win types	1	2	4	9	20	46	106	248	W_n
types	1	1	2	3	6	11	23	46	U_n
tournaments	1	3	15	105	945	10395	135135	2027025	$K_n = (2n-3)!!$

Here we denote the numbers of tournaments, types, win types, patterns by K_n, U_n, W_n and T_n, respectively, for the tournaments of n teams. Then we have the following recurrence relations.

$$\text{(patterns)} \qquad T_n = \sum_{k=1}^{n-1} T_k \cdot T_{n-k}, \quad T_1 = 1,$$

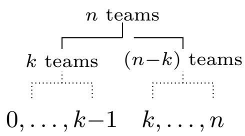
$$\text{(win types)} \qquad W_n = \sum_{k=1}^{n-1} W_k \cdot U_{n-k}, \quad W_1 = 1,$$

$$\text{(types)} \qquad U_n = \frac{1}{2}\Big(\sum_{k=1}^{n-1} U_k \cdot U_{n-k} \,\big(+U_{\frac{n}{2}} \text{ if } n \text{ is even}\big)\Big), \quad U_1 = 1,$$

$$\text{(tournaments)} \qquad K_n = \frac{1}{2}\sum_{k=1}^{n-1} {}_nC_k \cdot K_k \cdot K_{n-k}, \quad K_1 = 1.$$

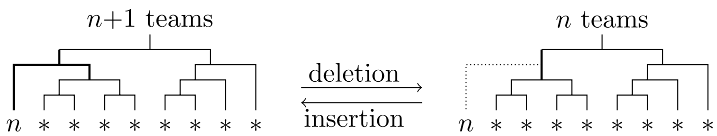
Consider a tournament of n teams. The players of the final game are the winner of k teams and that of the other $n-k$ teams with $k = 1, \dots, n-1$. They are determined by the tournament of k teams and that of $n-k$ teams. Considering the chart of the tournament under a suitable permutation of teams, we have the corresponding numbers of cases as follows.

$$T_n \leftarrow T_k \cdot T_{n-k} \qquad (1 \le k < n),$$
$$W_n \leftarrow W_k \cdot U_{n-k} \qquad (1 \le k < n),$$
$$U_n \leftarrow \begin{cases} U_k \cdot U_{n-k} & (1 \le k < n-k), \\ \frac{1}{2}U_k(U_k - 1) + U_k & (k = n-k), \end{cases}$$
$$K_n \leftarrow \begin{cases} {}_nC_k \cdot K_k \cdot K_{n-k} & (1 \le k < n-k), \\ {}_nC_k\big(\frac{1}{2}K_k(K_k-1)\big) + \big(\frac{1}{2}{}_nC_k\big)K_k & (k = n-k), \end{cases}$$



Hence, the recurrence relations of T_n and W_n are easily obtained because the first k teams and the last $n-k$ teams are distinguished. To get the recurrence relations of U_n and K_n, we may assume $k \le n-k$. Then taking care of the symmetry of the first k teams and the last $n-k$ teams when $k = n-k$, we have their recurrence relations.

Consider a tournament of $n+1$ teams labeled by $0, 1, \dots, n$. If the team n does not participate the tournament, we delete the first game that the team is expected to play and we naturally have a tournament of n teams as the following charts.



On the other hand, for a tournament of n teams labeled by $0, 1, \dots, n-1$, we can insert the first game of the team labeled by n in any one of $2n-1$ vertical line segments in the chart. Namely, the new game is the first game of one of n teams or the opponent of the game is a winner of one of the games of the tournament of n teams.

The pair of a chart of a tournament of n teams and one of its vertical line segments and the chart of a tournament of $n+1$ teams corresponds to each other by this operations, namely, deletion and insertion, respectively. Hence we have

$$K_{n+1} = (2n-1)K_n. \tag{2.3}$$

Moreover we have the following equalities by recurrence relations.

$$K_n = (2n-3)!! = \frac{1}{2}\sum_{k=1}^{n-1} {}_nC_k \cdot (2k-1)!! \cdot (2n-2k-1)!!,$$
$$T_n = \frac{(2n-2)!}{(n-1)!n!} = \sum_{k=1}^{n-k} \frac{(2k-2)!}{(k-1)!k!}\frac{(2(n-k)-2)!}{(n-k-1)!(n-k)!},$$
$$1 = \Bigl(1-\sum_{k=1}^{\infty} U_n x^k\Bigr)\Bigl(\sum_{k=0}^{\infty} W_{k+1}x^k\Bigr).$$

Remark 2.1. (i) We say a deletion of a team is *basic* when the first game of the team is also the first game of the opponent. An insertion of a team to a tournament is called *basic* if the inserted game is the first game for the both players. Note that the inversion of a basic deletion is a basic insertion and moreover at least one basic deletion is possible for any tournament. Hence every tournament of n teams is constructed by a successive application of basic insertions to a tournament of two teams. Consequently, for given two tournaments of n teams, a successive application of suitable $n-1$ pairs of a deletion and a basic insertion transforms one of the tournaments to the other. Here we have only to look at the inserted games in this procedure.

(ii) We consider the insertion which inserts a game played by the new team and the winner of the original tournament, and call it a top insertion. Applying successive top insertions to the game of two teams labeled by 0 and 1 we have a tournament of n teams labeled by $0,\ldots,n-1$ whose n-th game is played by the team with label $n-1$ and the selected team by the former games.

(iii) The 4 diagrams given just before Remark 5.12 show examples of transformations of a tournament by a pair of a deletion and an insertion. The last two diagrams correspond to a basic insertion and a top insertion, respectively.

Let L be a finite set with $|L| > 1$. Here $|L|$ denotes the cardinality of L.

Definition 2.2. $\mathcal{I} = \{I_\nu \mid \nu = 1,\ldots.r\}$ is *a commuting family* of L if $|I_\nu| > 1$ and $I_\nu \subset L$ for $\nu = 1,\ldots,r$ and

$$I_\nu \cap I_{\nu'} = \emptyset \text{ or } I_\nu \subsetneqq I_{\nu'} \text{ or } I_\nu \supsetneqq I_{\nu'} \quad\text{for }\ 1 \le \nu < \nu' \le r.$$

Moreover if there is no commuting family $\mathcal{I}'$ of L satisfying $\mathcal{I} \subsetneqq \mathcal{I}'$, $\mathcal{I}$ is called a *maximal commuting family* of L.

Let $\mathcal{I}$ be a maximal commuting family of L with $|L| > 2$. If there is an element I_0 of $\mathcal{I}$ with $|I_0| = |L|-1$, then $\mathcal{I}\setminus\{L\}$ is a maximal commuting family of I_0. If such L_0 does not exist in $\mathcal{I}$, there exist I_1 and I_2 in $\mathcal{I}$ which satisfy $I_1 \cap I_2 = \emptyset$ and

$$\mathcal{I}\setminus\{L\} = \mathcal{I}_1 \cup \mathcal{I}_2,\ \mathcal{I}_i := \{I \in \mathcal{I} \mid I \subset I_i\} \quad (i = 1, 2).$$

Then $\mathcal{I}_i$ are maximal families of I_i for $i = 1$ and 2. Thus we have the following theorem.

Theorem 2.1. *There is a natural bijection of the set of maximal commuting families of L onto the set of single-elimination tournaments of the teams labelled by L.*

Through the expression of the games of a tournament by the labels of the related teams as in (2.1), a maximal commuting family $\mathcal{I}$ of L corresponds to a single-elimination tournament of teams labeled by the elements of L. In particular

$$|\mathcal{I}| = |L| - 1. \tag{2.4}$$

Definition 2.3. For a maximal commuting family $\mathcal{I}$ of L, we put

$$\widetilde{\mathcal{I}} := \mathcal{I} \cup \bigcup_{\nu \in L} \{\{\nu\}\} \tag{2.5}$$

and define

$$b(I) = \{\bar{b}(I), \bar{b}'(I)\} \qquad (I \in \mathcal{I},\ \bar{b}(I),\, \bar{b}'(I) \in \widetilde{\mathcal{I}}) \tag{2.6}$$

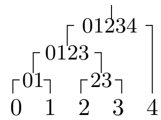
so that $I = \bar{b}(I) \sqcup \bar{b}'(I)$, namely,

$$\bar{b}(I) \cap \bar{b}'(I) = \emptyset, \ \ \bar{b}(I) \cup \bar{b}'(I) = I.$$

Suppose the tournament corresponding to $\mathcal{I}$ is finished. We consider that $\bar{b}$ specifies the losing side for each game in the tournament matches. Moreover, suppose $i \in L$ is the final winner. Then $i \notin \bar{b}(I)$ for $I \in \mathcal{I}$. The map $\bar{b}$ satisfying this condition is denoted by b^i.

Here, we give an example with $L = \{0, 1, 2, 3, 4\}$ and

$$\mathcal{I} = \big\{\{0, 1\}, \{0, 1, 2, 3\}, \{2, 3\}, \{0, 1, 2, 3, 4\}\big\}$$

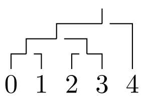


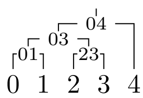
and

$$\begin{aligned} b(\{0,1,2,3\}) &= \big\{\{2,3\},\{0,1\}\big\}, & b^0(\{0,1,2,3\}) &= \{2,3\},\\ b(\{0,1,2,3,4\}) &= \big\{\{4\},\{0,1,2,3\}\big\}, & b^0(\{0,1,2,3,4\}) &= \{4\},\\ b(\{2,3\}) &= \big\{\{2\},\{3\}\big\}, & b^0(\{2,3\}) &= \{2\}. \end{aligned}$$

Indicating the losing side by a gap of segment in a tournament chart, we obtain the following left chart. Then the players of each game is determined by $\bar{b}$ as is indicated in the following right chart. We denote the labels of the players of the game corresponding to $I \in \mathcal{I}$ by $I_{\bar{b}}$. Then

$$I_{\bar{b}} := I \setminus \bigcup_{I \supsetneq J \in \mathcal{I}} \bar{b}(J), \tag{2.7}$$





$$\begin{cases} \{0,1\}_{\bar{b}} = \{0,1\},\\ \{2,3\}_{\bar{b}} = \{2,3\},\\ \{0,1,2,3\}_{\bar{b}} = \{0,3\},\\ \{0,1,2,3,4\}_{\bar{b}} = \{0,4\}. \end{cases}$$

3. Spectra of KZ-type equations

A system of the equations

$$\mathcal{M} : \frac{\partial u}{\partial x_i} = \sum_{\substack{0 \le \nu \le n-1\\ \nu \ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u \quad (i = 0, \ldots, n-1) \tag{3.1}$$

with a vector $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ and matrices $A_{ij} = A_{ji} \in M(N, \mathbb{C})$ is called a Knizhnik-Zamolodchikov-type (KZ-type) equation of rank N (cf. [KZ]). Here $M(N, \mathbb{C})$ denote the space of square matrices of size N with elements in $\mathbb{C}$ and A_{ij} is called the residue matrix along $x_i = x_j$ and they satisfy the integrability condition

$$[A_{ij}, A_{k\ell}] = 0 \qquad \big(\forall \{i, j, k, \ell\} \subset L_n\big), \tag{3.2}$$

$$[A_{ij}, A_{ik} + A_{jk}] = 0 \qquad \big(\forall \{i, j, k\} \subset L_n\big), \tag{3.3}$$

where i, j, k and ℓ are distinct indices. Here and hereafter we use the notation

$$L_n := \{0, 1, \ldots, n-1\}, \quad \tilde{L}_n := L_n \cup \{\infty\}, \quad L_n^i := L_n \setminus \{i\}. \tag{3.4}$$

Definition 3.1. General residue matrices are defined by

$$\begin{aligned} A_{i\infty} &= -(A_{i0} + A_{i1} + \cdots + A_{i,n-1}) \quad (i \in L_n),\\ A_{ii} &= A_\emptyset = A_i = 0,\\ A_{i_1 i_2 \cdots i_k} &:= \sum_{1\le p<q\le k} A_{i_p i_q} \quad \bigl(\{i_1,\dots,i_k\} \subset \tilde L_n\bigr). \end{aligned}$$

The matrix $A_{i\infty}$ is called the residue matrices of $\mathcal{M}$ along $x_i = \infty$.

This definition and the integrability condition implies the following lemma.

Lemma 3.2 ([10, §2])**.** (i) $\displaystyle\sum_{i=0}^{n-1} A_{i\infty} = -2A_{L_n}$,

(ii) *For subsets I and J of $\tilde L_n$, we have*

$$A_I - A_{\tilde L_n \setminus I} = A_{L_n} \qquad \bigl(I \subset L_n\bigr), \tag{3.5}$$

$$[A_I, A_J] = 0 \qquad \bigl(I \cap J = \emptyset \text{ or } I \subset J \text{ or } I \supset J\bigr), \tag{3.6}$$

$$[A_{L_n}, A_I] = 0. \tag{3.7}$$

Since $[A_{L_n}, A_{ij}] = 0$ for $0 \le i < j < n$, we have the following corollary.

Corollary 3.3. *Suppose the system $\mathcal{M}$ is irreducible, namely, there is no nontrivial proper subspace $V \subset \mathbb{C}^N$ satisfying $A_{ij}V \subset V$ for $1 \le i < j < n$. Then*

$$A_{L_n} = \kappa I_N \tag{3.8}$$

with a suitable $\kappa \in \mathbb{C}$.

Definition 3.4. The transformation of $\mathcal{M}$ induced from the map $u \mapsto (x_p - x_q)^\lambda u$ is denoted by $\mathrm{Ad}\bigl((x_p - x_q)^\lambda\bigr)$, which transforms the residue matrix A_{pq} to $A_{pq} + \lambda$ and does not change the other residue matrices $A_{i\nu}$ in (3.1).

If $\mathcal{M}$ is irreducible, $\mathrm{Ad}\bigl((x_p - x_q)^{-\tau}\bigr)$ maps $\mathcal{M}$ to the equation with $A_{L_n} = 0$.

Definition 3.5. A KZ-type equation $\mathcal{M}$ is called **homogeneous** if $A_{L_n} = 0$. In this case, we have

$$\begin{aligned} A_I &= A_{\tilde L_n \setminus I} \qquad (I \subset L_n),\\ A_{i\infty} &= A_{\tilde L_n \setminus \{i\}} \quad (i \in L_n). \end{aligned} \tag{3.9}$$

Remark 3.6. Let $\mathcal{I}$ be a commuting family of subsets of $\tilde L_n$. Then

$$\{I \in \mathcal{I} \mid \infty \notin I\} \cup \{\tilde L_n \setminus I \mid \infty \in I \in \mathcal{I}\}$$

is a commuting family of subsets of L_n.

Remark 3.7. Since the residue matrices are invariant by the coordinate transformation $x_0 \mapsto ax_0 + b$, we may specialize $(x_1, x_2) = (0, 1)$ without loss of generality. By this specialization the number of variables is reduced to $n-2$. Appell's F_1 and F_2 etc. are realized as solutions to certain KZ-type equations with $n = 4$ (cf. [5]).

Definition 3.8. The set $\bigl\{A_I \mid I \in \mathcal{I}\bigr\}$ defined by the maximal commuting family $\mathcal{I}$ of L_n is called a **maximal commuting family of residue matrices** of $\mathcal{M}$.

The result in the former section implies the following.

Corollary 3.9. (i) $|\mathcal{I}| = |L| - 1$.

(ii) *$\mathcal{M}$ has $(2n-3)!!$ maximal commuting families of residue matrices.*

Example 3.10. We give examples of maximal commuting families of residue matrices. Here i, j, k, ℓ, m, p, q are distinct indices. Since the maximal family of residue matrices always contains A_{L_n}, we give the family omitted A_{L_n}

(i) When $n = 4$, the families are

$$\{A_{ij}, A_{k\ell}\},\ \{A_{ij}, A_{ijk}\} \qquad \big(\{i,j,k,\ell\} = \{0,1,2,3\}\big).$$

Moreover if $\mathcal{M}$ is homogeneous, we have $A_{0\infty} = A_{123}$ etc. and these $W_4 = 15$ families cf. [10]) are

$$\{A_{ij}, A_{k\ell}\} \qquad \big(\{i,j,k,\ell\} \subset \{0,1,2,3,\infty\}\big).$$

(ii) When $n = 5$, there are $W_5 = 105$ maximal commuting families of residue matrices (cf. [8, p.94]) :

$$\{A_{ij}, A_{k\ell}, A_{ijk\ell}\},\ \{A_{ij}, A_{ijk}, A_{\ell m}\},\ \{A_{ij}, A_{ijk}, A_{ijk\ell}\}$$
$$\big(\{i,j,k,\ell,m\} = \{0,1,2,3,4\}\big).$$

Moreover if $\mathcal{M}$ is homogeneous, they are

$$\{A_{ij}, A_{k\ell}, A_{pq}\},\ \{A_{ij}, A_{ijk}, A_{pq}\} \qquad (\{i,j,k,\ell,p,q\} = \{0,1,2,3,4,5,\infty\}).$$

Now we review the notation introduced by [10]. The (generalized) eigenvalues and their multiplicities of a matrix $A \in M(N,\mathbb{C})$ is written by

$$[A] = \{[\lambda_1]_{m_1}, \ldots, [\lambda_r]_{m_r}\}$$

with $m_1 + \cdots + m_r = N$ and $\prod_{i=1}^{r}(A - \lambda_i) = 0$.

Sometimes we assume $m_i \geq m_j$ if $\lambda_i = \lambda_j$ and moreover

$$\operatorname{rank} \prod_{\nu=1}^{k}(A - \lambda_\nu) = N - (m_1 + \cdots + m_k) \quad (k = 1, \ldots, r),$$

or $A \in M(N,\mathbb{C})$ is a limit of matrices satisfying this equality, but we do not assume them in this paper.

When two matrices A, $B \in M(N,\mathbb{C})$ commute to each other, we can consider simultaneous eigenspace decompositions and we introduce the notation to the simultaneous eigenvalues and their multiplicities. For example, when

$$A = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix},\ B = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 3 \end{pmatrix},$$

we have

$$[A] = \{[0]_3, [-1]_1\} = \{[0]_3, -1\},\ [B] = \{[1]_1, [2]_2, [3]_1\} = \{1, [2]_2, 3\}$$

and the simultaneous eigenvalues and their multiplicities are written by

$$[A : B] = \big\{[0:1]_1, [0:2]_2, [-1:3]_1\big\} = \big\{[0:1], [0:1]_2, [-1:3]\big\}.$$

In general, when $[B_i, B_j] = 0$ $(i, j = 1, .., k)$, we use the notation

$$[B_1 : \cdots : B_k] = \big\{[\lambda_{1,1} : \cdots : \lambda_{k,1}]_{m_1}, \cdots, [\lambda_{1,r} : \cdots : \lambda_{k,r}]_{m_r}\big\}$$

and for a positive integer p, we put

$$[B_1 : \cdots : B_k]_p = \big\{[\lambda_{1,1} : \cdots : \lambda_{k,1}]_{pm_1}, \cdots, [\lambda_{1,r} : \cdots : \lambda_{k,r}]_{pm_r}\big\}.$$

The matrices which commute to each other can be simultaneously transformed to upper triangular matrices under a suitable base.

Definition 3.11. Let $\mathcal{L}_n$ be the set of maximal commuting families of L_n. The **spectra** of $\mathcal{M}$ is defined by

$$\begin{aligned}\operatorname{Sp}\mathcal{M} &:= \big\{[A_{I_1}:\cdots:A_{I_{n-1}}] \mid \mathcal{I}=\{I_1,\dots,I_{n-1}\}\big\}_{\mathcal{I}\in\mathcal{L}_n},\\ \operatorname{Sp}'\mathcal{M} &:= \big\{[A_{I_1}:\cdots:A_{I_{n-2}}] \mid \mathcal{I}=\{I_1,\dots,I_{n-2},L_n\}\big\}_{\mathcal{I}\in\mathcal{L}_n}.\end{aligned}$$

In most cases, A_{L_n} is a scalar matrix or 0 and we use $\operatorname{Sp}'\mathcal{M}$ in place of $\operatorname{Sp}\mathcal{M}$. For example, the transformation used in [5, §5] keeps the homogeneity and therefore we may assume $A_{L_n}=0$.

Remark 3.12. Let $\{I_1,\dots,I_k\}$ be a maximal commuting family of L_n. Then $[\sum_i c_iA_{I_i}]$ ($c_i\in\mathbb{C}$) is obtained from $\operatorname{Sp}\mathcal{M}$. For example, since $A_{01}+\cdots+A_{0k}=A_{01\cdots k}-A_{1\cdots k}$, we get $[A_{01}+\cdots+A_{0k}]$ from $\operatorname{Sp}\mathcal{M}$ (cf. [11], §7.4).

4. Blowing up of singular points

Note that the KZ-type equation (3.1) is written in the Pfaffian form

$$du=\Omega u,\quad \Omega=\sum_{0\le i<j<n}A_{ij}\,d\log(x_i-x_j). \tag{4.1}$$

Let $\mathcal{I}=\{I_1,\dots,I_{n-2},I_{n-1}=L_n\}$ be an element of $\mathcal{L}_n$. Namely, $\mathcal{I}$ is a maximally commuting family of subsets of $L_n=\{0,1,\dots,n-1\}$. Using the notation given in Definition 2.3 and (2.7), we put

$$b(L_n)=\{J,\,J'\},\quad J=\{j_0,\dots,j_k\},\quad J'=\{j'_0,\dots,j'_{k'}\}, \tag{4.2}$$

$$(I_i)_{\bar b}=\{n_i,n'_i\},\quad n_i,\,n'_i\in L_n. \tag{4.3}$$

Then $k+k'=n-2$ and $k\ge0$, $k'\ge0$, and J and J' correspond semi-final matches of the tournament $\mathcal{I}$. Note that n_i and n'_i correspond to the players of the match I_i, which are determined by the result of games of the tournament. Moreover $b(L_n)$ corresponds to the singular point

$$x_{j_0}=\cdots=x_{j_k}\ \text{and}\ x_{j'_0}=\cdots=x_{j'_{k'}} \tag{4.4}$$

of the KZ-type equation and the local coordinate system $(x_{j_1},\dots,x_{j_k},x_{j'_1},\dots,x_{j'_{k'}})$ with putting $x_{j_0}=0$ and $x_{j'_0}=1$ is valid in a neighborhood of the point. Note that $x_{j_\nu}-x_{j'_{\nu'}}$ does not vanish at the neighborhood. We give a local coordinate system which gives a resolution of the singular point as follows.

Definition 4.1. Define local coordinate system $(X_1,\dots,X_{n-2})$ by

$$x_{n_i}-x_{n'_i}=\prod_{I_i\subset I_j\ne L_n}X_j. \tag{4.5}$$

Remark 4.2. We may assume $n_i\in\bar b(I_i)$ and $n'_i\in\bar b'(I_i)$. Then the condition

$$\begin{aligned}&|x_{n_i}-x_\nu|\le\epsilon|x_{n_i}-x_{n'_i}|,\ |x_{n'_i}-x_{\nu'}|\le\epsilon|x_{n_i}-x_{n'_i}|\\ &\qquad(\nu\in\bar b(I_i),\ \nu'\in\bar b'(I_i),\ i\in L_n)\end{aligned} \tag{4.6}$$

with $0<\epsilon\ll1$ corresponds to a neighborhood of the origin of the local coordinate $(X_1,\dots,X_{n-2})$. Considering $\mathcal{I}\in\mathcal{L}$ satisfying (4.2), the following theorem gives a resolution of the singular point (4.4).

Theorem 4.1. *The 1-form* $\Omega-\sum_{i=1}^{n-2}A_{I_i}d\log(X_i)$ *is non-singular around the origin of* $(X_1,\dots,X_{n-2})$.

Proof. Under the notation in Lemma 4.3 for $1 \le i < j < n$, the condition $I_{i,j} \subset I$ for $I \in \mathcal{I}$ is equals to $\{i,j\} \subset I$ and therefore the lemma shows

$$x_i - x_j = f_{i,j}(X) \cdot \prod_{I_{i,j} \subset I_\nu \in \mathcal{I} \setminus \{L_n\}} X_\nu. \tag{4.7}$$

Here $f_{i,j}$ is a polynomial of X_ν satisfying $I_\nu \subsetneqq I_{i,j}$. Note that the constant term of $f_{i,j}$ equals 1 or -1. Hence

$$d\log(x_i - x_j) - \sum_{\{i,j\} \subset I_\nu \in \mathcal{I} \setminus \{L_n\}} A_I \, d\log X_\nu \tag{4.8}$$

has no singularity around the origin. □

Lemma 4.3. *Let $\mathcal{I}$ be a maximal commuting family of $L = \{0, \dots, n-1\}$ and $\bar{b}$ is a map of $\mathcal{I}$ given in Definition 2.3. Put*

$$x_I := x_{n_I} - x_{n'_I} \quad \textit{with} \quad I_{\bar{b}} = \{n_I, n'_I\} \subset L \tag{4.9}$$

for $I \in \mathcal{I}$. Moreover, for $i, j \in L$ with $i \neq j$, we define $I_{i,j}$ the minimal subset in $\mathcal{I}$ containing both i and j. Then x_I $(I \in \mathcal{I})$ are linearly independent over $\mathbb{C}$ and

$$x_i - x_j = \sum_{I \in \mathcal{I}} \epsilon_{i,j}^I x_I \quad \textit{with} \quad \begin{cases} \epsilon_{i,j}^I = 0 & (I \supsetneqq I_{i,j} \text{ or } I \cap I_{i,j} = \emptyset), \\ \epsilon_{i,j}^I \in \{1, -1\} & (I = I_{i,j}), \\ \epsilon_{i,j}^I \in \mathbb{Z} & (I \subsetneqq I_{i,j}). \end{cases} \tag{4.10}$$

Proof. Put $(I_{i,j})_{\bar{b}} = \{k, \ell\}$. Note that the lemma is clear when $|I_{i,j}| = 2$. We will prove (4.10) by the induction on the cardinality of $I_{i,j}$. We may assume $i \in \bar{b}(I_{i,j})$ and $j \in \bar{b}'(I_{i,j})$ by swapping i and j if necessary. Similarly we may moreover assume $k \in \bar{b}(I_{i,j})$ and $\ell \in \bar{b}'(I_{i,j})$. Then $i = k$ or $I_{i,k} \subsetneqq I_{i,j}$. Moreover $j = \ell$ or $I_{j,\ell} \subsetneqq I_{i,j}$. Since $x_i - x_j = (x_k - x_\ell) + (x_i - x_k) - (x_j - x_\ell)$, the hypothesis of the induction proves (4.10). Since the dimension of $\sum_{i,j \in L} \mathbb{C}(x_i - x_j)$ equals $n-1$ and $|\mathcal{I}| = n-1$, x_I $(I \in \mathcal{I})$ are linearly independent. □

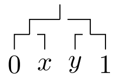
Examples

Put $\Omega' := \sum_{i=1}^{n-2} A_{I_i} \, d\log X_i$ in the theorem.

$\mathbf{n = 4}$

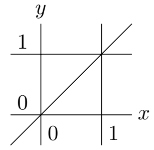
$$\Omega = A_{x0} \frac{dx}{x} + A_{y0} \frac{dy}{y} + A_{x1} \frac{d(x-1)}{x-1} + A_{y1} \frac{d(y-1)}{y-1} + A_{xy} \frac{d(x-y)}{x-y}.$$

$(x, y) = (0, 1)$:



$0 \ x \ y \ 1$
$x_0 \, x_1 \, x_2 \, x_3$

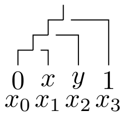
$$\begin{cases} x_1 - x_0 = x = X, \\ x_3 - x_2 = 1 - y = Y, \end{cases}$$



$$\begin{cases} X = x, \\ Y = 1 - y, \end{cases} \qquad \begin{cases} \frac{dx}{x} = \frac{dX}{X}, \\ \frac{dy}{y} = \frac{dY}{Y}, \end{cases}$$

$$\Omega' = A_{x0} \frac{dX}{X} + A_{y1} \frac{dY}{Y} \qquad (|x|, \ |y-1| \ll 1)$$

$(x, y) = (0, 0)$:



$$\begin{cases} x_2 - x_0 = y = Y, \\ x_1 - x_0 = x = XY, \\ x_2 - x_1 = y - x = (1 - X)Y, \end{cases}$$

$$\begin{cases} X = \frac{x}{y}, \\ Y = y, \end{cases} \qquad \begin{cases} \frac{dx}{x} = \frac{dX}{X} + \frac{dY}{Y}, \\ \frac{d(x-y)}{x-y} = \frac{dY}{Y} + \frac{d(X-1)}{X-1}, \end{cases}$$

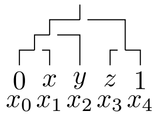
$$\Omega' = A_{x0}\frac{dX}{X} + A_{xy0}\frac{dY}{Y}, \qquad A_{xy0} := A_{x0} + A_{y0} + A_{xy} \quad (|x| \ll |y| \ll 1).$$

Remark 4.4. In [5, 13], the local coordinate $(X, Y) = (\frac{y}{x}, y)$ is used for a desingularization of the origin, where (X, Y) is in a neighborhood of $(\infty, 0)$. This coordinate transformation keeps KZ-type equations and the point $(\infty, 0)$ is a normal crossing singular point of the equations.

n = 5

$$\begin{aligned} \Omega = {} & A_{x0}\frac{dx}{x} + A_{y0}\frac{dy}{y} + A_{z0}\frac{dz}{z} + A_{x1}\frac{d(x-1)}{x-1} + A_{y1}\frac{d(y-1)}{y-1} + A_{z1}\frac{d(z-1)}{z-1} \\ & + A_{xy}\frac{d(x-y)}{x-y} + A_{yz}\frac{d(y-z)}{y-z} + A_{xz}\frac{d(x-z)}{x-z}. \end{aligned}$$

$(x, y, z) = (0, 0, 1)$:

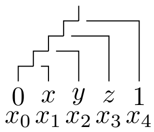


$$\begin{cases} x_2 - x_0 = y = Y, \\ x_1 - x_0 = x = XY, \\ x_4 - x_3 = 1 - z = Z, \end{cases}$$

$$\begin{cases} X = \frac{x}{y} \\ Y = y, \\ Z = 1 - z, \end{cases} \qquad \begin{cases} x - y = (X - 1)Y, \\ z = 1 - Z, \end{cases}$$

$$\Omega' = A_{x0}\frac{dX}{X} + A_{xy0}\frac{dY}{Y} + A_{z1}\frac{dZ}{Z} \qquad (|x| \ll |y| \ll 1,\ |z - 1| \ll 1)$$

$(x, y, z) = (0, 0, 0)$:

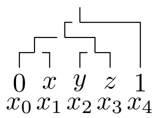


$$\begin{cases} x_3 - x_0 = z = Z, \\ x_2 - x_0 = y = YZ, \\ x_1 - x_0 = x = XYZ, \end{cases}$$

$$\begin{cases} X = \frac{x}{y}, \\ Y = \frac{y}{z}, \\ Z = z, \end{cases} \qquad \begin{cases} y - z = (Y - 1)Z, \\ x - y = (X - 1)YZ, \\ x - z = (XY - 1)Z, \end{cases}$$

$$\Omega' = A_{x0}\frac{dX}{X} + A_{xy0}\frac{dY}{Y} + A_{xyz0}\frac{dZ}{Z} \qquad (|x| \ll |y| \ll |z| \ll 1)$$

$(x, y, z) = (0, 0, 0)$:



$$\begin{cases} x_3 - x_0 = z = Z, \\ x_1 - x_0 = x = XZ, \\ x_3 - x_2 = z - y = YZ, \end{cases}$$

$$\begin{cases} X = \frac{x}{z}, \\ Y = \frac{z-y}{z}, \\ Z = z, \end{cases} \qquad \begin{cases} y = (1 - Y)Z, \\ x - y = (X + Y - 1)Z, \\ x - z = (X - 1)Z, \end{cases}$$

$$\Omega' = A_{x0}\frac{dX}{X} + A_{yz}\frac{dY}{Y} + A_{xyz0}\frac{dZ}{Z} \qquad (|x|,\ |y - z| \ll |z| \ll 1).$$

5. Middle convolutions of KZ-type equations

The convolution $\widetilde{\mathrm{mc}}_{x_0,\mu}\,\mathcal{M}$ of $\mathcal{M}$ (3.1) with $\mu \in \mathbb{C}$ is defined by

$$\widetilde{\mathcal{M}} \, : \, \frac{\partial \tilde{u}}{\partial x_i} = \sum_{0 \le \nu < n} \frac{\tilde{A}_{i\nu}}{x_i - x_\nu} \tilde{u} \quad (0 \le i < n),$$

$$\tilde{A}_{0k} = \begin{array}{cc} & \begin{array}{ccccc} \hphantom{A_{01}} & \hphantom{\cdots} & k & \hphantom{\cdots} & \hphantom{A_{0n-1}} \end{array} \\ k & \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{01} & \cdots & A_{0k} + \mu & \cdots & A_{0n-1} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array} \in M\big((n-1)N, \mathbb{C}\big)$$

(Dettweiler-Reiter [1]),

$$\tilde{A}_{ij} = \begin{array}{cc} & \begin{array}{ccccccc} \hphantom{A_{ij}} & \hphantom{\ddots} & i & \hphantom{\ddots} & j & \hphantom{\ddots} & \hphantom{A_{ij}} \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} A_{ij} & & & & & & \\ & \ddots & & & & & \\ & & A_{ij} + A_{0j} & & -A_{0j} & & \\ & & & \ddots & & & \\ & & -A_{0i} & & A_{ij} + A_{0i} & & \\ & & & & & \ddots & \\ & & & & & & A_{ij} \end{pmatrix} \end{array} \in M\big((n-1)N, \mathbb{C}\big)$$

(Haraoka [2]).

In particular, the compatibility condition of $\mathcal{M}$ assures that of $\widetilde{\mathcal{M}}$.

We prepare a notation to give the definition of the middle convolution and analyze it.

Definition 5.1. For an integer n greater than 1 and a positive integer N, we define

$$L_n^0 = \{1, 2, \ldots, n-1\}, \quad L_n = \{0, 1, 2, \ldots, n-1\},$$

$$\iota_j(v) := (v)_j := \begin{array}{cc} j & \begin{pmatrix} 0 \\ \vdots \\ v \\ \vdots \\ 0 \end{pmatrix} \end{array} \in \mathbb{C}^{(n-1)N}, \quad \iota_j : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N} \quad (v \in \mathbb{C}^N,\ j \in L_n^0),$$

$$\iota_I := \sum_{i \in I} \iota_i : \mathbb{C}^N \hookrightarrow \mathbb{C}^{(n-1)N},\ (v)_I := \iota_I(v) \in \mathbb{C}^{(n-1)N} \quad (v \in \mathbb{C}^N,\ I \subset L_n^0),$$

$$V_I := \iota_I(\mathbb{C}^N) \simeq \mathbb{C}^N.$$

For the equation $\mathcal{M}$ given by (3.1), we define subspaces of $\mathbb{C}^{(n-1)N}$:

$$\mathcal{K}_i := \iota_i(\ker A_{0i}) = \begin{array}{cc} i & \begin{pmatrix} 0 \\ \vdots \\ \ker A_{0i} \\ \vdots \\ 0 \end{pmatrix} \end{array} \subset \mathbb{C}^{(n-1)N},$$

$$\mathcal{K}_\infty := \ker \tilde{A}_{0\infty} \overset{\mu \ne 0}{=} \iota_{L_n^0}\big(\ker(A_{0\infty} - \mu)\big) = \left\{ \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \mid A_{0\infty} v = \mu v \right\} \subset \mathbb{C}^{(n-1)N},$$

$$\mathcal{K} := \mathcal{K}_\infty + \bigoplus_{i=1}^{n} \mathcal{K}_i.$$

Here we remark that, if $\mu \ne 0$, then $\ker \tilde{A}_{0\infty} = \iota_{L_n^0}\big(\ker(A_{0\infty} - \mu)\big)$ and $\mathcal{K}$ is the direct sum of $\mathcal{K}_1, \ldots, \mathcal{K}_{n-1}$ and $\mathcal{K}_\infty$.

Since $\tilde{A}_I\mathcal{K}\subset\mathcal{K}$, $\tilde{A}_I$ induce linear transformations on the quotient space $\mathbb{C}^{(n-1)N}/\mathcal{K}$. These linear transformation are expressed by matrices

$$\bar{A}_I\in M\big((n-1)N-\dim\mathcal{K},\mathbb{C}\big) \tag{5.1}$$

under a base of the quotient space.

Definition 5.2 ([1], [2]). The **middle convolution** $\overline{\mathcal{M}}=\mathrm{mc}_{x_0,\mu}\mathcal{M}$ of $\mathcal{M}$ is defined by

$$\overline{\mathcal{M}}:\frac{\partial\bar{u}}{\partial x_i}=\sum_{\nu\in L_n\setminus\{i\}}\frac{\bar{A}_{i\nu}}{x_i-x_\nu}\bar{u}\quad(i\in L_n).$$

If the ordinary differential equation with the variable x_0 defined by $\mathcal{M}$ is irreducible and μ is generic, $\mathrm{mc}_{x_0,\mu}\mathcal{M}$ is also irreducible and it is proved by [1] that

$$\mathrm{mc}_{x_0,\mu'}\circ\mathrm{mc}_{x_0,\mu}=\mathrm{mc}_{x_0,\mu+\mu'},\ \mathrm{mc}_{x_0,0}=\mathrm{id}. \tag{5.2}$$

In most cases, the irreducibility of this ODE coincides with that of $\mathcal{M}$ (cf. [9]).

When $n=4$, the convolution of $A_{123}=A_{12}+A_{13}+A_{23}$ is

$$\begin{aligned}\tilde{A}_{123}&=\begin{pmatrix}A_{12}+A_{02}&-A_{02}&0\\-A_{01}&A_{12}+A_{01}&0\\0&0&A_{12}\end{pmatrix}+\begin{pmatrix}A_{13}+A_{03}&0&-A_{03}\\0&A_{13}&0\\-A_{01}&0&A_{13}+A_{01}\end{pmatrix}\\&\quad+\begin{pmatrix}A_{23}&0&0\\0&A_{23}+A_{03}&-A_{03}\\0&-A_{02}&A_{23}+A_{02}\end{pmatrix}\\&=\begin{pmatrix}A_{0123}-A_{01}&-A_{02}&-A_{03}\\-A_{01}&A_{0123}-A_{02}&-A_{03}\\-A_{01}&-A_{02}&A_{0123}-A_{03}\end{pmatrix}\in M(3N,\mathbb{C}).\end{aligned}$$

Let m be a positive integer smaller than n. Then

$$\tilde{A}_{1\cdots m}=\begin{pmatrix}A_{0\cdots m}-A_{01}&\cdots&-A_{0m}&&&\\\vdots&\ddots&\vdots&&&\\-A_{01}&\cdots&A_{0\cdots m}-A_{0m}&&&\\&&&A_{1\cdots m}&&\\&&&&\ddots&\\&&&&&A_{1\cdots m}\end{pmatrix}\in M((n-1)N).$$

Lemma 5.3. *Retain the notation in Definition 5.1. For $I\subset L_n$ $(|I|>1)$ and $j,\,k\in L_n^0$ and $v\in\mathbb{C}^N$, we have*

$$\begin{aligned}\tilde{A}_I(v)_J&=(A_Iv)_J\in V_J&&(I\subset J\subset L_n^0),\\\tilde{A}_I(v)_j&=(A_{0I}v)_j-(A_{0j}v)_I&&(I\ni j),\\\tilde{A}_I(v)_k&=(A_Iv)_k\in V_j&&(I\not\ni k),\\[\tilde{A}_I]&=[A_I]_{n-|I|}\cup[A_{0I}]_{|I|-1},&&\\\tilde{A}_I(v)_j&=(A_{0I}v)_j\in\mathcal{K}_j&&(v\in\ker A_{0j},\ I\ni j),\\\tilde{A}_I(v)_k&=(A_Iv)_k\in\mathcal{K}_k&&(v\in\ker A_{0k},\ I\not\ni k),\\\tilde{A}_I(v)_{L_n^0}&=(A_Iv)_{L_n^0}\in\mathcal{K}_\infty&&(v\in\ker(A_{0\infty}-\mu)).\end{aligned}$$

By the symmetry of coordinate $(x_1,\dots,x_n)$, we have only to prove this lemma in the case $I=\{1,\dots,m\}$, but the lemma is clear by the above expression of A_I. Note that the last three equalities in the above follow from the relation $[A_{0I},A_{0j}]=[A_I,A_{0k}]=[A_I,A_{0\infty}]=0$ $(j\in I,\ k\notin I)$.

Since $\tilde{A}_{0\cdots m} = \tilde{A}_{1\cdots k} + \tilde{A}_{01} + \cdots + \tilde{A}_{0m}$, we have

$$\tilde{A}_{0\cdots m} = \begin{pmatrix} A_{01\cdots m}+\mu & & & A_{0,m+1} & \cdots & A_{0,n-1} \\ & \ddots & & \vdots & \cdots & \vdots \\ & & A_{01\cdots m}+\mu & A_{0,m+1} & \cdots & A_{0,n-1} \\ & & & A_{1\cdots m} & & \\ & & & & \ddots & \\ & & & & & A_{1\cdots m} \end{pmatrix}.$$

Then $\widetilde{A}_{0\dots n-1}$ is a block diagonal matrix with the diagonal element $A_{0\dots n-1} + \mu$.

Lemma 5.4. (i) *For $I \subset L_n^0$ and $j,\, k \in L_n^0$ and $v \in \mathbb{C}^N$, we have*

$$\begin{aligned}
\tilde{A}_{0I}(v)_j &= ((A_{0I}+\mu)v)_j \in V_{\{j\}} && (I \ni j),\\
\tilde{A}_{0I}(v)_k &= (A_I v)_k + (A_{0k}v)_I && (I \not\ni k),\\
[\tilde{A}_{0I}] &= [A_{0I}+\mu]_{n-1-|I|} \cup [A_I]_{|I|},\\
\tilde{A}_{0I}(v)_j &= ((A_{0I}+\mu)v)_j \in \mathcal{K}_j && (v \in \ker A_{0j},\ I \ni j),\\
\tilde{A}_{0I}(v)_k &= (A_I v)_k \in \mathcal{K}_k && (v \in \ker A_{0k},\ I \not\ni k),\\
\tilde{A}_{0I}(v)_{L_n^0} &= (A_I v)_{L_n^0} \in \mathcal{K}_\infty && (v \in \ker(A_{0\infty}-\mu)).
\end{aligned}$$

Here the last equality in the above follows from the relation

$$(5.3) \qquad (A_{0I}+\mu)v + \sum_{\nu\in L_n^0\setminus I} A_{0\nu}v = A_I v + \Bigl(\sum_{\nu=1}^{n-1} A_{0\nu}+\mu\Bigr)v = A_I v - (A_{0\infty}-\mu)v.$$

Let $\mathcal{I}$ be a maximal commuting family of L_n. We denote by $\mathcal{I}_0$ the subset of $\mathcal{I}$ consisting the elements containing 0. The elements of I_0 are naturally ordered by the inclusion relationship and they are labelled as $I_{1,0} \subset I_{2,0} \subset \cdots \subset I_{m,0} = L_n$ with $m = |\mathcal{I}_||$.

Moreover we put $\mathcal{I}_k = \{I \in \mathcal{I} \mid I \subset I_{k,0} \setminus I_{k-1,0}\}$ and the elements of $\mathcal{I}_k$ are labelled as $I_{k,\nu}$ with $0 < \nu \le m_k = |\mathcal{I}_k|$ so that $I_{k,\nu} \supset I_{k,\nu'}$ implies $I_{k,\nu} \le I_{k,\nu'}$.

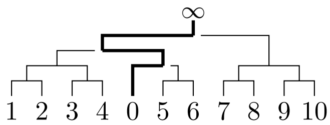
Definition 5.5. We define labels and an order to the elements of a maximal commuting family $\mathcal{I}$ of $L_n = \{0,\dots,n{-}1\}$:

$$\begin{aligned}
\mathcal{I}_0 &:= \{I \in \mathcal{I} \mid 0 \in I\}\\
&= \bigl\{I_{k,0} \mid 1 \le k \le m,\ I_{1,0} \subset I_{2,0} \subset \cdots \subset I_{m,0}\bigr\},\\
\mathcal{I}_k &:= \{I \in \mathcal{I} \mid I \subset I_{k,0} \setminus I_{k-1,0}\},\\
&= \{I_{k,\nu} \mid \nu = 1,\dots,m_k,\ I_{k,\nu} \supset I_{k,\nu'} \text{ or } I_{k,\nu} \cap I_{k,\nu'} = \emptyset \quad (\nu \le \nu')\},\\
I_{k,\nu} &\le I_{k',\nu'} \overset{\text{def.}}{\Longleftrightarrow} k < k' \text{ or } (k = k' \text{ and } \nu \le \nu'),\\
I^{(\ell)} &:= I_{k,\nu} \text{ with } \ell = |\{I \in \mathcal{I} \mid I \le I_{k,\nu}\}| \quad (1 \le \ell < n).
\end{aligned}$$

Here the numbers ℓ indicate the order of the elements of $\mathcal{I}$. Namely,

$$\mathcal{I} = \{I^{(\ell)} \mid \ell = 1,\dots,n{-}1\}, \quad I^{(1)} < I^{(2)} < \cdots < I^{(n-1)}.$$

Thus a maximal commuting family $\mathcal{I}$ of L_n becomes a totally ordered set. Here we note that the order is not uniquely determined. It may be easy to see this order if $\mathcal{I}$ is expressed by a figure of the corresponding tournament of the teams labeled by the elements of L_n where the team with the label 0 is the final winner.

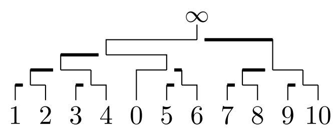
Example 5.6. In the case of the tournament 

,

$$\begin{aligned}
&\mathcal{I}_0 = \bigl\{I_{1,0} = \{0,5,6\},\ I_{2,0} = \{0,1,2,3,4,5,6\},\ I_{3,0} = \{0,1,2,\ldots,9,10\}\bigr\},\\
&\mathcal{I}_1 = \bigl\{I_{1,1} = \{5,6\}\bigr\},\\
&\mathcal{I}_2 = \bigl\{I_{2,1} = \{1,2,3,4\},\ I_{2,2} = \{1,2\},\ I_{2,3} = \{3,4\}\bigr\},\\
&\mathcal{I}_3 = \bigl\{I_{3,1} = \{7,8,9,10\},\ I_{3,2} = \{7,8\},\ I_{3,3} = \{9,10\}\bigr\},
\end{aligned}$$

$I_{1,0} < I_{1,1} < I_{2,0} < I_{2,1} < I_{2,2} < I_{2,3} < I_{3,0} < I_{3,1} < I_{3,2} < I_{3,3}$.

Now we recall the map b^0 in Definition 2.3. The set of teams $b^0(I)$ is not necessarily uniquely determined by a tournament. But if we identify $b^0(I)$ with the teams failed the game labeled by I, the definition of b^0 corresponds to the tournament figure where the winner of each game is indicated. An example of the figure is

(5.4) 

Here $n = 11$ and the corresponding $b^0(I^{(1)}), b^0(I^{(2)}), \ldots, b^0(I^{(n-1)})$ are

$$\overset{\{0,5,6\}}{\underset{6}{\{5,6\}}},\ \overset{\{5,6\}}{\underset{5}{\{5\}}},\ \overset{\{0,1,2,3,4,5,6\}}{\underset{4}{\{1,2,3,4\}}},\ \overset{\{1,2,3,4\}}{\underset{2}{\{1,2\}}},\ \overset{\{1,2\}}{\underset{1}{\{1\}}},\ \overset{\{3,4\}}{\underset{3}{\{3\}}},\ \overset{\{0,1,\ldots,10\}}{\underset{10}{\{7,8,9,10\}}},\ \overset{\{7,8,9,10\}}{\underset{8}{\{7,8\}}},\ \overset{\{7,8\}}{\underset{7}{\{7\}}},\ \overset{\{9,10\}}{\underset{9}{\{9\}}}$$

and the number under the set $b^0(I^{(i)})$ means that the corresponding team failed in the game $I^{(i)}$.

Definition 5.7. Define subspaces of $\mathbb{C}^{(n-1)N}$ as follows (cf. Definitions 2.3, 5.1).

$$\begin{aligned}
W^{(\ell)} &:= W_I := \sum_{\nu=1}^{\ell} V_{b^0(I^{(\nu)})} \quad (I = I^{(\ell)} \in \mathcal{I}),\\
U_{b^0(\mathcal{I})} &:= \bigl(U_{ij}\bigr)_{\substack{1\le i<n-1\\ 1\le j\le n-1}} \in GL((n-1)N, \mathbb{C}) \quad \text{with}\\
U_{ij} &= \begin{cases} E_N & (j \in b^0(I^{(i)})),\\ O_N & (j \notin b^0(I^{(i)})). \end{cases}
\end{aligned}$$

Here E_N is the identity matrix and O_N is the zero matrix in $M(N,\mathbb{C})$.

Lemma 5.8. (i) *For $0 \notin I \in \mathcal{I}$, we have*

$$V_I \subset W_I. \tag{5.5}$$

$$V_{\bar b(I)}, V_{\bar b'(I)} \subset W_I. \tag{5.6}$$

(ii) $\dim W^{(\ell)} = \ell N \quad (\ell = 1,\ldots,n-1)$

(iii) *Let I, $K \in \mathcal{I}$ and $v \in \mathbb{C}^N$. Putting $J = b^0(K)$, we have the following.*

If $0 \notin I$, then

$$\tilde A_I(v)_J = \begin{cases} (A_I v)_J & (I \not\supset K),\\ (A_{0I} v)_J - \bigl(\sum_{\nu\in J} A_{0\nu} v\bigr)_I & (I \supset K). \end{cases}$$

If $0 \in I$*, then*

$$\tilde{A}_I(v)_J = \begin{cases} \big((A_I + \mu)v\big)_J & (I \supset K), \\ (A_{I\setminus\{0\}}v)_J + \big(\sum_{\nu\in J} A_{0\nu}v\big)_{I\setminus\{0\}} & (I \not\supset K). \end{cases}$$

(iv) $\tilde{A}_I W^{(\ell)} \subset W^{(\ell)} \quad (I \in \mathcal{I},\ 1 \le \ell < n).$

Proof. Putting $I = I_{k,i}$, we will first prove (5.5) and (5.6) by the induction with respect to i. Since $I = \bar{b}(I) \sqcup \bar{b}'(I)$, we remark that (5.6) follows from (5.5). When $i = 1$, (5.5) is clear because $b^0(I_{k,0}) = I_{k,1}$. When $i > 1$, there exists $J \in \mathcal{I}$ satisfying $I \subset J \in \mathcal{I}_k$ and $I \in b(J)$ (cf. (2.6)). Then $V_I \subset W_J \subset W_I$ by the induction hypothesis for (5.6) with replacing I by J.

Hence

$$\begin{aligned}\sum_{k=1}^{m}\sum_{\nu=0}^{m_k} V_{b^0(I_{k,\nu})} &= \sum_{k=1}^{m}\sum_{\nu=0}^{m_k} W_{I_{k,\nu}} = \sum_{k=1}^{m}\Big(V_{b^0(I_{k,0})} + \sum_{\nu=1}^{m_k}(V_{\bar{b}(I_{k,i})} + V_{\bar{b}'(I_{k,i})})\Big)\\ &\supset \sum_{\nu=1}^{n-1} V_{\{\nu\}} \simeq \mathbb{C}^{(n-1)N}.\end{aligned}$$

Since $\sum_{k=1}^{m}(m_k+1) = n-1$, $\sum_{k=1}^{m}\sum_{\nu=0}^{m_k} V_{b^0(I_{k,\nu})}$ is a direct sum decomposition of $\mathbb{C}^{(n-1)N}$, which implies (ii).

Then Lemma 5.3 and Lemma 5.4 show (iii). In particular we have (iv). □

Definition 5.9. Let L be a finite set. For $i \in L$ and nonempty subsets I and J of L, we define

$$\mathrm{md}_{i,J}(I) := \begin{cases} I \cup \{i\} & (I \supset J), \\ I \setminus \{i\} & (I \not\supset J), \end{cases} \quad \mathrm{me}_{i,J}(I) := \begin{cases} 1 & (i \in I \supset J), \\ 0 & (i \notin I \text{ or } I \not\supset J). \end{cases}$$

Remark 5.10. (i) If $i \in I \supset J$ or $i \notin I \not\supset J$, then $\mathrm{md}_{i,J}(I) = I$.

(ii) Let $K = I^{(\ell)} \in \mathcal{I}$. Then $\widetilde{A}_I$ induces a linear transformation on the quotient space $W^{(\ell)}/W^{(\ell-1)} \simeq V_{b^0(K)} \simeq \mathbb{C}^N$, which is identified with A_I^K given by (5.7).

(iii) If the elements of a family $\{I_\nu \mid \nu = 0, 1, \ldots, r\}$ of subsets of L commute with each other (cf. (2.2)) and $|I_\nu| > 1$ ($\nu = 1, \ldots, r$), so do the elements of $\{I_0 \cup \{i\}\} \cup \{\mathrm{md}_{i,I_0}(I_\nu) \mid (\nu = 1, \ldots, r)\}$.

We have the following theorem from Lemma 5.3, Lemma 5.4 and Lemma 5.8.

Theorem 5.1. (i) *Let $\mathcal{I}$ be a maximal commuting family of $L_n = \{0, 1, \ldots, n-1\}$. Put $\mathcal{I} = \{I^{(1)}, \ldots, I^{(n-1)}\}$. Let $I \in \mathcal{I}$. Then under the notation in Definition 5.9, we have*

$$\begin{aligned}[\tilde{A}_I] &= [A_{I\cup\{0\}} + \mu]_{|I|-1} \sqcup [A_{I\setminus\{0\}}]_{n-|I|},\\ [\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}] &= \bigsqcup_{J\in\mathcal{I}} [A^J_{I^{(1)}} : \cdots : A^J_{I^{(n-1)}}],\\ [\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_j} &= [A^{\{j\}}_{I^{(1)}} : \cdots : A^{\{j\}}_{I^{(n-1)}}]|_{\ker A_{0j}} \quad (j \in L_n^0),\\ [\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]|_{\mathcal{K}_\infty} &= [A^{L_n}_{I^{(1)}} : \cdots : A^{L_n}_{I^{(n-1)}}]|_{\ker(A_{0\infty}-\mu)}\end{aligned}$$

with denoting

$$A_I^K := A_{\mathrm{md}_{0,K}(I)} + \mathrm{me}_{0,K}(I)\cdot\mu. \tag{5.7}$$

(ii) *By a conjugation, $\widetilde{A}_{I^{(i)}}$ are simultaneously changed into upper triangular block matrices $U^{-1}_{b^0(\mathcal{I})}\widetilde{A}_{I^{(i)}}U_{b^0(\mathcal{I})}$.*

Remark 5.11. (i) In the above theorem, we have

$$A_{L_n}^J = A_{0\cdots n-1}^j = \mu + A_{0\cdots n-1} \quad \text{and} \quad A_{L_n}^{L_n} = A_{L_n^0} \tag{5.8}$$

and therefore we often omit the term $\tilde{A}_{L_n}$ in $[\tilde{A}_{I^{(1)}} : \cdots : \tilde{A}_{I^{(n-1)}}]$.

Moreover we remark

$$A_{0\infty} = A_{L_n^0} - A_{L_n}$$

and Corollary 3.3 and

$$\mathrm{mc}_{x_0,\mu} = \mathrm{Ad}\big((x_p - x_q)^{-\lambda}\big) \circ \mathrm{mc}_{x_0,\mu} \circ \mathrm{Ad}\big((x_p - x_q)^{\lambda}\big) \quad (1 \le p < q \le n-1). \tag{5.9}$$

(ii) The generalized Riemann scheme

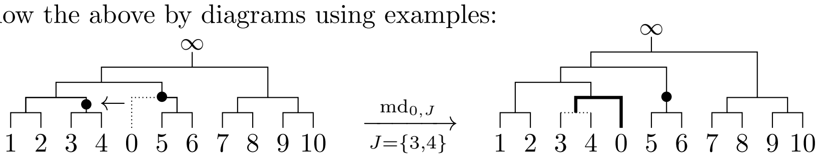
$$\big\{[\bar{A}_{ij}] \mid \{i,j\} \subset \tilde{L}_n\big\} \tag{5.10}$$

of the residue matrices $\mathrm{mc}_{x_0,\mu}\mathcal{M}$ is obtained by Theorem 5.1 and $\mathrm{Sp}\,\mathcal{M}$.

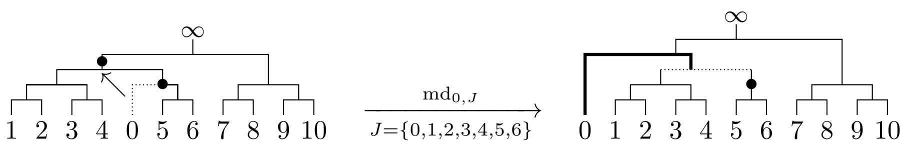
We examine the relation between the theorem and tournaments. We write $\mathcal{I}$ by the corresponding tournament and the team with label i by (i). The theorem gives the transformation of $\mathrm{Sp}\,\mathcal{M}$ using A_I^K in (5.7) for $I \in \mathcal{I}$. The corresponding transformation $\mathrm{md}_{x_0,I}^K$ of the tournament $\mathcal{I}$ is an insertion of (0) after the deletion of (0). The deleted game is replaced by the preceding game. The inserted game which is the first game of (0) is as follows.

- $(A_I^J)_{I\in\mathcal{I}}$: the new game is played with the winner of the game J and the preceding game of the opponent is replaced by the new game.
- $(A_I^j|_{\ker A_{0j}})_{I\in\mathcal{I}}$: the new game is played with (j) as a basic insertion.
- $(A_I^\infty|_{\ker(A_{0\infty}-\mu)})_{I\in\mathcal{I}}$: the new game is played with the winner of the tournament as a top insertion. Here we put $A_I^\infty = A_I^{L_n}$.

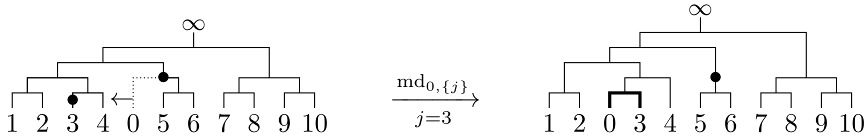
We show the above by diagrams using examples:



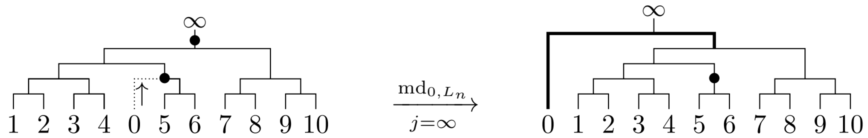
$J = \{3,4\} : \{3,4\} \to \{0,3,4\},\ \{1,2,3,4\} \to \{0,1,2,3,4\},\ \{0,5,6\} \to \{5,6\}$



$J = \{0,1,2,3,4,5,6\} : \{0,5,6\} \to \{5,6\}$



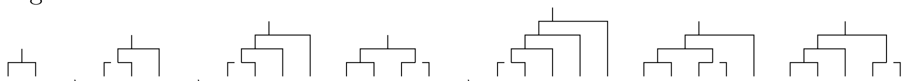
$j = 3 : \{3,4\} \to \{0,3,4\},\ \{1,2,3,4\} \to \{0,1,2,3,4\},\ \{0,5,6\} \to \{5,6\},\ +\{0,3\}$



$j = \infty : \{0,5,6\} \to \{5,6\},\ \{0,1,\ldots,6\} \to \{1,\ldots,6\},\ \{0,1,\ldots,10\} \to \{1,\ldots,10\}$

Remark 5.12. Middle convolutions $\mathrm{mc}_{x_0,\mu}$, additions $\mathrm{Ad}\big((x_i - x_j)^\lambda\big)$ and permutations of suffices $\{0, 1, \dots, n-1\}$ define transformations on the space of KZ-type equations. The change of spectra $\mathrm{Sp}\,\mathcal{M}$ is obtained by Theorem 5.1.

We examine the necessary data of $\mathcal{M}$ to get $[A_{12}]$ of a equation which is obtained by a successive application of these transformation to the original KZ-type equation $\mathcal{M}$. Since $[\tilde{A}_{12}|_{\mathcal{K}_1}] = [A_{012}|_{\ker A_{01}}]$, we need $[A_{01} : A_{012}]$ considering additions. Considering more permutations, we need $[A_{12} : A_{123}]$ and $[A_{23} : A_{123}]$ in general. Considering a middle convolution with respect to x_0, we moreover need $[A_{01} : A_{012} : A_{0123}]$ and $[A_{01} : A_{23} : A_{0123}]$. These considerations correspond to the following diagram.



1 2 $\to$ 0 1 2 $\to$ 0 1 2 3 2 3 1 0 $\to$ 0 1 2 3 4 2 3 1 0 4 2 3 4 1 0

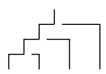
These changes of patterns correspond to basic insertions. Since any tournament is obtained by a successive application of basic insertions (cf. Remark 2.1), we need $\mathrm{Sp}\,\mathcal{M}$ in general.

On the other hand, convolutions, additions and permutations of suffixes define transformation on the space $\{[A_I] \mid I \subset L_n\}$ which does not contain simultaneous eigenspace decompositions. To get the eigenspace decomposition of a residue matrix or a commuting family of residue matrices of the equation obtained by applying middle convolutions and additions to an original KZ-type equation $\mathcal{M}$, the necessary data contained in $\mathrm{Sp}\,\mathcal{M}$ is depend on the procedure of the application, for example, whether $\ker A_j$ is zero or not. It may be good to check the necessary data for the real calculation, simultaneous eigenspace decompositions of families of commuting residue matrices of $\mathcal{M}$ For example, we refer to [10, Theorem 4.1] or §7.2 when the middle convolutions are restricted only on some variables.

6. Examples

The transformation of $\mathrm{Sp}\,\mathcal{M}$ by a middle convolution of $\mathcal{M}$ is obtained by Theorem 5.1. Since the transformation is symmetric with respect to the suffices $\{1, \dots, n-1\}$ of the variables $(x_0, \dots, x_{n-1})$, we have only to examine the transformation of the maximal commuting families of residue matrices corresponding to the representatives of win types. In the cases $n = 3, 4, 5, 6, 7, \dots$, the number of the win types are $W_n = 2, 4, 9, 20, 46, \dots$, respectively.

In this section, we assume $n = 4$ and examine $W_4 = 4$ cases. The results are kept valid by permutations of suffixes $\{1, 2, 3\}$. Note that the total number of maximal commuting families of residue matrices equals $K_4 = 15$. For simplicity, we assume $\mathcal{M}$ is homogeneous. Then $A_{0123} = 0$ and $\tilde{A}_{0123} = \mu$.

1.  0 1 2 3 $\quad \mathcal{I} = \big\{\{0,1\}, \{0,1,2\}, \{0,1,2,3\}\big\} \xrightarrow{b^0} \big\{\{1\}, \{2\}, \{3\}\big\}$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{01} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{012} = \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{012}+\mu & 0 & A_{03} \\ 0 & A_{012}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix}$$

$$
\begin{aligned}
[\tilde{A}_{01} : \tilde{A}_{012}] &= \{[A_{01}+\mu : A_{012}+\mu], [0 : A_{012}+\mu], [0 : A_{12}]\}\\
[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_1} &= [A_{01}+\mu : A_{012}+\mu]|_{\ker A_{01}}\\
[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_2} &= [0 : A_{012}+\mu]|_{\ker A_{02}}\\
[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_3} &= [0 : A_{12}]|_{\ker A_{03}}\\
[\tilde{A}_{01} : \tilde{A}_{012}]|_{\mathcal{K}_\infty} &= [0 : A_{12}]|_{\ker (A_{0\infty}-\mu)}
\end{aligned}
$$

Here $\tilde{A}_{ij}$, U and V are block matrices.

2. 1 2 0 3 $\quad \mathcal{I} = \big\{\{0,1,2\},\{1,2\},\{0,1,2,3\}\big\} \xrightarrow{b^0} \big\{\{1,2\},\{1\},\{3\}\big\}$

$$
U = \begin{pmatrix} 1 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U
$$

$$
\tilde{A}_{012} = \begin{pmatrix} A_{012}{+}\mu & 0 & A_{03}\\ 0 & A_{012}{+}\mu & A_{03}\\ 0 & 0 & A_{12}\end{pmatrix} \to \begin{pmatrix} A_{012}{+}\mu & 0 & A_{03}\\ 0 & A_{012}{+}\mu & 0\\ 0 & 0 & A_{12}\end{pmatrix}
$$

$$
\tilde{A}_{12} = \begin{pmatrix} A_{012}{-}A_{01} & -A_{02} & 0\\ -A_{01} & A_{012}{-}A_{02} & 0\\ 0 & 0 & A_{12}\end{pmatrix} \to \begin{pmatrix} A_{12} & -A_{01} & 0\\ 0 & A_{012} & 0\\ 0 & 0 & A_{12}\end{pmatrix}
$$

$$
\begin{aligned}
[\tilde{A}_{012} : \tilde{A}_{12}] &= \{[A_{012}+\mu : A_{12}], [A_{012}+\mu : A_{012}], [A_{12} : A_{12}]\}\\
[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_1} &= [A_{012}+\mu : A_{012}]|_{\ker A_{01}}\\
[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_2} &= [A_{012}+\mu : A_{012}]|_{\ker A_{02}}\\
[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_3} &= [A_{12} : A_{12}]|_{\ker A_{03}}\\
[\tilde{A}_{012} : \tilde{A}_{12}]|_{\mathcal{K}_\infty} &= [A_{12} : A_{12}]|_{\ker (A_{0\infty}-\mu)}
\end{aligned}
$$

A_I^J (cf. Theorem 5.1)

$J\backslash I$	$\widetilde{012}$	$\widetilde{12}$	$\widetilde{0123}$
012	012+μ	12	μ
12	012+μ	012	μ
0123	12	12	μ

$j\backslash I$	$\widetilde{012}$	$\widetilde{12}$	$\widetilde{0123}$
1	012+μ	012	μ
2	012+μ	012	μ
3	12	12	μ
∞	12	12	μ

Remark 6.1. The simultaneous eigenspace decomposition of $\big(\tilde{A}_I\big)_{I\in\mathcal{I}}$ is obtained by A_I^J in Theorem 5.1. The above left table is the $(n{-}1)\times(n{-}1)$ matrix whose (J,I)-element with $I\in\mathcal{I}$ and $J\in\mathcal{I}$ is the suffix K of $A_K = A_I^J$. Moreover $012+\mu$ in the table means $A_{012}+\mu$. Then K contains 0 if and only if $I\supset J$ and the term "$+\mu$" exists if and only if $0\in I\supset J$. Here the last column corresponding to $\tilde{A}_{0123}$ is omitted. Similarly, the above right table shows $A_I^{\{j\}}$ and $A_I^{L_n}$ which describe $\big(\tilde{A}_I|_{\mathcal{K}_j}\big)_{I\in\mathcal{I}}$ and $\big(\tilde{A}_I|_{\mathcal{K}_\infty}\big)_{I\in\mathcal{I}}$, respectively.

3. 1 2 3 0 $\quad \mathcal{I} = \big\{\{0,1,2,3\},\{1,2,3\},\{1,2\}\big\} \xrightarrow{b^0} \big\{\{1,2,3\},\{1,2\},\{1\}\big\}$

$$
U = \begin{pmatrix} 1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & -1\\ 1 & -1 & 0\end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U
$$

$$\tilde{A}_{123} = \begin{pmatrix} -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \end{pmatrix} \to \begin{pmatrix} A_{123} & -A_{01}-A_{02} & -A_{01} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} A_{012}{-}A_{01} & -A_{02} & 0 \\ -A_{01} & A_{012}{-}A_{02} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \to \begin{pmatrix} A_{12} & 0 & 0 \\ 0 & A_{12} & -A_{01} \\ 0 & 0 & A_{012} \end{pmatrix}$$

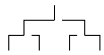
$$[\tilde{A}_{123} : \tilde{A}_{12}] = \{[A_{123} : A_{12}], [0 : A_{12}], [0 : A_{012}]\}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_1} = [0 : A_{012}]|_{\ker A_{01}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_2} = [0 : A_{012}]|_{\ker A_{02}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_3} = [0 : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}]|_{\mathcal{K}_\infty} = [A_{123} : A_{12}]|_{\ker (A_{0\infty}-\mu)}$$

4.  0 1 2 3 $\quad \mathcal{I} = \big\{\{0,1\},\{0,1,2,3\},\{2,3\}\big\} \xrightarrow{b^0} \big\{\{1\},\{2,3\},\{2\}\big\}$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \tilde{A}_* \to V\tilde{A}_*U$$

$$\tilde{A}_{01} = \begin{pmatrix} A_{01}{+}\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} A_{01}{+}\mu & A_{03}+A_{02} & A_{02} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{023}{-}A_{02} & -A_{03} \\ 0 & -A_{02} & A_{023}{-}A_{03} \end{pmatrix} \to \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23} & -A_{02} \\ 0 & 0 & A_{023} \end{pmatrix}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}] = \{[A_{01}+\mu : A_{23}], [0 : A_{23}], [0 : A_{023}]\}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_1} = [A_{01}+\mu : A_{23}]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_2} = [0 : A_{023}]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_3} = [0 : A_{023}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{K}_\infty} = [0 : A_{23}]|_{\ker (A_{0\infty}-\mu)}$$

Case **1**

$\widetilde{A}$	$\widetilde{01}$	$\widetilde{012}$
01	01+μ	012+μ
012	0	012+μ
0123	0	12
1	01+μ	012+μ
2	0	012+μ
3	0	12
∞	0	12

Case **3**

$\widetilde{A}$	$\widetilde{123}$	$\widetilde{12}$
0123	123	12
123	0	12
12	0	012
1	0	012
2	0	012
3	0	12
∞	123	12

Case **4**

$\widetilde{A}$	$\widetilde{01}$	$\widetilde{23}$
01	01+μ	23
0123	0	23
23	0	023
1	01+μ	23
2	0	023
3	0	023
∞	0	23

We show a computer program displaying the result in this section which uses functions in a library [6] of the computer algebra `Risa/Asir`. Then the result including the figures of tournaments is displayed through a PDF file created by TEX.

```
N=4;    /* N-2=2 variables HG */
T=os_md.symtournament(N|to="T");        /* T: types */
for(S="";T!=[];T=cdr(T)){               /* R: win types */
  R=os_md.xytournament(car(T),0|verb=21,winner="all");
```

```
  for(;R!=[];R=cdr(R)){
    C=car(R);
    S=S+"\\raisebox{-2mm}{"             /* S: source of TeX */
    +os_md.xytournament(C[0],0|teams=C[1],winner=0) /* figure */
    +"}\\qquad"+rtostr(C[4])+"\\ $\\to$ \\ "+rtostr(C[3])
    +os_md.midKZ(C[2],C[3]);            /* spectra */
  }
}
os_md.dviout(S);                        /* display */
```

In the above program

- In the first line, the number of variables $n = 4$ is given by `N=4`.
- In the 2-nd line, all the types `T` for `N` teams are obtained.
- In the 4-th line, all win types $\mathcal{I}$ are obtained in `R`.
- From 5-th line, $\mathrm{Sp}(\mathrm{mc}_{x_0,\mu}\,\mathcal{M})$ and the findings as presented in this section are transformed into a source text `S` in TEX and in the last line it is displayed using a PDF file transformed from the source text.

Remark 6.2. The top insertion imbeds the tournaments of $n-1$ teams in those of n teams. The image of this imbedding is the tournaments of n teams with $b^0(L_n) = \{n-1\}$. This corresponds to the KZ-type equation $\mathcal{M}$ with $A_{i,n-1} = 0$ $(0 \le i \le n-2)$. Hence our result of KZ-type equations with $n-1$ variables follows from that of KZ-type equations with n variables.

The first two examples in this section correspond to this imbedding and the results for $n = 3$ are obtained by omitting $b^0(\{0, 1, 2, 3\}) = \{3\}$. Namely, we get them by the first 2×2 blocks of the matrices in these examples. Moreover we omit the term $\mathcal{K}_3$ and the last terms of the simultaneous eigenspace decompositions. The term A_{012} can remain.

7. Further considerations

7.1. **Infinite point.** KZ-type equation $\mathcal{M}$ in §3 is considered to be defined on the configuration space of $n+1$ points of $\mathbb{P}^1$. By a linear fractional transformation transforming the infinite point to a finite point, we have a KZ-type equation with $n+1$ variables which has no singularity at infinite point. Then all the singular points are finite points and it may be easier to understand a symmetry among singular points. If the original equation has $n-1$ variables, the resulting KZ-type equation has n variables and is characterized by the condition

$$A_{i\infty} := \sum_{\nu=0}^{n-1} A_{i\nu} = 0 \quad (0 \le i < n) \tag{7.1}$$

on the residue matrices A_{ij}. Hence we assume the following.

Definition 7.1. ∞ is a **pseudo-singular point**, namely, there exist $\mu_i \in \mathbb{C}$ such that

$$A_{i\infty} = \mu_i \quad (0 \le i < n). \tag{7.2}$$

Here μ_i mean scalar matrices.

If we apply $\mathrm{Ad}\big((x_0 - x_1)^\lambda (x_0 - x_2)^\lambda (x_1 - x_2)^{-\lambda}\big)$ to $\mathcal{M}$, $A_{0\infty}$ is changed into $A_{0\infty} - \lambda$ and $A_{i\infty}$ for $i \ne 0$ are unchanged. Hence the KZ-tye equation with a pseudo-singular infinite point can be changed to a equation satisfying (7.1). We examine the middle convolution of $\tilde{A}_{i\infty}$. Note that

$$\tilde{A}_{0\infty} = -\sum_{\nu=1}^{n-1} A_{0\nu} = \begin{pmatrix} -A_{01}-\mu & -A_{02} & \cdots & -A_{0,n-1} \\ -A_{01} & -A_{02}-\mu & \cdots & -A_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{01} & -A_{02} & \cdots & -A_{0,n-1}-\mu \end{pmatrix},$$

$$\tilde{A}_{1\infty} = -\sum_{\nu=0}^{n-1} A_{1\nu} = \begin{pmatrix} A_{0\infty}+A_{1\infty}+A_{01}-\mu & 0 & \cdots & 0 \\ A_{01} & A_{1\infty} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{01} & 0 & \cdots & A_{1\infty} \end{pmatrix}.$$

Hence for the variable x_0, the middle convolution of the KZ-type equation $\mathcal{M}$ satisfying (7.2) is defined by mc_{x_0,μ_0}. Then

$$\begin{aligned} \widetilde{A}_{0\infty}(v)_{L_n^0} &= \big((A_{0\infty}-\mu)v\big)_{L_n^0}, & \widetilde{A}_{1\infty}(v)_{L_n^0} &= \big((A_{01}+A_{1\infty})v\big)_{L_n^0}, \\ \widetilde{A}_{0\infty} &= 0 \mod \mathcal{K}_\infty, & \widetilde{A}_{i\infty} &= \mu_i \mod \mathcal{K}_i \quad (1 \le i < n), \end{aligned}$$

and therefore

$$\mathcal{K}_\infty = V_{L_n^0},\ \ \overline{A}_{0\infty} = 0,\ \ \overline{A}_{i\infty} = \mu_i \quad (0 < i < n) \tag{7.3}$$

and $\mathrm{mc}_{x_0,\mu_0}\mathcal{M}$ also has a pseudo-singular infinite point.

7.2. **Fixed singular points.** We examine the KZ-type equation

$$\mathcal{M} : \frac{\partial u}{\partial x_i} = \sum_{\substack{0 \le \nu \le n-1 \\ \nu \ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u + \sum_{q=1}^{m} \frac{B_{iq}}{x_i - y_q} u \quad (i = 0, \dots, n-1) \tag{7.4}$$

which have fixed singular points $y_1, \dots, y_m$ together with $x_i = x_j$.

We may assume $\mathcal{M}$ has a pseudo-singular infinite point without loss of generality.

For $\{i_1, \dots, i_p\} \subset \{0, \dots, n-1\}$ and $\{j_1, \dots, j_q\} \subset \{n, \dots, m+n-1\}$, put

$$A_{i_1,\dots,i_p;j_1,\dots,j_q} := \sum_{1 \le \nu < \nu' \le p} A_{i_\nu i_{\nu'}} + \sum_{\substack{1 \le \nu \le p \\ 1 \le \nu' \le q}} B_{i_\nu j_{\nu'}-n+1}.$$

We may think that we put $y_j = x_{n-1+j}$ for $j = 1, \dots, m$. Note that

$$A_{i_1,\dots,i_p;j_1,\dots,j_q} = A_{i_1,\dots,i_p,j_1,\dots,j_q} - A_{j_1,\dots,j_q}.$$

Here the terms $A_{j_\nu j_{\nu'}}$ in the above right hand side are cancelled.

The integrability condition of $\mathcal{M}$ is

$$\begin{aligned} &[A_{ij}, A_{k\ell}] = [A_{i;q}, A_{j;q'}] = [A_{ij}, A_{k;q}] = 0, \\ &[A_{ij}, A_{ijk}] = [A_{ij}, A_{ij;q}] = [A_{i;q}, A_{ij;q}] = 0. \end{aligned} \tag{7.5}$$

Here i, j, k, $\ell \in L_n$ and q, $q' \in \{n, n+1, \dots, n+m1\}$ are distinct numbers. Since $[A_{01}, A_{01\cdots k;q}] = [A_{01}, \sum_{0 \le i < j \le k} A_{i,j} + A_{01;q} + \sum_{i=2}^{k} A_{i;q}] = 0$ etc., we have

$$\begin{aligned} [A_I, A_J] &= 0 \quad (I \cap J = \emptyset \text{ or } I \subset J \text{ or } I \supset J), \\ [A_I, A_{J;q}] &= 0 \quad (I \cap J = \emptyset \text{ or } I \subset J), \\ [A_{I;q}, A_{J;q'}] &= 0 \quad (I \cap J = \emptyset \text{ and } q \ne q') \end{aligned} \tag{7.6}$$

for I, $J \subset \{0, 1, \dots, n-1\}$ and $\{q, q'\} \subset \{n, n+1, \dots, n+m-1\}$.

Hereafter in §7.2, we assume (7.2).

Definition 7.2. For a pair of finite sets (L, L') with $L \cap L' = \emptyset$, a maximal commuting $\mathcal{I}$ of (L, L') is defined as follows. Namely, $\mathcal{I} = \{\mathcal{I}^{(\nu)} \mid \nu \in L'\}$,

$$L_n = \bigsqcup_{j \in L'} S_j \tag{7.7}$$

and $\mathcal{I}^{(j)}$ are maximal commuting family of $S_j \cup \{j\}$, respectively.

Moreover putting $L' = \{r_1, \ldots, r_m\}$, we define

$$\widehat{\mathcal{I}} := \mathcal{I} \cup \bigcup_{j=2}^{m} \{\widehat{S}_j\}, \ \ \widehat{S}_j := \bigcup_{\nu=1}^{j} \bigl(S_\nu \cup \{\nu\}\bigr). \tag{7.8}$$

Here n are numbers of the variables and the fixed points, respectively, and $n = |S_1| + \cdots + |S_m|$.

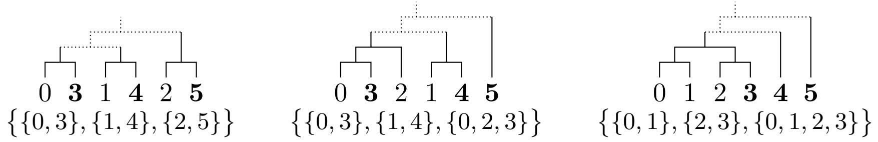
We can consider the middle convolution of $\mathcal{M}$ with respect to any one of the variables $x_0, \ldots, x_{n-1}$ and the resulting change of the eigenspace decompositions of residue matrices are obtained from Theorem 5.1. We consider residue matrices in (7.3) and those corresponding to $\mathcal{I}$ and define the base of the matrices by $\widehat{\mathcal{I}}$. In particular, if $n = 1$, this coincides with the result given in [1].

Remark 7.3. The tournament corresponds to $\widehat{\mathcal{I}}$ is characterized as follows. The team $(n + m - j)$ and the team $(n + m - 1 - j)$ will face in a match $j-1$ games before the final if they have won the preceding matches $(j = 1, \ldots, m)$.

Example 7.4. The representatives of the maximal commuting family of $(L, L') = (\{0,1,2\}, \{3,4,5\})$ under the permutations of the elements of L and those of L' are

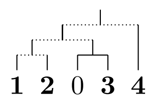
$$\begin{aligned}
1+1+1 &: \bigl\{\{0,3\},\{1,4\},\{2,5\}\bigr\},\\
2+1+0 &: \bigl\{\{0,3\},\{1,4\},\{0,2,3\}\bigr\},\ \bigl\{\{0,1\},\{2,3\},\{0,1,4\}\bigr\},\\
3+0+0 &: \bigl\{\{0,1\},\{2,3\},\{0,1,2,3\}\bigr\},\ \bigl\{\{0,1\},\{0,1,2\},\{0,1,2,3\}\bigr\},\\
&\quad \bigl\{\{0,1\},\{0,1,3\},\{0,1,2,3\}\bigr\},\ \bigl\{\{0,3\},\{0,1,3\},\{0,1,2,3\}\bigr\}.
\end{aligned}$$

When $S_3 = \{0,1,2\}$ and $S_4 = S_5 = \emptyset$, we have 4 families indicated by $3+0+0$ $(= |S_1| + |S_2| + |S_3|)$ in the above because $W_{3+1} = 4$. The numbers of maximal commuting families are $3K_4 = 45$, $3! \cdot 3K_3 = 54$, 6 according to the cases $3+0+0$, $2+1+0$, $1+1+1$, respectively. The total number of them equals 105.



Example 7.5. Suppose $n = 1$. Then there are m maximal commuting families of (L, L'). For example, when $m = 4$, we have

$$\{\mathcal{I}\} = \bigl\{\{\{0,1\}\},\{\{0,2\}\},\{\{0,3\}\},\{\{0,4\}\}\bigr\},\ \ \widehat{\mathcal{I}} = \widehat{\{\{0,3\}\}}\ :$$



Remark 7.6. Suppose $m = 1$. The integrability condition of the KZ-type equation $\mathcal{M}$ equals to that of the extended KZ-type equation with $n+1$ variables which are imposed on residue matrices A_{ij} $(0 \le i < j \le n)$. Hence the KZ-type equation adding the partial derivation with respect to the variable $x_n = y_1$ satisfies the integrability condition. On the other hand, the product of a solution to the extended equation and any function of x_n satisfies the original equation. Hence the dimension of the solutions to the original equation with variables $(x_0, \ldots, x_n)$ is infinite.

7.3. **Spectra and Accessory parameters.** Suppose $n = 1$ and $m = 2$. Then we have the Fuchsian ststem

$$\mathcal{N} : \frac{du}{dx_0} = \frac{A_{01}}{x_0 - x_1} u + \frac{A_{02}}{x_0 - x_2} u \tag{7.9}$$

with three singular points x_1, x_2 and ∞. Putting

$$A_{12} = -A_{01} - A_{02}, \tag{7.10}$$

namely, $A_{12} = A_{0,\infty}$, the equation $\mathcal{N}$ is extended to the KZ-type equation $\mathcal{M}$ with the three variables (x_0, x_1, x_2). Conversely, we may assume (7.10) for the irreducible KZ-type equation with $n = 3$ (cf. Definition 3.5).

In general, $\mathcal{N}$ is not necessarily rigid and has $2r$ accessory parameters ($r = 0, 1, \ldots$). For example, A_{01} and A_{02} are generic matrices in $M(3, \mathbb{C})$, $\mathcal{N}$ has 2 accessory parameters. Since additions and middle convolutions are invertible transformations which dos not change the number of accessory parameters, we get KZ-type equations with n variables which have $2r$ accessory parameters ($n = 3, 4, \ldots$). Conversely, we know the number of accessory parameters of a KZ-type equation if it can be transformed into a KZ-type equation with three variables.

We define that a KZ-type equation $\mathcal{M}$ is **rigid** if the equation is uniquely determined by $\operatorname{Sp}\mathcal{M}$ with no accessory parameter. It is an interesting problem to examine that a KZ-type equation is transformed into another KZ-type equation by a successive application of the transformations we have considered. In particular, the problem is quite interesting if the equations are rigid and so is the problem determining irreducible KZ-type equations which cannot be reduced the rank by any application of these transformations.

A KZ-type equation $\mathcal{M}$ obtained by applying these applications to the trivial equation $u' = 0$ is rigid. In this case, owing to symmetries of $\operatorname{Sp}\mathcal{M}$, we may get several relations between the solutions to the equation as in the case of Kummer's relation for Gauss hypergeometric functions (cf. [5, Remark 5.17]).

In general, for a holonomic system $\mathcal{M}$, we blow up its singular locus to normal crossing singular points and get commuting residue matrices attached to normal crossing divisors (cf. [3]). The **spectra** $\operatorname{Sp}\mathcal{M}$ is the set of conjugacy classes of commuting residue matrices at the normal crossing singular points. When the commuting matrices are semisimple, the conjugacy class are the set of simultaneous eigenvalues and their multiplicities.

7.4. **Semilocal monodromy.** For example, [11] calculates $[\widetilde{A}_{03} + \widetilde{A}_{04}]$ from $[A_{0i}]$ and $[A_{03} + A_{04}]$ etc. in the case $L = \{0\}$ and $L' = \{1, 2, 3, 4\}$. Here we examine to calculate $[\widetilde{A}_{03} + \widetilde{A}_{04}]$ in the case $L = \{0, 1, 2\}$ and $L' = \{3, 4, 5\}$. Formally we have $A_{03} + A_{04} = A_{034} - A_{34}$ and we may calculate as if A_{34} etc. exist.

This corresponds to the tournaments such that the teams (3) and (4) play the final game and so do the teams (4) and (5) if they have won the former games. Moreover we may restrict to the tournaments such that teams (3) and (4) play a semi-final or quarter final game if they have won the preceding games.

For $\{0, 1, 2\} \supset I_1 \supset I_2$, $\widetilde{A}_{I_1 34} - \widetilde{A}_{I_2 34}$ is expressed by residue matrices in (7.4). To get the eigenspace decomposition of $\widetilde{A}_{I_1 34} - \widetilde{A}_{I_2 34}$ from the residue matrices A_{ij} of (7.4), we examine the tournaments containing the games corresponding to $I_1 34$ and $I_2 34$ and apply Theorem 5.1 to maximal commuting residue matrices expressed by the tournaments. Then the simultaneous eigenspace decompositions containing the residue matrices $A_{I_1' 34}$ and $A_{I_2' 34}$ with $I_1' \supset I_2'$ appear but we have only to calculate the eigenspace decompositions containing the matrices $A_{I_1' 34} - A_{I_2' 34}$. In this case, we may have $I_1' = I_2'$ even if $I_1 \supsetneqq I_2$.

7.5. **single-elimination tournaments.** The following table shows the relation between KZ-type equations and single-elimination tournaments discussed in this paper.

KZ-type equation with n variables	Tournament of n teams
Family of maximal commuting residue matrices	Tournament
Spectra of a KZ-type equation	Set of all tournaments
Singular points	Semi-final matches
Local coordinate for desingularization	Result of a tournament before semi-final
Variable of middle convolution	Winner of tournament
Base of upper triangulation of the family	Result of all matches
Middle convolution	Deletion and insertion of the winner
Kernels to define middle convolution	Basic/Top insertion of the winner
With other m fixed singular points	Divide n teams into m groups

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