Differential equations attached to generalized flag manifolds and their applications to integral geometry

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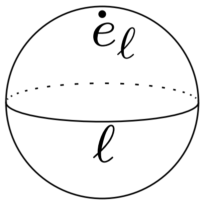
§ Introduction

Radon transform

P. Funk (1916, Math. Ann.)



$$C(S^2) \ni f \mapsto \left(\ell \mapsto R_f(\ell) = \textstyle\int_\ell f(\ell(\theta))d\theta\right)$$

$$\ell(\theta) = e'_\ell \cos\theta + e''_\ell \sin\theta \qquad (\{e_\ell, e'_\ell, e''_\ell\} : \text{orthonormal})$$

$$\Longrightarrow \ C^\infty(S^2/\mathbb{Z}_2) \overset{\sim}{\to} C^\infty(S^2/\mathbb{Z}_2)$$

$\mathbb{P}^2(\mathbb{R}) = S^2/\mathbb{Z}_2$: 2-dimensional real projective space

Grassmann manifolds

$$\mathrm{Gr}_k(\mathbb{R}^n) := \{V_k : k\text{-dimensional subspaces } \subset \mathbb{R}^n\} \qquad (\mathbb{P}^{n-1}(\mathbb{R}) = \mathrm{Gr}_1(\mathbb{R}^n))$$

$$M^o(n,k;\mathbb{R}) := \left\{ X = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \cdots & \vdots \\ x_{n1} & \cdots & x_{nk} \end{pmatrix} \in M(n,k;\mathbb{R});\ \operatorname{rank} X = k \right\}$$

$$= M^o(n,k;\mathbb{R})/GL(k,\mathbb{R}) \ni \begin{pmatrix} I_k \\ x \end{pmatrix} \qquad (x \in M(n-k,k;\mathbb{R}))$$

$\dim \mathrm{Gr}_k(\mathbb{R}^n) = nk - k^2 = (n-k)k$

$$\mathcal{R}^k_\ell : C(\mathrm{Gr}_k(\mathbb{R}^n)) \ni f \mapsto (\mathcal{R}^k_\ell f)(x) \in C(\mathrm{Gr}_\ell(\mathbb{R}^n)) \qquad (0 < k < \ell < n)$$

$$(\mathcal{R}^k_\ell f)(x) = \int_{x \ni t: k\text{-dimensional subspaces}} f(t)dt$$

$G = GL(n,\mathbb{R})$ transitively acts on $\mathrm{Gr}_k(\mathbb{R}^n)$ from the left.

For a point $\mathrm{Gr}_k(\mathbb{R}^n) \ni \begin{pmatrix} I_k \\ 0 \end{pmatrix} GL(k,\mathbb{R}) = \begin{pmatrix} GL(k,\mathbb{R}) \\ 0 \end{pmatrix}$.

Put $P_{k,n} := \{g \in G;\ {}^t g^{-1}\binom{GL(k,\mathbb{R})}{0} = \binom{GL(k,\mathbb{R})}{0}\}$. Then

$$\mathrm{Gr}_k(\mathbb{R}^n) = GL(n,\mathbb{R})/P_{k,n} \qquad (= O(n)/O(k) \times O(n-k))$$

$$P_{k,n} = \left\{ p = \begin{pmatrix} g_1 & 0 \\ y & g_2 \end{pmatrix};\ g_1 \in GL(k,\mathbb{R}),\ g_2 \in GL(n-k,\mathbb{R}),\ y \in M(n-k,k;\mathbb{R}) \right\}$$

$$\supset P := \{ \begin{pmatrix} * & & & \\ * & * & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & * \end{pmatrix} \in GL(n,\mathbb{R})\}$$

$$\mathcal{B}(G/P_{k,n};L_\lambda) := \{f \in \mathcal{B}(G);\ f(xp) = f(x)|\det g_1|^{\lambda_1}|\det g_2|^{\lambda_2}, \quad \forall p \in P_{k,n}\}$$

$$(= \mathcal{B}(O(n)/O(k) \times O(n-k)))$$

$$= \{f \in \mathcal{B}(M^o(n,k;\mathbb{R}));\ f(Xg_1) = f(X)|\det g_1|^{-\lambda_1}, \quad \forall g_1 \in GL(k,\mathbb{R})\}$$

$$(x = \left(x_{ij}\right)_{\substack{1\le i\le n \\ 1\le j\le n}} \in G \subset \mathbb{R}^{n^2}, \quad x \mapsto {}^t x^{-1} \mapsto X)$$

$$\Longrightarrow (\mathcal{R}_\ell^k f)(x) = \int_{O(\ell)/O(k)\times O(\ell-k)} f(xk)dk$$

Studied by Gelfand, Helgason, Rubin, Grinberg, Gonzalez, Kakehi etc.

Problem. Characterize the image of $\mathcal{R}_\ell^k$! $\quad (\ell + k < n)$

Poisson transform

$\mathcal{P}^n_{k,\lambda} : \mathcal{B}(G/P_{k,n}; L_\lambda) \ni f \mapsto (\mathcal{P}^n_{k,\lambda} f)(x) = \int_{O(n)} f(xk)dk \in \mathcal{B}(G/K) \quad (K = O(n))$

G/K: a Riemannian symmetric space

$$\mathcal{P}^n_\lambda : \mathcal{B}(G/P; L_\lambda) := \{f \in \mathcal{B}(G);\ f(g \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \cdots & \cdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix})$$
$$= f(g)|a_{11}|^{\lambda_1}|a_{22}|^{\lambda_2} \cdots |a_{nn}|^{\lambda_n}\}$$
$$\ni f \mapsto (\mathcal{P}^n_\lambda f)(x) = \int_{O(n)} f(xk)dk \in \mathcal{B}(G/K)$$

$$\Longrightarrow \mathcal{P}^n_{k,\lambda} : \mathcal{B}(G/P_{k,n}; L_\lambda) \hookrightarrow \mathcal{B}(G/P; L_\lambda) \overset{\mathcal{P}^n_\lambda}{\to} \mathcal{B}(G/K)$$

The inverse map of $\mathcal{P}^n_\lambda$: *the boundary value map*.

Problem. Characterize the image of $\mathcal{P}^n_{k,\lambda}$! (In some cases, the boundary value problem of a Hermitian symmetric space and its Shilov boundary)

Remark. $(\mathcal{P}^n_\lambda f)(cx) = |c|^{\lambda_1+\cdots+\lambda_n}(\mathcal{P}^n_\lambda f)(x) \Rightarrow$ We may restrict on $SL(n,\mathbb{R})$.

Example 1. $(n = 2,\ k = 1)$
$SL(2,\mathbb{R})/SO(2) \simeq D := \{z \in \mathbb{C}\,;\ |z| < 1\}$, $SL(2,\mathbb{R})/(P \cap SL(2,\mathbb{R})) \simeq \partial D$ and

$$\mathcal{P}^2_{1,\lambda} = \mathcal{P}^2_\lambda : \mathcal{B}(\partial D) \ni f \mapsto \int_0^{2\pi} f(e^{i\theta}) \left(\frac{1-|z|^2}{|e^{i\theta}-z|^2}\right)^{1+\lambda} \frac{d\theta}{2\pi} \in \mathcal{A}(D;\mathcal{M}_\lambda),$$

$$\mathcal{M}_\lambda : \frac{(1-|z|^2)^2}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u = \lambda(\lambda+1)u \qquad (z = x+iy)$$

$\lambda \notin \{-1,-2,-3,\ldots\} \Rightarrow \mathcal{P}^2_\lambda : \mathcal{B}(\partial D) \overset{\sim}{\to} \mathcal{A}(D;\mathcal{M}_\lambda)$ a topological G-isomorphism (Helgason, 1970).

§ Differential Equations

For $X \in M(n,\mathbb{R})$, $E_{ij} := \left(\delta_{pi}\delta_{qj}\right)_{\substack{1\le p\le n\\1\le q\le n}}$, $x = \left(x_{ij}\right)_{\substack{1\le i\le n\\1\le j\le n}} \in G$ and $\phi \in C^\infty(G)$, put

$$(X\phi)(x) := \frac{d}{dt}\phi(xe^{tX})\Big|_{t=0}, \qquad E_{ij} = \sum_{\nu=1}^{n} x_{\nu i}\frac{\partial}{\partial x_{\nu j}},$$

$$(\pi(X)\phi)(x) := \frac{d}{dt}\phi(e^{-tX}x)\Big|_{t=0} = \frac{d}{dt}\phi(xe^{-tx^{-1}Xx})\Big|_{t=0}, \qquad \pi(E_{ij}) = -\sum_{\nu=1}^{n} x_{j\nu}\frac{\partial}{\partial x_{i\nu}},$$

$$(\pi(X)\phi)(x) = (-(\mathrm{Ad}(x^{-1})X)\phi)(x), \qquad \mathrm{Ad}(x)X = xXx^{-1} \in M(n,\mathbb{C})$$

$$\mathfrak{g} = M(n,\mathbb{R}) \text{ is a Lie algebra}, \qquad [E_{ij}, E_{k\ell}] = \epsilon(\delta_{jk}E_{i\ell} - \delta_{\ell i}E_{kj})$$

$U^\epsilon(\mathfrak{g})$: the universal enveloping algebra ($\supset \mathfrak{g}$) $\simeq$ a subalgebra of $\mathcal{D}(G)$ (the ring of differential operators on G) generated by X ($X \in \mathfrak{g}$). ($\epsilon = 1$)

Definition 2. $\mathbf{E}_{n,k,\lambda} := \{P \in U(\mathfrak{g});\ \pi(P)\mathcal{B}(G/P_{k,n}; L_\lambda) = 0\}$.

Fact. i) The image of the Radon transform $\mathcal{R}^n_k$, Poisson transform $\mathcal{P}^n_{k,\lambda}$, a Penrose transform or Whittaker integrals, intertwining operators etc. (any G-homomorphism of $\mathcal{B}(G/P_{k,n}; L_\lambda)$ or a certain cohomology space of sections of a line bundle over the complexification of $G/P_{k,n}$) satisfies the differential equations given by $\mathbf{E}_{n,k,\lambda}$.

ii) $\mathbf{E}_{n,k,\lambda} \simeq a(\bigcap_{g\in G} \mathrm{Ad}(g) J_{k,\lambda})$

$$J_{k,\lambda} = \sum_{\substack{i\neq j \\ i>k \text{ or } j\leq k}} U(\mathfrak{g})E_{ij} + \sum_{i=1}^{k} U(\mathfrak{g})(E_{ii}-\lambda_1) + \sum_{j=k+1}^{n} U(\mathfrak{g})(E_{jj}-\lambda_2)$$

by $a : U(\mathfrak{g}) \ni XY \mapsto (-Y)(-X) \ni U(\mathfrak{g})$ (anti-automorphism, $X, Y \in \mathfrak{g}$)

Problem. Find a good generator system of $\mathbf{E}_{n,k,\lambda}$.

Fact. $\mathbf{E}_{n,k,\lambda}$ is a *quantization* of the defining ideal of the conjugacy class of the matrices $A_k(\lambda_1, \lambda_2) := \begin{pmatrix} \lambda_1 I_k & 0 \\ * & \lambda_2 I_{n-k} \end{pmatrix}$ in $M(n, \mathbb{C})$.

Quantize *characteristic polynomials*, *minimal polynomials*, *elementary divisors* in the linear algebra!

Theorem 3 (Minimal Polynomial). The quantization of the minimal polynomial $(x-\lambda_1)(x-\lambda_2)$ of $A_k(\lambda_1,\lambda_2)$ equals $(x-\lambda_1)(x-k\epsilon-\lambda_2)$ and for a generic λ $a(\mathbf{E}_{n,k,\lambda})$ is generated by

$$\left\langle \Big((\mathbb{E}-\lambda_1)(\mathbb{E}-k\epsilon-\lambda_2) \Big)_{ij}, \ \sum_{i=1}^{n} E_{ii} - k\lambda_1 - (n-k)\lambda_2 \right\rangle$$

Here $\mathbb{E} = \Big(E_{ij}\Big)_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} = X\partial \in M(n, U(\mathfrak{g}))$ with $X = \Big(x_{ij}\Big)_{\substack{1\leq i\leq n \\ 1\leq j\leq n}}$, $\partial = \Big(\frac{\partial}{\partial x_{ij}}\Big)_{\substack{1\leq i\leq n \\ 1\leq j\leq n}}$.

Cor. $\lambda_1 - \lambda_2 \notin \{1, 2, 3, \ldots\} \Rightarrow$ the image of $\mathcal{P}^n_{k,\lambda}$ is characterized by this system

Remark. $\mathrm{rank}(A_k(\lambda_1,\lambda_2)-\lambda_1) \le n-k \Rightarrow (n-k+1)$-minors of $A_k(\lambda_1,\lambda_2)-\lambda_1$ vanish. ($\epsilon = 0$)

Theorem 4 (minors, elementary divisors). Assume $2k \le n$ (for simplicity).
$\lambda_1 - \lambda_2 \notin \{\epsilon, \cdots, (n-k)\epsilon\} \;\Rightarrow$

$$a(\mathbf{E}_{n,k,\lambda}) = \Big\langle \det\Big(E_{i_\mu j_\nu} - (\lambda_1 + (\nu - n + k - 1)\epsilon)\delta_{i_\mu j_\nu}\Big)_{\substack{1\le\mu\le n-k+1\\ 1\le\nu\le n-k+1}}, \det\Big(E_{i'_\mu j'_\nu} - (\lambda_2 + (\nu-1)\epsilon)\delta_{i'_\mu j'_\nu}\Big)_{\substack{1\le\mu\le k+1\\ 1\le\nu\le k+1}}\Big\rangle.$$

$\lambda_1 - \lambda_2 \in \{k\epsilon, \cdots, (n-k)\epsilon\} \;\Rightarrow$

$$a(\mathbf{E}_{n,k,\lambda}) = \Big\langle \frac{d}{dt}\det\Big(E_{i_\mu j_\nu} - (t + \lambda_1 + (\nu - n + k - 1)\epsilon)\delta_{i_\mu j_\nu}\Big)_{\substack{1\le\mu\le n-k+1\\ 1\le\nu\le n-k+1}}\Big|_{t=0}, \det\Big(E_{i'_\mu j'_\nu} - (\lambda_2 + (\nu-1)\epsilon)\delta_{i'_\mu j'_\nu}\Big)_{\substack{1\le\mu\le k+1\\ 1\le\nu\le k+1}}\Big\rangle,$$

Here

$$\epsilon = \begin{cases} 0 & \text{(classical)} \\ 1 & \text{(quantum)} \end{cases}$$

$$\det\big(A_{ij}\big) = \sum_\sigma \mathrm{sign}(\sigma) A_{\sigma(1)1} A_{\sigma(2)2}\cdots, \qquad I = \{i_1, \ldots, i_{n-k-1}\} \text{ etc.}$$

This also characterizes the image of $\mathcal{P}^n_{k,\lambda}$ under the same condition.

Fact. $\mathcal{R}_\ell^k$ is lifted to the G-map

$$\mathcal{R}_\ell^k : \mathcal{B}(G/P_{k,n}; L_{\ell,0}) \to \mathcal{B}(G/P_{\ell,n}; L_{k,0}).$$

Theorem 5 ([O, 1996]). $0 < k < k+\ell < n \Rightarrow \mathcal{R}_\ell^k$ is a *topological G-isomorphism* onto

$$\begin{aligned}\Big\{\Phi\Big((x_{ij})_{\substack{1\le i\le \ell\\ 1\le j\le n}}\Big) \in \mathcal{B}(M^0(n,\ell;\mathbb{R}));&\\ \Phi(xg) = |\det g|^{-k}\Phi(x) \quad &\text{for } g \in GL(\ell,\mathbb{R}),\\ \det\Big(\frac{\partial}{\partial x_{i_\mu j_\nu}}\Big)_{\substack{1\le\mu\le k+1\\ 1\le\nu\le k+1}}\Phi(x) = 0 \quad &\text{(Capell type)}\\ \text{for } \ 1 \le i_1 < \cdots < i_{k+1} \le n,\ 1 \le j_1 < \cdots < j_{k+1} \le \ell&\Big\}.\end{aligned} \tag{1}$$

§ Generalization

The elementary divisors for every element A of $M(n,\mathbb{C}) \xrightarrow{\text{quantization}}$ *the annihilator of any generalized Verma module of the scalar type* for $\mathfrak{gl}(n)$ (with its classical limit) ([O, Adv. in Math., 2005]).

Classical limits: the nilpotent conjugacy class (a generator system of its defining ideal) by Kostant (regular nilpotent) and Weyman (any nilpotent, 1989; conjectured by Tanisaki).

$$A \sim A_{\Theta,\lambda} := \left\{ \begin{pmatrix} \lambda_1 I_{n'_1} & & & & \mathbf{0} \\ A_{21} & \lambda_2 I_{n'_2} & & & \\ A_{31} & A_{32} & \lambda_3 I_{n'_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ A_{L1} & A_{L2} & A_{L3} & \cdots & \lambda_L I_{n'_L} \end{pmatrix} \in M(n,\mathbb{C})\,;\ A_{ij} \in M(n'_i, n'_j; \mathbb{C}) \right\},$$

$$J_\Theta(\lambda) = \sum_{\nu=1}^{L} \sum_{\substack{n_{\nu-1}<i\le n_L \\ n_{\nu-1}<j\le n_\nu}} U(\mathfrak{g})^\epsilon (E_{ij} - \lambda_\nu \delta_{ij}) \qquad (n_0 = 0,\ n_\nu := n'_1 + \cdots + n'_\nu),$$

$$\mathbf{E}_\Theta(\lambda) := \bigcap_{g \in GL(n,\mathbb{C})} \mathrm{Ad}(g) J_\Theta(\lambda) \qquad (\leftarrow \text{Construct its generator system!}),$$

$$z^{(\ell)} := \begin{cases} z\big(z-\epsilon\big)\cdots\big(z-(\ell-1)\epsilon\big) & \text{if } \ell > 0, \\ 1 & \text{if } \ell \le 0, \end{cases}$$

$$d_m^\epsilon(x) = \prod_{j=1}^{L} \big(x - \lambda_j - n_{j-1}\epsilon\big)^{(n'_j + m - n)} \qquad (d_n^\epsilon(x)\text{: characteristic polynomial}),$$

$$e_m^\epsilon(x) = d_m^\epsilon(x)/d_{m-1}^\epsilon(x) \qquad (\text{elementary divisors}),$$

$$q^\epsilon(x) = \prod_{j=1}^{L} \big(x - \lambda_j - n_{j-1}\epsilon\big) \qquad (\text{minimal polynomial})$$

The quantization of the minimal polynomial $q_\Theta(x)$ is defined and calculated for any simple Lie algebra $\mathfrak{g}$ and its non-trivial finite-dimensional representation $(\pi, \mathbb{C}^N)$ and for the generalized Verma module $\mathcal{M}_\Theta(\lambda)$ of the scalar type ([O], [O-Oda, J. of Lie Theory, 2006]).
([Gould, 1985] for the characteristic polynomial)

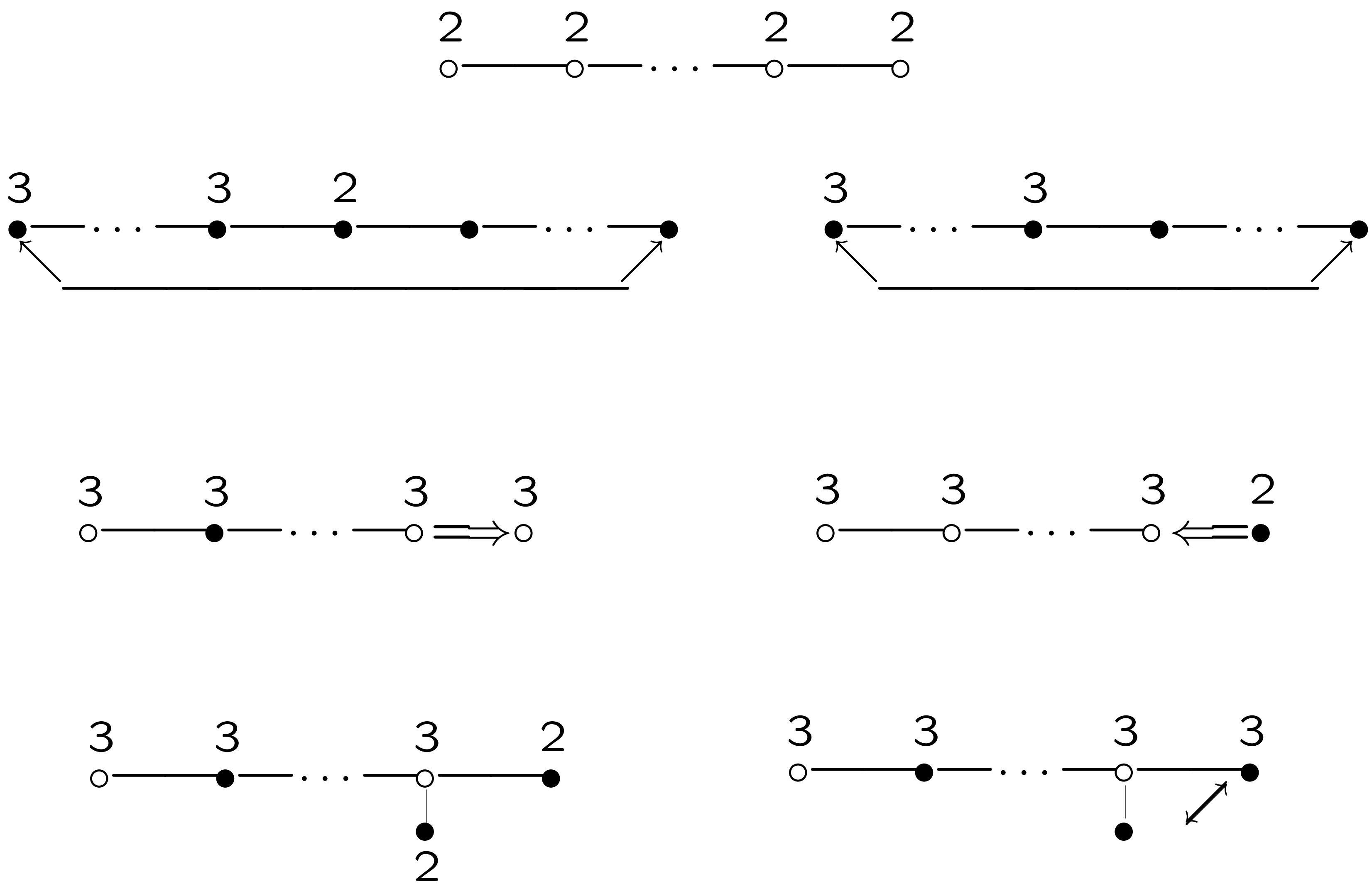
ϖ: the projection of $M(N;\mathbb{C})$ to $\mathfrak{g}$ with respect to
$\langle X, Y\rangle = \mathrm{Trace}\, XY \ (X, Y \in M(N,\mathbb{C}))$
$\mathbb{E}_\pi := \Big(\varpi(E_{ij})\Big)_{\substack{1\le i\le N,\\ 1\le j\le N}} \in M(N; U(\mathfrak{g}))$,
$q_{\pi,\Theta}(\mathbb{E}_\pi)\mathcal{M}_\Theta(\lambda) = 0$.

Theorem 6 ([O], [Oda-O, J. of Lie Theory, 2006]).
$\bar\pi$: the lowest weight of π.
$\mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$
$\mathcal{W}(\pi)$: the set of the weights of $(\pi, \mathbb{C}^N)$.
$\overline{\mathcal{W}}_\mp(\pi)$: the set of the lowest weights of $(\pi|_{\mathfrak{l}_\Theta}, \mathbb{C}^N)$.

$$q_{\pi,\Theta}(x) := \prod_{\varpi\in\overline{\mathcal{W}}_\mp(\pi)}{}' \Big(x - \langle\lambda,\varpi\rangle - \frac{1}{2}\langle\bar\pi - \varpi, \bar\pi + \varpi - 2\rho\epsilon\rangle\Big)$$

$$\pi = \begin{cases} \text{the natural representation if } \mathfrak{g} \text{ is classical,} \\ \text{the minimal dimensional representation if } \mathfrak{g} \text{ is exceptional.} \end{cases}$$

Degree of minimal polynomial for the natural representation



§. Applications.

Radon transformation.

1. We can generalize Gelfand's hypergeometric functions based on the Radon transformations on projective spaces.

2. The image of the Radon transforms of the functions $\{0 \subset V_1^{(k_1)} \subset V_2^{(k_2)} \subset \mathbb{R}^n\}$ to those of $\{0 \subset U_1^{(\ell_1)} \subset U_2^{(\ell_2)} \subset \mathbb{R}^n\}$ is characterized by our system (not by the differential equations defined by Kakehi etc. They are K-invariant operators restricting K-spectrum).

3. Radon transformations are special cases of intertwining operators between degenerate principal series.

Poisson transformation.

1. Our system characterize the image for the generic eigenvalues.
($\Leftarrow$ The infinitesimal character doesn't satisfy integral condition or G is of Type A, B, C or BC and the infinitesimal character is in an open Weyl chamber or).

2. If the symmetric space is realized with a boundary $G/P_{\ominus}$ (such as Satake compactification) and a simultaneous eigenfunction of the invariant differential operator has a natural weighted boundary value on $G/P_{\ominus}$ (in the sense of [Ben Said-Oshima-Shimeno, Int. Math. Res. Not., 2003]), then the function automatically satisfies our system.

3. We can generalized to the space of sections of an associated line bundle on a Riemannian symmetric space.

4. Suppose $p > q$ and the boundary is $U(p,q)/P_k$ with $p > q$ (Here the Levi part of P_k ($1 \leq k \leq q-1$) is $U(p-q) \times GL(q-k) \times \mathbb{T}^k$). Then $\mathfrak{g}$-stable generators defined from minimal polynomials are generated by $2k+1$-th order operators. But we have K-stable generators with degree $2k$.

$$GL(p+q) \supset \begin{pmatrix} GL(p) & \\ & GL(q) \end{pmatrix}, \ \exists q_{2k}(t) : \text{a polynomial}, \ q_{2k}(\mathbb{E}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow B \text{ or } C$$

Whittaker models (Realization in $\mathrm{Ind}_N^G(\varpi)$)
ϖ : a unitary character of the maximal nilpotent subgroup of G.

The regularity condition of ϖ for the existence (or multiplicity) of Whittaker models for a degenerate principal series $\mathrm{Ind}_{P_\Theta}^G(\tau)$ of G induced from a finite dimensional irreducible representation τ of P_Θ with a generic parameter is obtained by using our system.

For example, when $G = GL(n, \mathbb{R})$
"(the partition $\Leftarrow$ non-vanishing parts of ϖ) is dual of (the partition $\Leftarrow$ the Levi part of P_Θ)"
$\Leftrightarrow$ the Whittaker model with moderate growth is of multiplicity free.

In this case, the Whittaker vector is reduced to the usual Whittaker function
$\Leftrightarrow$ P_Θ is maximal.

This is the end of my Talk.
Thank you!