

## Middle convolutions of KZ-type equations

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### Single-elimination tournaments (cf. [2])

We consider single-elimination tournaments of  $n$  teams. The tournament of 3 teams distinguished by the labels 0, 1 and 2 are expressed by . This chart means that team 0 and team 1 play first and the winner of this game and team 2 play the final game. If the teams are not distinguished, there are two patterns . In the case of the tournaments of  $n$  teams, the patterns correspond binary trees with  $n - 1$  non-leaf nodes and a root and  $n$  leaves and the total number of them are given by the Catalan number  $C_{n-1}$ . In this case, there are  $n - 1$  games and each game determines a final winner of some teams.

The single-elimination tournament of three teams is determined by the teams playing the first game and there are 3 cases of the tournaments, which are given by

$$\{\{0, 1\}, \{0, 1, 2\}\}, \{\{1, 2\}, \{0, 1, 2\}\}, \{\{0, 2\}, \{0, 1, 2\}\}. \quad (1)$$

These are the sets of games in the tournaments and each game is shown by the set of teams such that the game determines the final winner of the teams.

We call an isomorphic class of patterns under suitable permutation of teams a type of the tournament. If we distinguish the final winner, there are two types of the tournaments of 3 teams, which we call win types. There is one type of the tournaments of 3 teams and there are two win types of them.

$$\begin{array}{c} \diagup \diagdown \\ 0 \quad 1 \quad 2 \end{array} = \begin{array}{c} \diagup \diagdown \\ 1 \quad 0 \quad 2 \end{array} = \begin{array}{c} \diagup \diagdown \\ 2 \quad 0 \quad 1 \end{array} \neq \begin{array}{c} \diagup \diagdown \\ 1 \quad 2 \quad 0 \end{array} \quad \begin{array}{c} \diagup \diagdown \\ 0 \quad 2 \quad 1 \end{array}, \quad \begin{array}{c} \diagup \diagdown \\ \circ \quad \diagup \diagdown \end{array} \quad \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \quad \circ \end{array}$$

The above left hand side shows the expression of the tournament and the thick segment indicates that the two tournaments selecting two players of the game are isomorphic. The right hand side gives the tournaments distinguished only by the final winner indicated by  $\circ$ . Thus we have

### 3 teams

2 patterns : 

[illegible]

1 type, 2 patterns, 3 (cases of) tournaments, 2 win types

4 teams

$$\begin{array}{c} \text{ } \\ | \\ \text{---} \\ / \quad \backslash \\ | \quad | \quad | \quad | \\ 0 \quad 1 \quad 2 \quad 3 \end{array} : 1 \text{ pattern}$$
$$0 \quad 1 \quad 2 \quad 3 : \frac{4!}{2^3} = 3 \text{ cases}$$

 : 4 patterns  
 $0 \underset{\circ}{\quad} 1 \underset{\circ}{\quad} 2 \underset{\circ}{\quad} 3 : \frac{4!}{2} = 12 \text{ cases}$

2 types, 5 patterns, 15 tournaments, 4 win types

5 teams


 : 2 patterns  
 :  $\frac{5!}{2^3} = 15$  ca


: 4 patterns  
0 1 2 3 4 :  $\frac{5!}{2^2} = 30$  cases,

: 8 patterns
   
 :  $\frac{5!}{2} = 60$  cases

3 types, 14 patterns, 105 tournaments, 9 win types

### Numbers of single-elimination tournaments

|                          |   |   |    |     |     |       |        |         |          |           |
|--------------------------|---|---|----|-----|-----|-------|--------|---------|----------|-----------|
| teams                    | 2 | 3 | 4  | 5   | 6   | 7     | 8      | 9       | 10       | 11        |
| variables as HG          | 0 | 1 | 2  | 3   | 4   | 5     | 6      | 7       | 8        | 9         |
| points in $\mathbb{P}^1$ | 3 | 4 | 5  | 6   | 7   | 8     | 9      | 10      | 11       | 12        |
| types                    | 1 | 1 | 2  | 3   | 6   | 11    | 23     | 46      | 98       | 207       |
| patterns                 | 1 | 2 | 5  | 14  | 42  | 132   | 429    | 1430    | 4862     | 16796     |
| tournaments              | 1 | 3 | 15 | 105 | 945 | 10395 | 135135 | 2027025 | 34459425 | 654729075 |
| win types                | 1 | 2 | 4  | 9   | 20  | 46    | 106    | 248     | 582      | 1376      |

### Middle convolution of KZ-type equation (cf. [2])

For a KZ-type equation

$$\mathcal{M} : \frac{\partial u}{\partial x_i} = \sum_{\substack{0 \leq \nu < n \\ \nu \neq i}} \frac{A_{i\nu}}{x_i - x_\nu} u \quad (A_{i\nu} \in M(N, \mathbb{C}), A_{ij} = A_{j,i}, i = 0, \dots, n-1)$$

Setting

$$\begin{aligned} A_{j,\infty} &:= -A_{j,0} - \dots - A_{j,j-1} - A_{j,j+1} - \dots - A_{j,n-1}, \\ A_I &:= \sum_{p,q \in I, p < q} A_{pq} \quad (I \subset \{0, \dots, n-1, \infty\}). \end{aligned}$$

the integrability condition for  $\mathcal{M}$  shows

$$[A_I, A_J] = 0 \quad (I \cap J = \emptyset \text{ or } I \subset J \text{ or } J \subset I).$$

For a matrix  $A \in M(N, \mathbb{C})$ , let  $\{[\lambda_1]_{m_1}, \dots, [\lambda_k]_{m_k}\}$  denote the eigenvalues with multiplicities of  $A$ , namely,  $N = m_1 + \dots + m_k$  and

$$\text{rank}(A - \lambda_1) \dots (A - \lambda_\nu) = N - m_1 - \dots - m_\nu \quad (\nu = 1, \dots, k)$$

if  $m_1 \geq m_2 \geq \dots \geq m_k$ . In this case, we write  $[A] = [[\lambda_1]_{m_1}, \dots, [\lambda_k]_{m_k}]$ . When  $A_1, \dots, A_L \in M(N, \mathbb{C})$  satisfies  $[A_i, A_j] = 0$  ( $1 \leq i < j \leq L$ ), we set

$$[A_1 : \dots : A_L] = [[\lambda_{1,1} : \dots : \lambda_{L,1}]_{m_1}, \dots, [\lambda_{1,k} : \dots : \lambda_{L,k}]_{m_k}].$$

Here  $(\lambda_{1,\nu}, \dots, \lambda_{L,\nu})$  are simultaneous eigenvalues with multiplisities  $m_\nu$  of  $(A_1, \dots, A_L)$ .

Let  $\mathcal{L}_n$  be the set of single-elimination tournaments of  $n$  teams labeled by  $L_n := \{0, 1, \dots, n-1\}$ . Then  $\mathcal{L}_n$  is the set of three element given in (1). Then for any  $\mathcal{I} \in \mathcal{L}_n$ ,  $|\mathcal{I}| = n-1$  and the residue matrices  $A_I$  ( $I \in \mathcal{I}$ ) commute to each other (cf. [1, 2]).

The *convolution* of  $\mathcal{M}$  with a parameter  $\mu \in \mathbb{C}$  change  $A_I$  into  $\tilde{A}_I \in M((n-1)N, \mathbb{C})$ . Then, there exist invariant subspaces  $\mathcal{L}_j$  for  $j = 1, \dots, n-1$  and  $j = \infty$  and the *middle convolutions*  $\text{mc}_{x_0, \mu} A_I$  are defined by  $\tilde{A}_I|_{\mathbb{C}^{(n-1)N}/(\mathcal{L}_1 + \dots + \mathcal{L}_{n-1} + \mathcal{L}_\infty)}$ . Here, we note that  $\mathcal{L}_1 + \dots + \mathcal{L}_{n-1} + \mathcal{L}_\infty$  is a direct sum, and  $\mathcal{L}_\nu \simeq \ker A_{0,\nu}$  and  $\mathcal{L}_\infty \simeq \ker(A_{0,\infty} - \mu)$ .

We can calculate  $[\text{mc}_{x_0, \mu} A_I]$  by Theorem 5.1 in [2], which gives the middle convolution  $\text{mc}_{x_0, \mu}$  of

$$\{[A_{I^{(1)}} : \dots : A_{I^{(n-1)}}] \mid (I^{(1)}, \dots, I^{(n-1)}) \in \mathcal{L}_n\}.$$

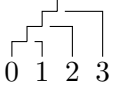
We may assume  $I^{(n-1)} = L_n$  and  $A_{L_n} = 0$  in the above, In this case,  $\text{mc}_{x_0, \mu} A_{L_n} = \mu$ .

In the following, we interpret the claim of [2, Theorem 5.1] in the case  $n \leq 6$ . See Section 6. Examples and Remark 6.2 in [2] for more details. We first show every win-type by a tournament diagram of  $n$ -teams. Then

$$[I^{(0)} : \dots : I^{(n-2)}], [b^0(I^{(0)}) : \dots : b^0(I^{(n-1)})], [I_b^{(0)} : \dots : I_b^{(n-1)}]$$

is shown. We start from the case when  $n = 4$ , which is a repetition of 6. Examples in [2].

4 teams, 2 variables, 4 cases (5 points in  $\mathbb{P}^1$ )



$$[[0,1],[0,1,2]] \quad [[1],[2],[3]] \quad [[0,1],[0,2],[0,3]]$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V \tilde{A}_* U$$

$$\tilde{A}_{01} = \begin{pmatrix} \mu + A_{01} & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{01} & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{012} = \begin{pmatrix} \mu + A_{012} & 0 & A_{03} \\ 0 & \mu + A_{012} & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{012} & 0 & A_{03} \\ 0 & \mu + A_{012} & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix}$$

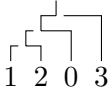
$$[\tilde{A}_{01} : \tilde{A}_{012}] = \{[\mu + A_{01} : \mu + A_{012}], [0 : \mu + A_{012}], [0 : A_{12}]\}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]_{\mathcal{L}_1} = [A_{01} + \mu : A_{012} + \mu]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]_{\mathcal{L}_2} = [0 : A_{012} + \mu]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]_{\mathcal{L}_3} = [0 : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012}]_{\mathcal{L}_\infty} = [0 : A_{12}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[0,1,2],[1,2]] \quad [[1,2],[1],[3]] \quad [[0,2],[1,2],[0,3]]$$

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V \tilde{A}_* U$$

$$\tilde{A}_{012} = \begin{pmatrix} \mu + A_{012} & 0 & A_{03} \\ 0 & \mu + A_{012} & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{012} & 0 & A_{03} \\ 0 & \mu + A_{012} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} -A_{01} + A_{012} & -A_{02} & 0 \\ -A_{01} & -A_{02} + A_{012} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} A_{12} & -A_{01} & 0 \\ 0 & A_{012} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}$$

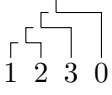
$$[\tilde{A}_{012} : \tilde{A}_{12}] = \{[\mu + A_{012} : A_{12}], [\mu + A_{012} : A_{012}], [A_{12} : A_{12}]\}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]_{\mathcal{L}_1} = [A_{012} + \mu : A_{012}]|_{\ker A_{01}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]_{\mathcal{L}_2} = [A_{012} + \mu : A_{012}]|_{\ker A_{02}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]_{\mathcal{L}_3} = [A_{12} : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12}]_{\mathcal{L}_\infty} = [A_{12} : A_{12}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[1,2,3],[1,2]] \quad [[1,2,3],[1,2],[1]] \quad [[0,3],[1,3],[1,2]]$$

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V\tilde{A}_*U$$

$$\tilde{A}_{123} = \begin{pmatrix} -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & -A_{02} & -A_{03} \end{pmatrix} \rightarrow \begin{pmatrix} -A_{01} - A_{03} - A_{02} & -A_{01} - A_{02} & -A_{01} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} -A_{01} + A_{012} & -A_{02} & 0 \\ -A_{01} & -A_{02} + A_{012} & 0 \\ 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} A_{12} & 0 & 0 \\ 0 & A_{12} & -A_{01} \\ 0 & 0 & A_{012} \end{pmatrix}$$

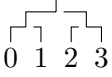
$$[\tilde{A}_{123} : \tilde{A}_{12}] = \{[A_{123} : A_{12}], [0 : A_{12}], [0 : A_{012}]\}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_1} = [0 : A_{012}]|_{\ker A_{01}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_2} = [0 : A_{012}]|_{\ker A_{02}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_3} = [0 : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_\infty} = [A_{123} : A_{12}]|_{\ker(A_{0\infty} - \mu)}$$



$$[[0,1],[2,3]] \quad [[1],[2,3],[2]] \quad [[0,1],[0,3],[2,3]]$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V\tilde{A}_*U$$

$$\tilde{A}_{01} = \begin{pmatrix} \mu + A_{01} & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{01} & A_{03} + A_{02} & A_{02} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & -A_{02} + A_{023} & -A_{03} \\ 0 & -A_{02} & -A_{03} + A_{023} \end{pmatrix} \rightarrow \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23} & -A_{02} \\ 0 & 0 & A_{023} \end{pmatrix}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}] = \{[\mu + A_{01} : A_{23}], [0 : A_{23}], [0 : A_{023}]\}$$

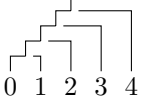
$$[\tilde{A}_{01} : \tilde{A}_{23}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{23}]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}|_{\mathcal{L}_2} = [0 : A_{023}]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}|_{\mathcal{L}_3} = [0 : A_{023}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{23}|_{\mathcal{L}_\infty} = [0 : A_{23}]|_{\ker(A_{0\infty} - \mu)}$$

5 teams, 3 variables, 9 cases (6 points in  $\mathbb{P}^1$ )



$[[0,1],[0,1,2],[0,1,2,3]]$

$[[1],[2],[3],[4]]$

$[[0,1],[0,2],[0,3],[0,4]]$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V \tilde{A}_* U$$

$$\tilde{A}_{01} = \begin{pmatrix} \mu + A_{01} & A_{02} & A_{03} & A_{04} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{01} & A_{02} & A_{03} & A_{04} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{012} = \begin{pmatrix} \mu + A_{012} & 0 & A_{03} & A_{04} \\ 0 & \mu + A_{012} & A_{03} & A_{04} \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{012} & 0 & A_{03} & A_{04} \\ 0 & \mu + A_{012} & A_{03} & A_{04} \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix}$$

$$\tilde{A}_{0123} = \begin{pmatrix} \mu + A_{0123} & 0 & 0 & A_{04} \\ 0 & \mu + A_{0123} & 0 & A_{04} \\ 0 & 0 & \mu + A_{0123} & A_{04} \\ 0 & 0 & 0 & A_{123} \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{0123} & 0 & 0 & A_{04} \\ 0 & \mu + A_{0123} & 0 & A_{04} \\ 0 & 0 & \mu + A_{0123} & A_{04} \\ 0 & 0 & 0 & A_{123} \end{pmatrix}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{0123}] = \{[\mu + A_{01} : \mu + A_{012} : \mu + A_{0123}], [0 : \mu + A_{012} : \mu + A_{0123}], [0 : A_{12} : \mu + A_{0123}], [0 : A_{12} : A_{123}]\}$$

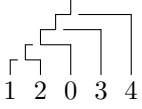
$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{0123}]_{\mathcal{L}_1} = [A_{01} + \mu : A_{012} + \mu : A_{0123} + \mu]_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{0123}]_{\mathcal{L}_2} = [0 : A_{012} + \mu : A_{0123} + \mu]_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{0123}]_{\mathcal{L}_3} = [0 : A_{12} : A_{0123} + \mu]_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{0123}]_{\mathcal{L}_4} = [0 : A_{12} : A_{123}]_{\ker A_{04}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{0123}]_{\mathcal{L}_\infty} = [0 : A_{12} : A_{123}]_{\ker (A_{0\infty} - \mu)}$$



$[[0,1,2],[1,2],[0,1,2,3]]$

$[[1,2],[1],[3],[4]]$

$[[0,2],[1,2],[0,3],[0,4]]$

$$U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V \tilde{A}_* U$$

$$\tilde{A}_{012} = \begin{pmatrix} \mu + A_{012} & 0 & A_{03} & A_{04} \\ 0 & \mu + A_{012} & A_{03} & A_{04} \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{012} & 0 & A_{03} & A_{04} \\ 0 & \mu + A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} -A_{01} + A_{012} & -A_{02} & 0 & 0 \\ -A_{01} & -A_{02} + A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} A_{12} & -A_{01} & 0 & 0 \\ 0 & A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix}$$

$$\tilde{A}_{0123} = \begin{pmatrix} \mu + A_{0123} & 0 & 0 & A_{04} \\ 0 & \mu + A_{0123} & 0 & A_{04} \\ 0 & 0 & \mu + A_{0123} & A_{04} \\ 0 & 0 & 0 & A_{123} \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{0123} & 0 & 0 & A_{04} \\ 0 & \mu + A_{0123} & 0 & 0 \\ 0 & 0 & \mu + A_{0123} & A_{04} \\ 0 & 0 & 0 & A_{123} \end{pmatrix}$$

$$[\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{0123}] = \{[\mu + A_{012} : A_{12} : \mu + A_{0123}], [\mu + A_{012} : A_{012} : \mu + A_{0123}], [A_{12} : A_{12} : \mu + A_{0123}], [A_{12} : A_{12} : A_{123}]\}$$

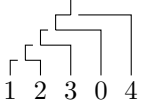
$$[\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{0123}]_{\mathcal{L}_1} = [A_{012} + \mu : A_{012} : A_{0123} + \mu]|_{\ker A_{01}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{0123}]_{\mathcal{L}_2} = [A_{012} + \mu : A_{012} : A_{0123} + \mu]|_{\ker A_{02}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{0123}]_{\mathcal{L}_3} = [A_{12} : A_{12} : A_{0123} + \mu]|_{\ker A_{03}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{0123}]_{\mathcal{L}_4} = [A_{12} : A_{12} : A_{123}]|_{\ker A_{04}}$$

$$[\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{0123}]_{\mathcal{L}_\infty} = [A_{12} : A_{12} : A_{123}]|_{\ker (A_{0\infty} - \mu)}$$



$[[0,1,2,3],[1,2,3],[1,2]]$

$[[1,2,3],[1,2],[1],[4]]$

$[[0,3],[1,3],[1,2],[0,4]]$

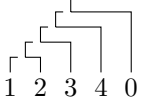
$$U = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V \tilde{A}_* U$$

$$\tilde{A}_{0123} = \begin{pmatrix} \mu + A_{0123} & 0 & 0 & A_{04} \\ 0 & \mu + A_{0123} & 0 & A_{04} \\ 0 & 0 & \mu + A_{0123} & A_{04} \\ 0 & 0 & 0 & A_{123} \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{0123} & 0 & 0 & A_{04} \\ 0 & \mu + A_{0123} & 0 & 0 \\ 0 & 0 & \mu + A_{0123} & 0 \\ 0 & 0 & 0 & A_{123} \end{pmatrix}$$

$$\tilde{A}_{123} = \begin{pmatrix} -A_{01} + A_{0123} & -A_{02} & -A_{03} & 0 \\ -A_{01} & -A_{02} + A_{0123} & -A_{03} & 0 \\ -A_{01} & -A_{02} & -A_{03} + A_{0123} & 0 \\ 0 & 0 & 0 & A_{123} \end{pmatrix} \rightarrow \begin{pmatrix} A_{123} & -A_{01} - A_{02} & -A_{01} & 0 \\ 0 & A_{0123} & 0 & 0 \\ 0 & 0 & A_{0123} & 0 \\ 0 & 0 & 0 & A_{123} \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} -A_{01} + A_{012} & -A_{02} & 0 & 0 \\ -A_{01} & -A_{02} + A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} A_{12} & 0 & 0 & 0 \\ 0 & A_{12} & -A_{01} & 0 \\ 0 & 0 & A_{012} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix}$$

$$\begin{aligned}
[\tilde{A}_{0123} : \tilde{A}_{123} : \tilde{A}_{12}] &= \{[\mu + A_{0123} : A_{123} : A_{12}], [\mu + A_{0123} : A_{0123} : A_{12}], [\mu + A_{0123} : A_{0123} : A_{012}], [A_{123} : A_{123} : A_{12}]\} \\
[\tilde{A}_{0123} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_1}] &= [A_{0123} + \mu : A_{0123} : A_{012}]|_{\ker A_{01}} \\
[\tilde{A}_{0123} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_2}] &= [A_{0123} + \mu : A_{0123} : A_{012}]|_{\ker A_{02}} \\
[\tilde{A}_{0123} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_3}] &= [A_{0123} + \mu : A_{0123} : A_{12}]|_{\ker A_{03}} \\
[\tilde{A}_{0123} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_4}] &= [A_{123} : A_{123} : A_{12}]|_{\ker A_{04}} \\
[\tilde{A}_{0123} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_\infty}] &= [A_{123} : A_{123} : A_{12}]|_{\ker (A_{0\infty} - \mu)}
\end{aligned}$$



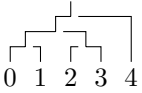
$$[[1, 2, 3, 4], [1, 2, 3], [1, 2]]$$

$$[[1, 2, 3, 4], [1, 2, 3], [1, 2], [1]]$$

$$[[0, 4], [1, 4], [1, 3], [1, 2]]$$

$$\begin{aligned}
U &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V \tilde{A}_* U \\
\tilde{A}_{1234} &= \begin{pmatrix} -A_{01} & -A_{02} & -A_{03} & -A_{04} \\ -A_{01} & -A_{02} & -A_{03} & -A_{04} \\ -A_{01} & -A_{02} & -A_{03} & -A_{04} \\ -A_{01} & -A_{02} & -A_{03} & -A_{04} \end{pmatrix} \rightarrow \begin{pmatrix} -A_{01} - A_{04} - A_{03} - A_{02} & -A_{01} - A_{03} - A_{02} & -A_{01} - A_{02} & -A_{01} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\tilde{A}_{123} &= \begin{pmatrix} -A_{01} + A_{0123} & -A_{02} & -A_{03} & 0 \\ -A_{01} & -A_{02} + A_{0123} & -A_{03} & 0 \\ -A_{01} & -A_{02} & -A_{03} + A_{0123} & 0 \\ 0 & 0 & 0 & A_{123} \end{pmatrix} \rightarrow \begin{pmatrix} A_{123} & 0 & 0 & 0 \\ 0 & A_{123} & -A_{01} - A_{02} & -A_{01} \\ 0 & 0 & A_{0123} & 0 \\ 0 & 0 & 0 & A_{0123} \end{pmatrix} \\
\tilde{A}_{12} &= \begin{pmatrix} -A_{01} + A_{012} & -A_{02} & 0 & 0 \\ -A_{01} & -A_{02} + A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} A_{12} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & A_{12} & -A_{01} \\ 0 & 0 & 0 & A_{012} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
[\tilde{A}_{1234} : \tilde{A}_{123} : \tilde{A}_{12}] &= \{[A_{1234} : A_{123} : A_{12}], [0 : A_{123} : A_{12}], [0 : A_{0123} : A_{12}], [0 : A_{0123} : A_{012}]\} \\
[\tilde{A}_{1234} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_1}] &= [0 : A_{0123} : A_{012}]|_{\ker A_{01}} \\
[\tilde{A}_{1234} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_2}] &= [0 : A_{0123} : A_{012}]|_{\ker A_{02}} \\
[\tilde{A}_{1234} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_3}] &= [0 : A_{0123} : A_{12}]|_{\ker A_{03}} \\
[\tilde{A}_{1234} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_4}] &= [0 : A_{123} : A_{12}]|_{\ker A_{04}} \\
[\tilde{A}_{1234} : \tilde{A}_{123} : \tilde{A}_{12}|_{\mathcal{L}_\infty}] &= [A_{1234} : A_{123} : A_{12}]|_{\ker (A_{0\infty} - \mu)}
\end{aligned}$$



$$[[0, 1], [0, 1, 2, 3], [2, 3]]$$

$$[[1], [2, 3], [2], [4]]$$

$$[[0, 1], [0, 3], [2, 3], [0, 4]]$$

$$\begin{aligned}
U &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V \tilde{A}_* U \\
\tilde{A}_{01} &= \begin{pmatrix} \mu + A_{01} & A_{02} & A_{03} & A_{04} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{01} & A_{03} + A_{02} & A_{02} & A_{04} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\tilde{A}_{0123} = \begin{pmatrix} \mu + A_{0123} & 0 & 0 & A_{04} \\ 0 & \mu + A_{0123} & 0 & A_{04} \\ 0 & 0 & \mu + A_{0123} & A_{04} \\ 0 & 0 & 0 & A_{123} \end{pmatrix} \rightarrow \begin{pmatrix} \mu + A_{0123} & 0 & 0 & A_{04} \\ 0 & \mu + A_{0123} & 0 & A_{04} \\ 0 & 0 & \mu + A_{0123} & 0 \\ 0 & 0 & 0 & A_{123} \end{pmatrix}$$

$$\tilde{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 & 0 \\ 0 & -A_{02} + A_{023} & -A_{03} & 0 \\ 0 & -A_{02} & -A_{03} + A_{023} & 0 \\ 0 & 0 & 0 & A_{23} \end{pmatrix} \rightarrow \begin{pmatrix} A_{23} & 0 & 0 & 0 \\ 0 & A_{23} & -A_{02} & 0 \\ 0 & 0 & A_{023} & 0 \\ 0 & 0 & 0 & A_{23} \end{pmatrix}$$

$$[\tilde{A}_{01} : \tilde{A}_{0123} : \tilde{A}_{23}] = \{[\mu + A_{01} : \mu + A_{0123} : A_{23}], [0 : \mu + A_{0123} : A_{23}], [0 : \mu + A_{0123} : A_{023}], [0 : A_{123} : A_{23}]\}$$

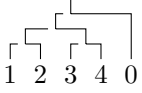
$$[\tilde{A}_{01} : \tilde{A}_{0123} : \tilde{A}_{23}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{0123} + \mu : A_{23}]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{0123} : \tilde{A}_{23}|_{\mathcal{L}_2} = [0 : A_{0123} + \mu : A_{023}]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{0123} : \tilde{A}_{23}|_{\mathcal{L}_3} = [0 : A_{0123} + \mu : A_{023}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{0123} : \tilde{A}_{23}|_{\mathcal{L}_4} = [0 : A_{123} : A_{23}]|_{\ker A_{04}}$$

$$[\tilde{A}_{01} : \tilde{A}_{0123} : \tilde{A}_{23}|_{\mathcal{L}_\infty} = [0 : A_{123} : A_{23}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[1, 2, 3, 4], [3, 4], [1, 2]]$$

$$[[1, 2, 3, 4], [1, 2], [3], [1]]$$

$$[[0, 4], [1, 4], [3, 4], [1, 2]]$$

$$U = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V \tilde{A}_* U$$

$$\tilde{A}_{1234} = \begin{pmatrix} -A_{01} & -A_{02} & -A_{03} & -A_{04} \\ -A_{01} & -A_{02} & -A_{03} & -A_{04} \\ -A_{01} & -A_{02} & -A_{03} & -A_{04} \\ -A_{01} & -A_{02} & -A_{03} & -A_{04} \end{pmatrix} \rightarrow \begin{pmatrix} -A_{01} - A_{04} - A_{03} - A_{02} & -A_{01} - A_{02} & -A_{03} & -A_{01} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{A}_{34} = \begin{pmatrix} A_{34} & 0 & 0 & 0 \\ 0 & A_{34} & 0 & 0 \\ 0 & 0 & -A_{03} + A_{034} & -A_{04} \\ 0 & 0 & -A_{03} & -A_{04} + A_{034} \end{pmatrix} \rightarrow \begin{pmatrix} A_{34} & 0 & -A_{03} & 0 \\ 0 & A_{34} & A_{03} & 0 \\ 0 & 0 & A_{034} & 0 \\ 0 & 0 & 0 & A_{34} \end{pmatrix}$$

$$\tilde{A}_{12} = \begin{pmatrix} -A_{01} + A_{012} & -A_{02} & 0 & 0 \\ -A_{01} & -A_{02} + A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} A_{12} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & -A_{01} \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{012} \end{pmatrix}$$

$$[\tilde{A}_{1234} : \tilde{A}_{34} : \tilde{A}_{12}] = \{[A_{1234} : A_{34} : A_{12}], [0 : A_{34} : A_{12}], [0 : A_{034} : A_{12}], [0 : A_{34} : A_{012}]\}$$

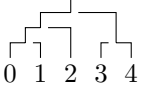
$$[\tilde{A}_{1234} : \tilde{A}_{34} : \tilde{A}_{12}|_{\mathcal{L}_1} = [0 : A_{34} : A_{012}]|_{\ker A_{01}}$$

$$[\tilde{A}_{1234} : \tilde{A}_{34} : \tilde{A}_{12}|_{\mathcal{L}_2} = [0 : A_{34} : A_{012}]|_{\ker A_{02}}$$

$$[\tilde{A}_{1234} : \tilde{A}_{34} : \tilde{A}_{12}|_{\mathcal{L}_3} = [0 : A_{034} : A_{12}]|_{\ker A_{03}}$$

$$[\tilde{A}_{1234} : \tilde{A}_{34} : \tilde{A}_{12}|_{\mathcal{L}_4} = [0 : A_{034} : A_{12}]|_{\ker A_{04}}$$

$$[\tilde{A}_{1234} : \tilde{A}_{34} : \tilde{A}_{12}|_{\mathcal{L}_\infty} = [A_{1234} : A_{34} : A_{12}]|_{\ker (A_{0\infty} - \mu)}$$



$[[0,1],[0,1,2],[3,4]]$

$[[1],[2],[3,4],[3]]$

$[[0,1],[0,2],[0,4],[3,4]]$

$$\begin{aligned}
U &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V\tilde{A}_*U \\
\tilde{A}_{01} &= \begin{pmatrix} \mu+A_{01} & A_{02} & A_{03} & A_{04} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \mu+A_{01} & A_{02} & A_{04}+A_{03} & A_{03} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\tilde{A}_{012} &= \begin{pmatrix} \mu+A_{012} & 0 & A_{03} & A_{04} \\ 0 & \mu+A_{012} & A_{03} & A_{04} \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \mu+A_{012} & 0 & A_{04}+A_{03} & A_{03} \\ 0 & \mu+A_{012} & A_{04}+A_{03} & A_{03} \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \\
\tilde{A}_{34} &= \begin{pmatrix} A_{34} & 0 & 0 & 0 \\ 0 & A_{34} & 0 & 0 \\ 0 & 0 & -A_{03}+A_{034} & -A_{04} \\ 0 & 0 & -A_{03} & -A_{04}+A_{034} \end{pmatrix} \rightarrow \begin{pmatrix} A_{34} & 0 & 0 & 0 \\ 0 & A_{34} & 0 & 0 \\ 0 & 0 & A_{34} & -A_{03} \\ 0 & 0 & 0 & A_{034} \end{pmatrix}
\end{aligned}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{34}] = \{[\mu+A_{01} : \mu+A_{012} : A_{34}], [0 : \mu+A_{012} : A_{34}], [0 : A_{12} : A_{34}], [0 : A_{12} : A_{034}]\}$$

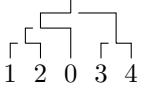
$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{34}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{012} + \mu : A_{34}]|_{\ker A_{01}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{34}|_{\mathcal{L}_2} = [0 : A_{012} + \mu : A_{34}]|_{\ker A_{02}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{34}|_{\mathcal{L}_3} = [0 : A_{12} : A_{034}]|_{\ker A_{03}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{34}|_{\mathcal{L}_4} = [0 : A_{12} : A_{034}]|_{\ker A_{04}}$$

$$[\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{34}|_{\mathcal{L}_\infty} = [0 : A_{12} : A_{34}]|_{\ker (A_{0\infty} - \mu)}$$



$[[0,1,2],[1,2],[3,4]]$

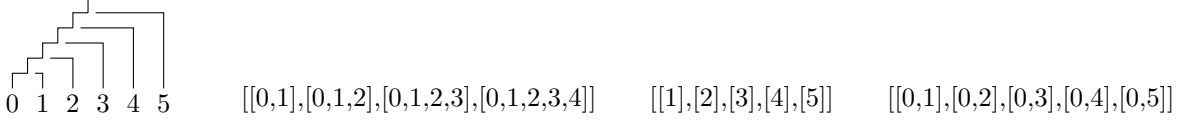
$[[1,2],[1],[3,4],[3]]$

$[[0,2],[1,2],[0,4],[3,4]]$

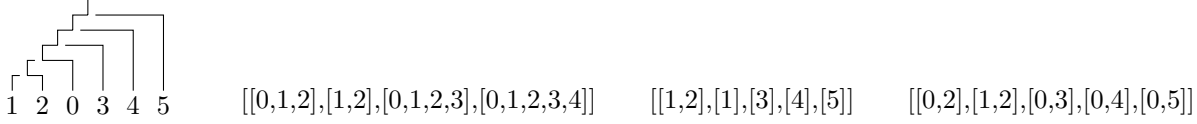
$$\begin{aligned}
U &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad V = U^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \tilde{A}_* \rightarrow V\tilde{A}_*U \\
\tilde{A}_{012} &= \begin{pmatrix} \mu+A_{012} & 0 & A_{03} & A_{04} \\ 0 & \mu+A_{012} & A_{03} & A_{04} \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \mu+A_{012} & 0 & A_{04}+A_{03} & A_{03} \\ 0 & \mu+A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \\
\tilde{A}_{12} &= \begin{pmatrix} -A_{01}+A_{012} & -A_{02} & 0 & 0 \\ -A_{01} & -A_{02}+A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \rightarrow \begin{pmatrix} A_{12} & -A_{01} & 0 & 0 \\ 0 & A_{012} & 0 & 0 \\ 0 & 0 & A_{12} & 0 \\ 0 & 0 & 0 & A_{12} \end{pmatrix} \\
\tilde{A}_{34} &= \begin{pmatrix} A_{34} & 0 & 0 & 0 \\ 0 & A_{34} & 0 & 0 \\ 0 & 0 & -A_{03}+A_{034} & -A_{04} \\ 0 & 0 & -A_{03} & -A_{04}+A_{034} \end{pmatrix} \rightarrow \begin{pmatrix} A_{34} & 0 & 0 & 0 \\ 0 & A_{34} & 0 & 0 \\ 0 & 0 & A_{34} & -A_{03} \\ 0 & 0 & 0 & A_{034} \end{pmatrix}
\end{aligned}$$



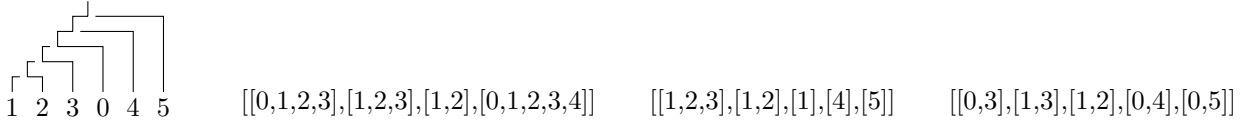
6 teams, 4 variables, 20 cases (7 points in  $\mathbb{P}^1$ )



$$\begin{aligned}\mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{0123} : \tilde{A}_{01234}] \quad [[1], [2], [3], [4], [5]] \\ &= \{[\mu + A_{01} : \mu + A_{012} : \mu + A_{0123} : \mu + A_{01234}], [0 : \mu + A_{012} : \mu + A_{0123} : \mu + A_{01234}], \\ &\quad [0 : A_{12} : \mu + A_{0123} : \mu + A_{01234}], [0 : A_{12} : A_{123} : \mu + A_{01234}], [0 : A_{12} : A_{123} : A_{1234}]\} \\ \mathcal{A}|_{\mathcal{L}_1} &= [A_{01} + \mu : A_{012} + \mu : A_{0123} + \mu : A_{01234} + \mu]|_{\ker A_{01}} \\ \mathcal{A}|_{\mathcal{L}_2} &= [0 : A_{012} + \mu : A_{0123} + \mu : A_{01234} + \mu]|_{\ker A_{02}} \\ \mathcal{A}|_{\mathcal{L}_3} &= [0 : A_{12} : A_{0123} + \mu : A_{01234} + \mu]|_{\ker A_{03}} \\ \mathcal{A}|_{\mathcal{L}_4} &= [0 : A_{12} : A_{123} : A_{01234} + \mu]|_{\ker A_{04}} \\ \mathcal{A}|_{\mathcal{L}_5} &= [0 : A_{12} : A_{123} : A_{1234}]|_{\ker A_{05}} \\ \mathcal{A}|_{\mathcal{L}_\infty} &= [0 : A_{12} : A_{123} : A_{1234}]|_{\ker (A_{0\infty} - \mu)}\end{aligned}$$



$$\begin{aligned}\mathcal{A} &:= [\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{0123} : \tilde{A}_{01234}] \quad [[1, 2], [1], [3], [4], [5]] \\ &= \{[\mu + A_{012} : A_{12} : \mu + A_{0123} : \mu + A_{01234}], [\mu + A_{012} : A_{012} : \mu + A_{0123} : \mu + A_{01234}], \\ &\quad [A_{12} : A_{12} : \mu + A_{0123} : \mu + A_{01234}], [A_{12} : A_{12} : A_{123} : \mu + A_{01234}], [A_{12} : A_{12} : A_{123} : A_{1234}]\} \\ \mathcal{A}|_{\mathcal{L}_1} &= [A_{012} + \mu : A_{012} : A_{0123} + \mu : A_{01234} + \mu]|_{\ker A_{01}} \\ \mathcal{A}|_{\mathcal{L}_2} &= [A_{012} + \mu : A_{012} : A_{0123} + \mu : A_{01234} + \mu]|_{\ker A_{02}} \\ \mathcal{A}|_{\mathcal{L}_3} &= [A_{12} : A_{12} : A_{0123} + \mu : A_{01234} + \mu]|_{\ker A_{03}} \\ \mathcal{A}|_{\mathcal{L}_4} &= [A_{12} : A_{12} : A_{123} : A_{01234} + \mu]|_{\ker A_{04}} \\ \mathcal{A}|_{\mathcal{L}_5} &= [A_{12} : A_{12} : A_{123} : A_{1234}]|_{\ker A_{05}} \\ \mathcal{A}|_{\mathcal{L}_\infty} &= [A_{12} : A_{12} : A_{123} : A_{1234}]|_{\ker (A_{0\infty} - \mu)}\end{aligned}$$



$$\begin{aligned}\mathcal{A} &:= [\tilde{A}_{0123} : \tilde{A}_{123} : \tilde{A}_{12} : \tilde{A}_{01234}] \quad [[1, 2, 3], [1, 2], [1], [4], [5]] \\ &= \{[\mu + A_{0123} : A_{123} : A_{12} : \mu + A_{01234}], [\mu + A_{0123} : A_{0123} : A_{12} : \mu + A_{01234}], \\ &\quad [\mu + A_{0123} : A_{0123} : A_{012} : \mu + A_{01234}], [A_{123} : A_{123} : A_{12} : \mu + A_{01234}], [A_{123} : A_{123} : A_{12} : A_{1234}]\} \\ \mathcal{A}|_{\mathcal{L}_1} &= [A_{0123} + \mu : A_{0123} : A_{012} : A_{01234} + \mu]|_{\ker A_{01}} \\ \mathcal{A}|_{\mathcal{L}_2} &= [A_{0123} + \mu : A_{0123} : A_{012} : A_{01234} + \mu]|_{\ker A_{02}} \\ \mathcal{A}|_{\mathcal{L}_3} &= [A_{0123} + \mu : A_{0123} : A_{12} : A_{01234} + \mu]|_{\ker A_{03}} \\ \mathcal{A}|_{\mathcal{L}_4} &= [A_{123} : A_{123} : A_{12} : A_{01234} + \mu]|_{\ker A_{04}} \\ \mathcal{A}|_{\mathcal{L}_5} &= [A_{123} : A_{123} : A_{12} : A_{1234}]|_{\ker A_{05}} \\ \mathcal{A}|_{\mathcal{L}_\infty} &= [A_{123} : A_{123} : A_{12} : A_{1234}]|_{\ker (A_{0\infty} - \mu)}\end{aligned}$$

$$[[0,1,2,3,4],[1,2,3,4],[1,2,3],[1,2]] \quad [[1,2,3,4],[1,2,3],[1,2],[1],[5]] \quad [[0,4],[1,4],[1,3],[1,2],[0,5]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01234} : \tilde{A}_{1234} : \tilde{A}_{123} : \tilde{A}_{12}] \quad [[1, 2, 3, 4], [1, 2, 3], [1, 2], [1], [5]] \\ &= \{[\mu + A_{01234} : A_{1234} : A_{123} : A_{12}], [\mu + A_{01234} : A_{01234} : A_{123} : A_{12}], \\ &\quad [\mu + A_{01234} : A_{01234} : A_{0123} : A_{12}], [\mu + A_{01234} : A_{01234} : A_{0123} : A_{012}], [A_{1234} : A_{1234} : A_{123} : A_{12}]\} \\ \mathcal{A}|_{\mathcal{L}_1} &= [A_{01234} + \mu : A_{01234} : A_{0123} : A_{012}]|_{\ker A_{01}} \\ \mathcal{A}|_{\mathcal{L}_2} &= [A_{01234} + \mu : A_{01234} : A_{0123} : A_{012}]|_{\ker A_{02}} \\ \mathcal{A}|_{\mathcal{L}_3} &= [A_{01234} + \mu : A_{01234} : A_{0123} : A_{12}]|_{\ker A_{03}} \\ \mathcal{A}|_{\mathcal{L}_4} &= [A_{01234} + \mu : A_{01234} : A_{123} : A_{12}]|_{\ker A_{04}} \\ \mathcal{A}|_{\mathcal{L}_5} &= [A_{1234} : A_{1234} : A_{123} : A_{12}]|_{\ker A_{05}} \\ \mathcal{A}|_{\mathcal{L}_\infty} &= [A_{1234} : A_{1234} : A_{123} : A_{12}]|_{\ker (A_{0\infty} - \mu)} \end{aligned}$$

$$[[1,2,3,4,5],[1,2,3,4],[1,2,3],[1,2]] \quad [[1,2,3,4,5],[1,2,3,4],[1,2,3],[1,2],[1]] \quad [[0,5],[1,5],[1,4],[1,3],[1,2]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{12345} : \tilde{A}_{1234} : \tilde{A}_{123} : \tilde{A}_{12}] \quad [[1, 2, 3, 4, 5], [1, 2, 3, 4], [1, 2, 3], [1, 2], [1]] \\ &= \{[-A_{01} - A_{05} - A_{04} - A_{03} - A_{02} : A_{1234} : A_{123} : A_{12}], [0 : A_{1234} : A_{123} : A_{12}], \\ &\quad [0 : A_{01234} : A_{123} : A_{12}], [0 : A_{01234} : A_{0123} : A_{12}], [0 : A_{01234} : A_{0123} : A_{012}]\} \\ \mathcal{A}|_{\mathcal{L}_1} &= [A_{012345} : A_{01234} : A_{0123} : A_{012}]|_{\ker A_{01}} \\ \mathcal{A}|_{\mathcal{L}_2} &= [A_{012345} : A_{01234} : A_{0123} : A_{012}]|_{\ker A_{02}} \\ \mathcal{A}|_{\mathcal{L}_3} &= [A_{012345} : A_{01234} : A_{0123} : A_{12}]|_{\ker A_{03}} \\ \mathcal{A}|_{\mathcal{L}_4} &= [A_{012345} : A_{01234} : A_{123} : A_{12}]|_{\ker A_{04}} \\ \mathcal{A}|_{\mathcal{L}_5} &= [A_{012345} : A_{1234} : A_{123} : A_{12}]|_{\ker A_{05}} \\ \mathcal{A}|_{\mathcal{L}_\infty} &= [A_{12345} : A_{1234} : A_{123} : A_{12}]|_{\ker (A_{0\infty} - \mu)} \end{aligned}$$

$$[[0,1],[0,1,2,3],[2,3],[0,1,2,3,4]] \quad [[1],[2,3],[2],[4],[5]] \quad [[0,1],[0,3],[2,3],[0,4],[0,5]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{0123} : \tilde{A}_{23} : \tilde{A}_{01234}] \quad [[1], [2, 3], [2], [4], [5]] \\ &= \{[\mu + A_{01} : \mu + A_{0123} : A_{23} : \mu + A_{01234}], [0 : \mu + A_{0123} : A_{23} : \mu + A_{01234}], \\ &\quad [0 : \mu + A_{0123} : A_{023} : \mu + A_{01234}], [0 : A_{123} : A_{23} : \mu + A_{01234}], [0 : A_{123} : A_{23} : A_{1234}]\} \\ \mathcal{A}|_{\mathcal{L}_1} &= [A_{01} + \mu : A_{0123} + \mu : A_{23} : A_{01234} + \mu]|_{\ker A_{01}} \\ \mathcal{A}|_{\mathcal{L}_2} &= [0 : A_{0123} + \mu : A_{023} : A_{01234} + \mu]|_{\ker A_{02}} \\ \mathcal{A}|_{\mathcal{L}_3} &= [0 : A_{0123} + \mu : A_{023} : A_{01234} + \mu]|_{\ker A_{03}} \\ \mathcal{A}|_{\mathcal{L}_4} &= [0 : A_{123} : A_{23} : A_{01234} + \mu]|_{\ker A_{04}} \\ \mathcal{A}|_{\mathcal{L}_5} &= [0 : A_{123} : A_{23} : A_{1234}]|_{\ker A_{05}} \\ \mathcal{A}|_{\mathcal{L}_\infty} &= [0 : A_{123} : A_{23} : A_{1234}]|_{\ker (A_{0\infty} - \mu)} \end{aligned}$$

$$[[0,1,2,3,4],[1,2,3,4],[3,4],[1,2]] \quad [[1,2,3,4],[1,2],[3],[1],[5]] \quad [[0,4],[1,4],[3,4],[1,2],[0,5]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01234} : \tilde{A}_{1234} : \tilde{A}_{34} : \tilde{A}_{12}] \quad [[1, 2, 3, 4], [1, 2], [3], [1], [5]] \\ &= \{[\mu + A_{01234} : A_{1234} : A_{34} : A_{12}], [\mu + A_{01234} : A_{01234} : A_{34} : A_{12}], \\ &\quad [\mu + A_{01234} : A_{01234} : A_{034} : A_{12}], [\mu + A_{01234} : A_{01234} : A_{34} : A_{012}], [A_{1234} : A_{1234} : A_{34} : A_{12}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{01234} + \mu : A_{01234} : A_{34} : A_{012}]|_{\ker A_{01}}$$

$$\mathcal{A}|_{\mathcal{L}_2} = [A_{01234} + \mu : A_{01234} : A_{34} : A_{012}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [A_{01234} + \mu : A_{01234} : A_{034} : A_{12}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [A_{01234} + \mu : A_{01234} : A_{034} : A_{12}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [A_{1234} : A_{1234} : A_{34} : A_{12}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [A_{1234} : A_{1234} : A_{34} : A_{12}]|_{\ker (A_{0\infty} - \mu)}$$

$$[[1,2,3,4,5],[1,2,3,4],[3,4],[1,2]] \quad [[1,2,3,4,5],[1,2,3,4],[1,2],[3],[1]] \quad [[0,5],[1,5],[1,4],[3,4],[1,2]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{12345} : \tilde{A}_{1234} : \tilde{A}_{34} : \tilde{A}_{12}] \quad [[1, 2, 3, 4, 5], [1, 2, 3, 4], [1, 2], [3], [1]] \\ &= \{[-A_{01} - A_{05} - A_{04} - A_{03} - A_{02} : A_{1234} : A_{34} : A_{12}], [0 : A_{1234} : A_{34} : A_{12}], \\ &\quad [0 : A_{01234} : A_{34} : A_{12}], [0 : A_{01234} : A_{034} : A_{12}], [0 : A_{01234} : A_{34} : A_{012}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [0 : A_{01234} : A_{34} : A_{012}]|_{\ker A_{01}}$$

$$\mathcal{A}|_{\mathcal{L}_2} = [0 : A_{34} : A_{012}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [0 : A_{01234} : A_{034} : A_{12}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [0 : A_{01234} : A_{034} : A_{12}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [0 : A_{1234} : A_{34} : A_{12}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [A_{12345} : A_{1234} : A_{34} : A_{12}]|_{\ker (A_{0\infty} - \mu)}$$

$$[[0,1],[0,1,2],[0,1,2,3,4],[3,4]] \quad [[1],[2],[3,4],[3],[5]] \quad [[0,1],[0,2],[0,4],[3,4],[0,5]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{01234} : \tilde{A}_{34}] \quad [[1], [2], [3, 4], [3], [5]] \\ &= \{[\mu + A_{01} : \mu + A_{012} : \mu + A_{01234} : A_{34}], [0 : \mu + A_{012} : \mu + A_{01234} : A_{34}], \\ &\quad [0 : A_{12} : \mu + A_{01234} : A_{34}], [0 : A_{12} : \mu + A_{01234} : A_{034}], [0 : A_{12} : A_{1234} : A_{34}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{012} + \mu : A_{01234} + \mu : A_{34}]|_{\ker A_{01}}$$

$$\mathcal{A}|_{\mathcal{L}_2} = [0 : A_{012} + \mu : A_{01234} + \mu : A_{34}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [0 : A_{12} : A_{01234} + \mu : A_{034}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [0 : A_{12} : A_{01234} + \mu : A_{034}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [0 : A_{12} : A_{1234} : A_{34}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [0 : A_{12} : A_{1234} : A_{34}]|_{\ker (A_{0\infty} - \mu)}$$

$$\begin{array}{cccc}
1 & 2 & 0 & 3 & 4 & 5 \\
[[0,1,2],[1,2],[0,1,2,3,4],[3,4]] & & [[1,2],[1],[3,4],[3],[5]] & & [[0,2],[1,2],[0,4],[3,4],[0,5]]
\end{array}$$

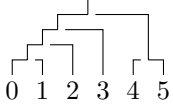
$$\begin{aligned}
\mathcal{A} &:= [\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{01234} : \tilde{A}_{34}] \quad [[1, 2], [1], [3, 4], [3], [5]] \\
&= \{[\mu + A_{012} : A_{12} : \mu + A_{01234} : A_{34}], [\mu + A_{012} : A_{012} : \mu + A_{01234} : A_{34}], \\
&\quad [A_{12} : A_{12} : \mu + A_{01234} : A_{34}], [A_{12} : A_{12} : \mu + A_{01234} : A_{034}], [A_{12} : A_{12} : A_{1234} : A_{34}]\} \\
\mathcal{A}|_{\mathcal{L}_1} &= [A_{012} + \mu : A_{012} : A_{01234} + \mu : A_{34}]|_{\ker A_{01}} \\
\mathcal{A}|_{\mathcal{L}_2} &= [A_{012} + \mu : A_{012} : A_{01234} + \mu : A_{34}]|_{\ker A_{02}} \\
\mathcal{A}|_{\mathcal{L}_3} &= [A_{12} : A_{12} : A_{01234} + \mu : A_{034}]|_{\ker A_{03}} \\
\mathcal{A}|_{\mathcal{L}_4} &= [A_{12} : A_{12} : A_{01234} + \mu : A_{034}]|_{\ker A_{04}} \\
\mathcal{A}|_{\mathcal{L}_5} &= [A_{12} : A_{12} : A_{1234} : A_{34}]|_{\ker A_{05}} \\
\mathcal{A}|_{\mathcal{L}_\infty} &= [A_{12} : A_{12} : A_{1234} : A_{34}]|_{\ker (A_{0\infty} - \mu)}
\end{aligned}$$

$$\begin{array}{cccc}
2 & 3 & 4 & 0 & 1 & 5 \\
[[0,1],[0,1,2,3,4],[2,3,4],[2,3]] & & [[1],[2,3,4],[2,3],[2],[5]] & & [[0,1],[0,4],[2,4],[2,3],[0,5]]
\end{array}$$

$$\begin{aligned}
\mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{01234} : \tilde{A}_{234} : \tilde{A}_{23}] \quad [[1], [2, 3, 4], [2, 3], [2], [5]] \\
&= \{[\mu + A_{01} : \mu + A_{01234} : A_{234} : A_{23}], [0 : \mu + A_{01234} : A_{234} : A_{23}], \\
&\quad [0 : \mu + A_{01234} : A_{0234} : A_{23}], [0 : \mu + A_{01234} : A_{0234} : A_{023}], [0 : A_{1234} : A_{234} : A_{23}]\} \\
\mathcal{A}|_{\mathcal{L}_1} &= [A_{01} + \mu : A_{01234} + \mu : A_{234} : A_{23}]|_{\ker A_{01}} \\
\mathcal{A}|_{\mathcal{L}_2} &= [0 : A_{01234} + \mu : A_{0234} : A_{023}]|_{\ker A_{02}} \\
\mathcal{A}|_{\mathcal{L}_3} &= [0 : A_{01234} + \mu : A_{0234} : A_{023}]|_{\ker A_{03}} \\
\mathcal{A}|_{\mathcal{L}_4} &= [0 : A_{01234} + \mu : A_{0234} : A_{23}]|_{\ker A_{04}} \\
\mathcal{A}|_{\mathcal{L}_5} &= [0 : A_{1234} : A_{234} : A_{23}]|_{\ker A_{05}} \\
\mathcal{A}|_{\mathcal{L}_\infty} &= [0 : A_{1234} : A_{234} : A_{23}]|_{\ker (A_{0\infty} - \mu)}
\end{aligned}$$

$$\begin{array}{cccc}
1 & 2 & 3 & 4 & 5 & 0 \\
[[1,2,3,4,5],[1,2,3],[4,5],[1,2]] & & [[1,2,3,4,5],[1,2,3],[1,2],[4],[1]] & & [[0,5],[1,5],[1,3],[4,5],[1,2]]
\end{array}$$

$$\begin{aligned}
\mathcal{A} &:= [\tilde{A}_{12345} : \tilde{A}_{123} : \tilde{A}_{45} : \tilde{A}_{12}] \quad [[1, 2, 3, 4, 5], [1, 2, 3], [1, 2], [4], [1]] \\
&= \{[-A_{01} - A_{05} - A_{04} - A_{03} - A_{02} : A_{123} : A_{45} : A_{12}], [0 : A_{123} : A_{45} : A_{12}], \\
&\quad [0 : A_{0123} : A_{45} : A_{12}], [0 : A_{123} : A_{045} : A_{12}], [0 : A_{0123} : A_{45} : A_{012}]\} \\
\mathcal{A}|_{\mathcal{L}_1} &= [0 : A_{0123} : A_{45} : A_{012}]|_{\ker A_{01}} \\
\mathcal{A}|_{\mathcal{L}_2} &= [0 : A_{0123} : A_{45} : A_{012}]|_{\ker A_{02}} \\
\mathcal{A}|_{\mathcal{L}_3} &= [0 : A_{0123} : A_{45} : A_{12}]|_{\ker A_{03}} \\
\mathcal{A}|_{\mathcal{L}_4} &= [0 : A_{123} : A_{045} : A_{12}]|_{\ker A_{04}} \\
\mathcal{A}|_{\mathcal{L}_5} &= [0 : A_{123} : A_{045} : A_{12}]|_{\ker A_{05}} \\
\mathcal{A}|_{\mathcal{L}_\infty} &= [A_{12345} : A_{123} : A_{45} : A_{12}]|_{\ker (A_{0\infty} - \mu)}
\end{aligned}$$



$$[[0,1],[0,1,2],[0,1,2,3],[4,5]] \quad [[1],[2],[3],[4,5],[4]] \quad [[0,1],[0,2],[0,3],[0,5],[4,5]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{0123} : \tilde{A}_{45}] \quad [[1],[2],[3],[4,5],[4]] \\ &= \{[\mu + A_{01} : \mu + A_{012} : \mu + A_{0123} : A_{45}], [0 : \mu + A_{012} : \mu + A_{0123} : A_{45}], \\ &\quad [0 : A_{12} : \mu + A_{0123} : A_{45}], [0 : A_{12} : A_{123} : A_{45}], [0 : A_{12} : A_{123} : A_{045}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{012} + \mu : A_{0123} + \mu : A_{45}]|_{\ker A_{01}}$$

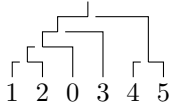
$$\mathcal{A}|_{\mathcal{L}_2} = [0 : A_{012} + \mu : A_{0123} + \mu : A_{45}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [0 : A_{12} : A_{0123} + \mu : A_{45}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [0 : A_{12} : A_{123} : A_{045}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [0 : A_{12} : A_{123} : A_{045}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [0 : A_{12} : A_{123} : A_{45}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[0,1,2],[1,2],[0,1,2,3],[4,5]] \quad [[1,2],[1],[3],[4,5],[4]] \quad [[0,2],[1,2],[0,3],[0,5],[4,5]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{0123} : \tilde{A}_{45}] \quad [[1,2],[1],[3],[4,5],[4]] \\ &= \{[\mu + A_{012} : A_{12} : \mu + A_{0123} : A_{45}], [\mu + A_{012} : A_{012} : \mu + A_{0123} : A_{45}], \\ &\quad [A_{12} : A_{12} : \mu + A_{0123} : A_{45}], [A_{12} : A_{12} : A_{123} : A_{45}], [A_{12} : A_{12} : A_{123} : A_{045}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{012} + \mu : A_{012} : A_{0123} + \mu : A_{45}]|_{\ker A_{01}}$$

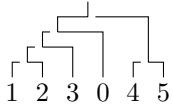
$$\mathcal{A}|_{\mathcal{L}_2} = [A_{012} + \mu : A_{012} : A_{0123} + \mu : A_{45}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [A_{12} : A_{12} : A_{0123} + \mu : A_{45}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [A_{12} : A_{12} : A_{123} : A_{045}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [A_{12} : A_{12} : A_{123} : A_{045}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [A_{12} : A_{12} : A_{123} : A_{45}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[0,1,2,3],[1,2,3],[1,2],[4,5]] \quad [[1,2,3],[1,2],[1],[4,5],[4]] \quad [[0,3],[1,3],[1,2],[0,5],[4,5]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{0123} : \tilde{A}_{123} : \tilde{A}_{12} : \tilde{A}_{45}] \quad [[1,2,3],[1,2],[1],[4,5],[4]] \\ &= \{[\mu + A_{0123} : A_{123} : A_{12} : A_{45}], [\mu + A_{0123} : A_{0123} : A_{12} : A_{45}], \\ &\quad [\mu + A_{0123} : A_{0123} : A_{012} : A_{45}], [A_{123} : A_{123} : A_{12} : A_{45}], [A_{123} : A_{123} : A_{12} : A_{045}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{0123} + \mu : A_{0123} : A_{012} : A_{45}]|_{\ker A_{01}}$$

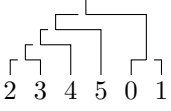
$$\mathcal{A}|_{\mathcal{L}_2} = [A_{0123} + \mu : A_{0123} : A_{012} : A_{45}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [A_{0123} + \mu : A_{0123} : A_{12} : A_{45}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [A_{123} : A_{123} : A_{12} : A_{045}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [A_{123} : A_{123} : A_{12} : A_{045}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [A_{123} : A_{123} : A_{12} : A_{45}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[0,1],[2,3,4,5],[2,3,4],[2,3]] \quad [[1],[2,3,4,5],[2,3,4],[2,3],[2]] \quad [[0,1],[0,5],[2,5],[2,4],[2,3]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{2345} : \tilde{A}_{234} : \tilde{A}_{23}] \quad [[1], [2, 3, 4, 5], [2, 3, 4], [2, 3], [2]] \\ &= \{[\mu + A_{01} : A_{2345} : A_{234} : A_{23}], [0 : A_{2345} : A_{234} : A_{23}], \\ &\quad [0 : A_{02345} : A_{234} : A_{23}], [0 : A_{02345} : A_{0234} : A_{23}], [0 : A_{02345} : A_{0234} : A_{023}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{2345} : A_{234} : A_{23}]|_{\ker A_{01}}$$

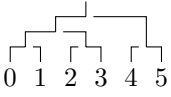
$$\mathcal{A}|_{\mathcal{L}_2} = [0 : A_{02345} : A_{0234} : A_{023}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [0 : A_{02345} : A_{0234} : A_{023}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [0 : A_{02345} : A_{0234} : A_{23}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [0 : A_{02345} : A_{234} : A_{23}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [0 : A_{2345} : A_{234} : A_{23}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[0,1],[0,1,2,3],[2,3],[4,5]] \quad [[1],[2,3],[2],[4,5],[4]] \quad [[0,1],[0,3],[2,3],[0,5],[4,5]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{0123} : \tilde{A}_{23} : \tilde{A}_{45}] \quad [[1], [2, 3], [2], [4, 5], [4]] \\ &= \{[\mu + A_{01} : \mu + A_{0123} : A_{23} : A_{45}], [0 : \mu + A_{0123} : A_{23} : A_{45}], \\ &\quad [0 : \mu + A_{0123} : A_{023} : A_{45}], [0 : A_{123} : A_{23} : A_{45}], [0 : A_{123} : A_{23} : A_{045}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{0123} + \mu : A_{23} : A_{45}]|_{\ker A_{01}}$$

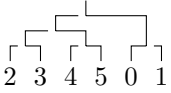
$$\mathcal{A}|_{\mathcal{L}_2} = [0 : A_{0123} + \mu : A_{023} : A_{45}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [0 : A_{0123} + \mu : A_{023} : A_{45}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [0 : A_{123} : A_{23} : A_{045}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [0 : A_{123} : A_{23} : A_{045}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [0 : A_{123} : A_{23} : A_{45}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[0,1],[2,3,4,5],[4,5],[2,3]] \quad [[1],[2,3,4,5],[2,3],[4],[2]] \quad [[0,1],[0,5],[2,5],[4,5],[2,3]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{2345} : \tilde{A}_{45} : \tilde{A}_{23}] \quad [[1], [2, 3, 4, 5], [2, 3], [4], [2]] \\ &= \{[\mu + A_{01} : A_{2345} : A_{45} : A_{23}], [0 : A_{2345} : A_{45} : A_{23}], \\ &\quad [0 : A_{02345} : A_{45} : A_{23}], [0 : A_{02345} : A_{045} : A_{23}], [0 : A_{02345} : A_{45} : A_{023}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{2345} : A_{45} : A_{23}]|_{\ker A_{01}}$$

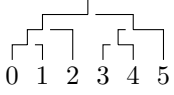
$$\mathcal{A}|_{\mathcal{L}_2} = [0 : A_{02345} : A_{45} : A_{023}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [0 : A_{02345} : A_{45} : A_{023}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [0 : A_{02345} : A_{045} : A_{23}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [0 : A_{02345} : A_{045} : A_{23}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [0 : A_{2345} : A_{45} : A_{23}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[0,1],[0,1,2],[3,4,5],[3,4]] \quad [[1],[2],[3,4,5],[3,4],[3]] \quad [[0,1],[0,2],[0,5],[3,5],[3,4]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{01} : \tilde{A}_{012} : \tilde{A}_{345} : \tilde{A}_{34}] \quad [[1],[2],[3,4,5],[3,4],[3]] \\ &= \{[\mu + A_{01} : \mu + A_{012} : A_{345} : A_{34}], [0 : \mu + A_{012} : A_{345} : A_{34}], \\ &\quad [0 : A_{12} : A_{345} : A_{34}], [0 : A_{12} : A_{0345} : A_{34}], [0 : A_{12} : A_{0345} : A_{034}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{01} + \mu : A_{012} + \mu : A_{345} : A_{34}]|_{\ker A_{01}}$$

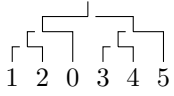
$$\mathcal{A}|_{\mathcal{L}_2} = [0 : A_{012} + \mu : A_{345} : A_{34}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [0 : A_{12} : A_{0345} : A_{034}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [0 : A_{12} : A_{0345} : A_{034}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [0 : A_{12} : A_{0345} : A_{34}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [0 : A_{12} : A_{345} : A_{34}]|_{\ker (A_{0\infty} - \mu)}$$



$$[[0,1,2],[1,2],[3,4,5],[3,4]] \quad [[1,2],[1],[3,4,5],[3,4],[3]] \quad [[0,2],[1,2],[0,5],[3,5],[3,4]]$$

$$\begin{aligned} \mathcal{A} &:= [\tilde{A}_{012} : \tilde{A}_{12} : \tilde{A}_{345} : \tilde{A}_{34}] \quad [[1,2],[1],[3,4,5],[3,4],[3]] \\ &= \{[\mu + A_{012} : A_{12} : A_{345} : A_{34}], [\mu + A_{012} : A_{012} : A_{345} : A_{34}], \\ &\quad [A_{12} : A_{12} : A_{345} : A_{34}], [A_{12} : A_{12} : A_{0345} : A_{34}], [A_{12} : A_{12} : A_{0345} : A_{034}]\} \end{aligned}$$

$$\mathcal{A}|_{\mathcal{L}_1} = [A_{012} + \mu : A_{012} : A_{345} : A_{34}]|_{\ker A_{01}}$$

$$\mathcal{A}|_{\mathcal{L}_2} = [A_{012} + \mu : A_{012} : A_{345} : A_{34}]|_{\ker A_{02}}$$

$$\mathcal{A}|_{\mathcal{L}_3} = [A_{12} : A_{12} : A_{0345} : A_{034}]|_{\ker A_{03}}$$

$$\mathcal{A}|_{\mathcal{L}_4} = [A_{12} : A_{12} : A_{0345} : A_{034}]|_{\ker A_{04}}$$

$$\mathcal{A}|_{\mathcal{L}_5} = [A_{12} : A_{12} : A_{0345} : A_{34}]|_{\ker A_{05}}$$

$$\mathcal{A}|_{\mathcal{L}_\infty} = [A_{12} : A_{12} : A_{345} : A_{34}]|_{\ker (A_{0\infty} - \mu)}$$

## References

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