不確定特異点をもつ 多項式係数常微分方程式の接続問題

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超幾何方程式研究会 2022
神戸大学
5 January, 2022

§ Generalized Riemann scheme

$$\mathcal{M} : \boxed{Pu = 0} \qquad (P = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_0(x) \in W[x])$$

$W[x] = \mathbb{C}[x, \partial]$, $\partial = \frac{d}{dx}$, $\vartheta = x\partial$

$c_0 = \infty, c_1, \ldots, c_p$: the singular points of $\mathcal{M}$ $\quad (a_n(c_j) = 0)$

If $c_1 = 0$, we assume $\mathcal{M}$ has (formal) local solutions

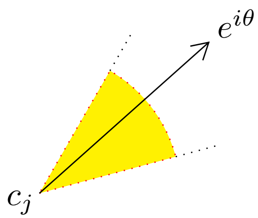
(E) $$\boxed{u_\nu(x) = \phi(x)x^{\lambda^o_{\nu,0}} \exp\Big(-\frac{\lambda^o_{\nu,r}}{rx^r} - \frac{\lambda^o_{\nu,r-1}}{(r-1)x^{r-1}} - \cdots - \frac{\lambda^o_{\nu,1}}{x}\Big)} \quad (\nu = 1, \ldots, n)$$

$\phi(x) = 1 + a_1x + a_2x^2 + \cdots \in \mathbb{C}[\![x]\!]$

$\lambda_\nu(x) := \lambda^o_{\nu,r}x^r + \cdots + \lambda^o_{\nu,1}x + \lambda^o_{\nu,0} \in \mathbb{C}[x]$, $\quad \lambda_\nu \neq \lambda_{\nu'}$ $(\Leftarrow \nu \neq \nu')$

Theorem [Hukuhara,...] $\theta \in \mathbb{R} \Rightarrow \exists\ R > 0$, θ_1, θ_2 and solutions u_ν to $\mathcal{M}$ such that $\theta \in (\theta_1, \theta_2)$ and u_ν have asymptotic (E) for $x \to c_j = 0$ on

$$V_{c_j,R,(\theta_1,\theta_2)} := \{x \in \mathbb{C} \mid |x - c_j| < R,\ \theta_1 < \arg(x - c_j) < \theta_2\}$$



Characteristic exponents $\lambda_{j,\nu} \in \mathbb{C}[x]$ of $\mathcal{M}$ at $x = c_j$ are defined through $c_j \mapsto 0$

- No logarithmic term ($\Leftarrow$ for simplicity)
- No ramified irregular singularity ($\Leftarrow$ essential)

$\{[\lambda_{j,1}]_{(m_1)}, \ldots, [\lambda_{j,n_j}]_{(m_{n_j})}\}$: characteristic exponents of $\mathcal{M}$ at $x = c_j$

$$\lambda_{j,\nu} \in \mathbb{C}[x], \quad n = m_{j,1} + \cdots + m_{j,n_j} \quad (j = 0, \ldots, p),$$

$$[\lambda]_{(m)} := \{\lambda, \lambda+1, \ldots, \lambda+m-1\} \quad (\lambda \in \mathbb{C}[x])$$

Generalized Riemann scheme (GRS)

$$(\bigstar): \left\{\begin{matrix} x = c_j \quad (j = 0, \ldots, p) \\ [\lambda_{j,1}]_{(m_{j,1})} \\ \vdots \\ [\lambda_{j,n_j}]_{(m_{j,n_j})} \end{matrix}\right\}, \quad \left\{\begin{matrix} x = 0 & x = 1 & x = \infty \\ 0 & [0]_{(n-1)} & \alpha_1 \\ 1 - \beta_1 & & \alpha_2 \\ \vdots & & \vdots \\ 1 - \beta_{n-1} & -\beta_n & \alpha_n \end{matrix}\right\} \quad \begin{matrix} 1^n, (n-1)1, 1^n \\ {}_nF_{n-1}(\alpha; \beta; x) \\ \sum a_\nu = \sum b_\nu \end{matrix}$$

- $\deg \lambda_{j,\nu} = 0$ $(\nu = 1, \ldots, n_j)$ $\Rightarrow$ c_j is a regular singular point

- Versal unfolding of (GRS) $\hookrightarrow$ (GRS) of a Fuchsian equation (cf. [Ov])

  $\Rightarrow$ spectral type

Example : spectral type $211|22|22,31$

$Pu=0$: $\mathrm{ord}\,P=4$ with $x=\infty$ (Poincaré rank 2) and 0 (regular singularity)

$$
\begin{aligned}
u(x) &\approx x^{-a_0}(1+o(x^{-1}))e^{-a_1x-\frac12 a_2x^2},\ x^{-a_0-1}e^{-a_1x-\frac12 a_2x^2} && (x\to+\infty)\\
&\approx x^{-b_0}e^{-b_1x},\ x^{-c_0}e^{-b_1x} && (x\to+\infty)\\
&\approx (1+o(x^2))x^{c_1},\ (x+o(x))^{c_1+1},\ x^{c_1+2},\ x^{c_2} && (x\to+0)
\end{aligned}
$$

Fuchs-Hukuhara relation : $2a_0+b_0+c_0+2c_1+c_2+c_3=4$

$$
\begin{Bmatrix}
x=\infty & x=0\\
[a_0+a_1x+a_2x^2]_{(2)} & [c_1]_{(3)}\\
b_0+b_1x & c_2\\
c_0+b_1x &
\end{Bmatrix}
=
\begin{Bmatrix}
x=\infty & (1) & (2) & x=0\\
[a_0]_{(2)} & [a_1]_2 & [a_2]_2 & [c_1]_{(3)}\\
b_0 & [b_1]_2 & [0]_2 & c_2\\
c_0 & & &
\end{Bmatrix}
$$

$\mathrm{idx}(211|22|22,31)=0 \quad\Rightarrow\quad 1\,\big(=1-\frac{\mathrm{idx}}{2}\big)$ accessory parameter

$$
\begin{Bmatrix}
x=\infty & x=\frac{1}{t_1} & x=\frac{1}{t_2} & x=0\\
[a_0-\frac{a_1}{t_1}+\frac{a_2}{t_1t_2}]_{(2)} & [\frac{a_1}{t_1}+\frac{a_2}{t_1(t_1-t_2)}]_{(2)} & [\frac{a_2}{t_2(t_2-t_1)}]_{(2)} & [c_1]_{(3)}\\
b_0-\frac{b_1}{t_1} & [\frac{b_1}{t_1}]_{(2)} & [0]_{(2)} & c_2\\
c_0-\frac{b_1}{t_1} & & &
\end{Bmatrix}
$$

§ Index of rigidity and operations

$$\operatorname{idx}\mathbf{m} = 2n^2 - \sum (n^2 - m_{j,\nu}^2) - \sum \deg(\lambda_{j,\nu} - \lambda_{j,\nu'}) \cdot m_{j,\nu} m_{j,\nu'}$$
$$= 2n^2 - \sum \bigl(n^2 - (m_{j,\nu}^{(r)})^2\bigr)$$

$$d_1(\mathbf{m}) = 2n - \sum (n - m_{j,1}) - \sum \deg(\lambda_{j,\nu} - \lambda_{j,1}) \cdot m_{j,\nu}$$
$$= 2n - \sum (n - m_{j,1}^{(r)})$$

(FC) $$\sum m_{j,\nu}\lambda_{j,\nu}(0) - \operatorname{ord}\mathbf{m} + \tfrac{1}{2}\operatorname{idx}\mathbf{m} := \Bigl|\{[\lambda_{j,\nu}^{(r)}]_{(m_{j,1}^{(r)})}\}\Bigr| = 0$$

Addition

$$u(x) \mapsto v(x) = \varphi(x)u(x) \;\Rightarrow\; \partial := \tfrac{d}{dx} \mapsto \operatorname{Ad}(\varphi)\partial = \varphi \circ \partial \circ \varphi^{-1} = \partial - \tfrac{\varphi'}{\varphi}$$
$$\operatorname{Ad}\bigl((x-c)^\lambda\bigr)\partial = \partial - \tfrac{\lambda}{x-c}, \quad \vartheta := x\partial \mapsto \operatorname{Ad}\bigl(x^\lambda\bigr)\vartheta = \vartheta - \lambda$$

Versal addition

$$\operatorname{Ad}\Biggl(\prod_{r=0}^{r_j}(x - c_j - t_r)^{\sum_{k=r}^{r_j} \frac{\lambda_{j,\nu,k}}{\prod_{\substack{0\le s\le k\\ s\ne r}}(t_r - t_s)}}\Biggr) = \operatorname{Ad}\Bigl(e^{\int \sum_{r=0}^{r_j} \frac{\lambda_{j,\nu,r}}{\prod_{k=0}^{r}(x-c_j-t_k)}dx}\Bigr)$$

$$\partial \mapsto \partial - \sum_{r=0}^{r_j} \frac{\lambda_{j,\nu,r}}{\prod_{k=0}^{r}(x - c_j - t_k)}, \; x \mapsto x$$

Middle convolution : a microlocal operator

$$u(x)\mapsto v(x)=\partial^{-\mu}u=\frac{1}{\Gamma(\mu)}\int_c^x u(t)(x-t)^{\mu-1}dt \quad (c:\text{ singular point, }\mu\in\mathbb{C})$$

$$Pu(x)=0 \;\Rightarrow\; \mathrm{mc}_\mu(P)v(x)=0 \quad (\mathrm{mc}_\mu(\partial)=\partial,\ \ \mathrm{mc}_\mu(\vartheta)=\vartheta-\mu)$$

$$P=\sum_{j=0}^n a_j(x)\partial^j=\sum_{j=0}^n\sum_{i=0}^N c_{ij}x^i\partial^j\in W[x] \quad (a_j(x)\in\mathbb{C}[x],\ \ c_{ij}\in\mathbb{C})$$

$$\partial^N P=\sum_{j=0}^n\sum_{i=0}^N \partial^N c_{ij}x^i\partial^j=\sum_{j=0}^n\sum_{i=0}^N c_{ij}\partial^{N-i}(\vartheta+1+i)_i\partial^j$$

$$\mathrm{mc}_\mu(P):=\partial^{-L}\Big(\sum_{j=0}^n\sum_{i=0}^N c_{ij}\partial^{N-i}(\vartheta+1+i-\mu)_i\partial^j\Big)=\partial^{N-L}\partial^{-\mu}P\partial^{\mu}\in W[x]$$

$(L\in\mathbb{Z}_{\ge0}$: largest) $\qquad (a)_i:=(a)(a+1)\cdots(a+i-1)$

$\mathrm{Ad}(\varphi^{-1})\circ\mathrm{Ad}(\varphi)=id,\ \ \mathrm{mc}_{-\mu}\circ\mathrm{mc}_\mu=id \;\Rightarrow\;$ invertible operators

They keep the index of rigidity $(\mathrm{idx}\,\mathbf{m})$

Spectral type $\mathbf{m}$ of $Pu=0$ is rigid $(\mathrm{idx}\,\mathbf{m}=2)$

$\Rightarrow$ $Pu=0$ is transformed into trivial Eq. $u'=0$ by these operations

$\Rightarrow$ Existence and construction of $Pu=0$ for a given (GRS)

§ Properties of middle convolution

$$(\bigstar):\left\{\begin{matrix} x=c_0=\infty & x=c_1 & \cdots & x=c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \cdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}$$

$$\mathrm{mc}_\mu : \{[\lambda_{j,\nu}]_{(m_{j,\nu})}\} \mapsto \{[\lambda'_{j,\nu}]_{(m_{j,\nu})}\},\ Pu=0 \mapsto P'v=0 \quad (P'=\mathrm{mc}_\mu(P))$$

Theorem. Suppose $\lambda_{0,1}=\mu+1$ $(\deg\lambda_{0,1}=0)$, $\lambda_{1,1}=\cdots=\lambda_{p,1}=0$ and other exponents are generic ($m_{j,1}$ may be 0). Then ([Of, Hi])

$$\begin{aligned} m'_{j,\nu} &= m_{j,\nu}-\delta_{\nu,1}\cdot d_1(\mathbf{m}) && (1\le\nu\le n_j,\ 0\le j\le p),\\ \lambda'_{j,1} &= \delta_{j,0}\cdot(1-\mu) && (j=0,\dots,p),\\ \lambda'_{j,\nu} &= \lambda_{j,\nu}+\big(\deg\lambda_{j,\nu}+(-1)^{\delta_{j,0}}\big)\cdot\mu && (2\le\nu\le n_j,\ 0\le j\le p) \end{aligned}$$

and the irreducibility is kept.

By suitable (versal) additions and transpositions of the indices, we may assume $(\bigstar)$ with $(\sharp)$ $m_{j,1}\ge m_{j,\nu}$ $(j=0,\dots,p,\ \nu=2,\dots,n_j)$.

Lemma. $(\sharp)$ and $\mathrm{idx}\,\mathbf{m}>0 \ \Rightarrow\ d_1(\mathbf{m})>0 \ \Rightarrow\ \mathrm{ord}\,P'=\mathrm{ord}\,P-d_1(\mathbf{m})<\mathrm{ord}\,P$

$$\boxed{(\partial^{-\mu}u)(x) = (I_a^{\mu}u)(x) := \frac{1}{\Gamma(\mu)}\int_a^x u(t)(x-t)^{\mu-1}dt}$$

(a : a singular point of $u(x)$)

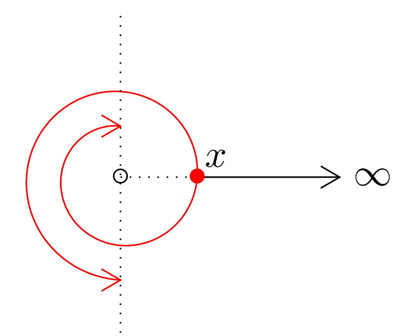
$$I_0^{\mu}(x^{\lambda}) = \frac{1}{\Gamma(\mu)}\int_0^x t^{\lambda}(x-t)^{\mu-1}dt = \frac{x^{\lambda+\mu}}{\Gamma(\mu)}\int_0^1 s^{\lambda}(1-s)^{\mu-1}ds \quad (t=xs)$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}$$

$$\tilde{I}_{\infty}^{\mu}(x^{\lambda}e^{-x}) = \frac{1}{\Gamma(\mu)}\int_x^{\infty} t^{\lambda}e^{-t}(t-x)^{\mu-1}dt = \frac{1}{\Gamma(\mu)}\int_0^{\infty}(x+s)^{\lambda}e^{-x-s}s^{\mu-1}ds$$

$$= \frac{x^{\lambda}e^{-x}}{\Gamma(\mu)}\int_0^{\infty}\sum_{n=0}^{\infty}(-\lambda)_n\Big(-\frac{s}{x}\Big)^n s^{\mu-1}e^{-t}ds$$

$$\sim \frac{x^{\lambda}e^{-x}}{\Gamma(\mu)}\sum_{n=0}^{\infty}(-\lambda)_n(-\tfrac{1}{x})^n\int_0^{\infty} s^{\mu+n}e^{-x}ds$$



$$\sim x^{\lambda}e^{-x}\sum_{n=0}^{\infty}\frac{(-\lambda)_n(\mu)_n}{n!}(-\tfrac{1}{x})^n = x^{\lambda}e^{-x}{}_2F_0(-\lambda,\mu;-\tfrac{1}{x})$$

$(|x| \to \infty,\ |\arg x| < \frac{3\pi}{2})$: (Whittaker function)

$$ {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;x) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!}x^n, $$ which satisfies

$$ \prod_{\nu=1}^{q}(\vartheta+\beta_\nu)\cdot\partial u = \prod_{\nu=1}^{p}(\vartheta+\alpha_\nu)u $$

$$ \text{Radius of the convergence } = \begin{cases} \infty & (p\le q) \\ 1 & (p=q+1) \\ 0 & (p>q+1) \end{cases} $$

$$ e^x = {}_0F_0(x), \quad (1-x)^{-\lambda} = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}x^n = {}_1F_0(\lambda;x), $$

$$ \begin{aligned} &I_0^\mu\bigl(x^\lambda\cdot{}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\dots,\beta_q;\pm x)\bigr) \\ &\quad = x^{\lambda+\mu}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}\cdot{}_{p+1}F_{q+1}(\alpha_1,\dots,\alpha_p,\lambda+1;\beta_1,\dots,\beta_q,\lambda+\mu+1;\pm x) \end{aligned} $$

$$ I_0^\mu(x^\lambda e^{-x}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}\cdot{}_1F_1(\lambda+1;\lambda+\mu+1;-x) \quad \text{(Kummer)} $$

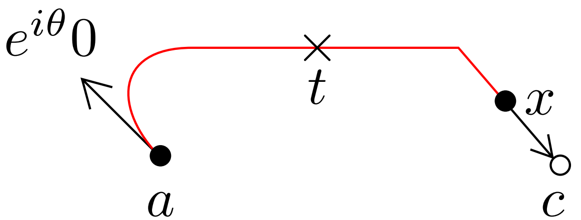
$$ I_0^\mu\bigl(x^\lambda(1-x)^{\lambda'}\bigr) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}\cdot{}_2F_1(\lambda+1,-\lambda';\lambda+\mu+1;x) \quad \text{(Gauss)} $$

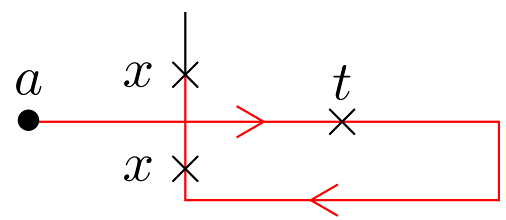
§ Riemann-Liouville transform

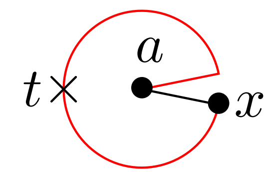
$$(I^{\mu}_{a+e^{i\theta}0}u)(x) := \frac{1}{\Gamma(\mu)}\int_{a+e^{i\theta}0}^{x} u(t)(x-t)^{\mu-1}dt = \frac{1}{\Gamma(\mu)}\int_{L} u(t)(x-t)^{\mu-1}dt$$

$L\ :\ [\alpha,\beta] \ni t \mapsto L(t) \in \mathbb{C},\ L(\alpha) = a,\ L(\beta) = x,\ \theta = \arg L'(\alpha)$,

$L(s) \neq L(t)$ for $\alpha \leq s < t \leq \beta$







$\tilde{L}\ :\ [\alpha,\gamma] \ni t \to \tilde{L}(t) \in \mathbb{C}$, $a = \tilde{L}(\alpha)$ and $c = \tilde{L}(\gamma)$,

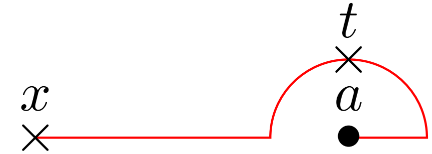
$\tilde{L}(s) \neq \tilde{L}(t)$ for $\alpha \leq s < t < \gamma$.

- $u(x)$ is holomorphic for $x = \tilde{L}(t_0)$ $(t_0 \in (\alpha,\gamma))$, $L := \tilde{L}|_{[\alpha,t_0]}$

Want to know the asymptotics of $(I^{\mu}_{a}u)(x)$ for $t_0 \to \alpha$ and $t_0 \to \gamma$

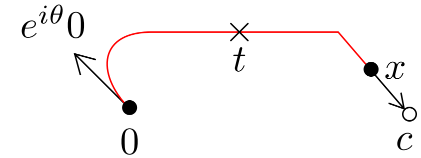
in terms of those of $u(x)$.

$$(\tilde{I}^{\mu}_{a-e^{i\theta_0}0}u)(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{a-e^{i\theta_0}0} u(t)(t-x)^{\mu-1}dt$$



§ Riemann-Liouville transform (Main Theorem)

$$\boxed{(I^{\mu}_{e^{i\theta}0}u)(x)=\frac{1}{\Gamma(\mu)}\int_{e^{i\theta}0}^{x}u(t)(x-t)^{\mu-1}dt}$$



(1) $u(x)\sim\Big(\sum_{n=0}^{\infty}a_nx^n\Big)x^{\lambda}\quad(x\to e^{i\theta}0)\qquad(\phi(x):=a_0+a_1x+\cdots\in\mathbb{C}[\![x]\!])$

$$\Rightarrow(I^{\mu}_{e^{i\theta}0}u)(x)\sim\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}\Big(\sum_{n=0}^{\infty}\frac{(\lambda+1)_na_n}{(\lambda+\mu+1)_n}x^n\Big)x^{\lambda+\mu}$$

(2) $u(x)\sim\phi(x)x^{\lambda}\exp\big(-\frac{C_0}{x^{m_0}}-\frac{C_1}{x^{m_1}}-\cdots\big)\quad(x\to e^{i\theta}0)$ and $\mathrm{Re}\,C_0e^{-im_0\theta}>0$

$$\Rightarrow(I^{\mu}_{e^{i\theta}0}u)(x)\sim(m_0C_0)^{-\mu}\psi(x)(c-x)^{\lambda+(m_0+1)\mu}\exp\big(-\frac{C_0}{x^{m_0}}-\frac{C_1}{x^{m_1}}-\cdots\big)$$

$\phi,\ \psi\in\mathbb{C}[\![x]\!]$, $\phi(0)=\psi(0)$, $m_0>m_1>\cdots>m_k>0$ and $m_j\in\mathbb{Z}_{>0}$

(3) $u(x)\approx(c-x)^{\lambda'}\quad(x\to c-e^{i\theta'}0)$ and $\mathrm{Re}\,(\lambda'+\mu)<0$

$$\Rightarrow(I^{\mu}_{e^{i\theta}0}u)(x)\approx\frac{\Gamma(-\lambda'-\mu)}{\Gamma(-\lambda')}(c-x)^{\lambda'+\mu}\quad\Leftarrow(\mathrm{Re}\,(\lambda-\lambda')>0\Leftarrow c=0,\ (1))$$

(4) $u(x)\approx(c-x)^{\lambda'}\exp\big(\frac{C'_0}{(c-x)^{m'_0}}+\frac{C'_1}{(c-x)^{m'_1}}+\cdots\big)\quad(x\to c-e^{i\theta'}0)$ and $\mathrm{Re}\,C'_0e^{-im'_0\theta}>0$

$$\Rightarrow(I^{\mu}_{e^{i\theta}0}u)(x)\approx(m'_0C'_0)^{-\mu}(c-x)^{\lambda'+(m'_0+1)\mu}\exp\big(\frac{C'_0}{(c-x)^{m'_0}}+\frac{C'_1}{(c-x)^{m'_1}}+\cdots\big)$$

§ Outline of the Proof of (4) ($\Leftarrow$ [Or])

$$\int_0^x (1-t)^{\lambda} e^{\frac{C}{(1-t)^m}+\frac{C_1}{(1-t)^{m_1}}+\cdots}(x-t)^{\mu-1}\phi(t)dt \quad (\phi\in \mathrm{C}[0,1],\ m>m_1>\cdots)$$

$$\big(t = x\big(1-(1-x)^{m+1}s\big),\ \operatorname{Re} C>0,\ \operatorname{Re}\mu>0 \Leftarrow I_0^{\mu}u = I_0^{\mu+k}\partial^k u\big)$$

$$= x^{\mu}(1-x)^{\lambda+(m+1)\mu} e^{\frac{C}{(1-x)^m}+\frac{C_1}{(1-x)^{m_1}}+\cdots}$$

$$\times \int_0^{\frac{1}{(1-x)^{m+1}}} \big(1+x(1-x)^m s\big)^{\lambda} s^{\mu-1} e^{-f(x,(1-x)^m s)} \phi\big(x-x(1-x)^{m+1}s\big)ds,$$

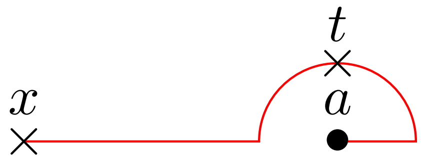
$$f(x,s_2) = \frac{C}{(1-x)^m}\Big(1-\frac{1}{(1+xs_2)^m}\Big) + \frac{C_1}{(1-x)^{m_1}}\Big(1-\frac{1}{(1+xs_2)^{m_1}}\Big)+\cdots,$$

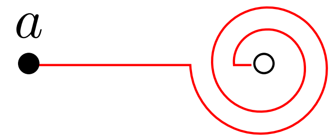
- $\displaystyle\int_{\frac{1}{(1-x)^m}}^{\frac{1}{(1-x)^{m+1}}} \cdots ds \xrightarrow{x\to 1-0} 0,$
- $\displaystyle\Big(f\big(x,(1-x)^m s\big),\ \phi\big(x-x(1-x)^{m+1}s\big)\Big) \xrightarrow{x\to 1-0} \big(mCs,\ \phi(1)\big) \ \Rightarrow$

$$\int_0^{\frac{1}{(1-x)^m}} \cdots ds \xrightarrow[\text{Lebergue's Dom. Cov. Th.}]{x\to 1-0} \int_0^{\infty} s^{\mu-1}e^{-mCs}\phi(1)ds = \Gamma(\mu)\phi(1)$$

(1), (3) $\Leftarrow$ [Of], (2) $\Leftarrow$ [Ov]

Several Path

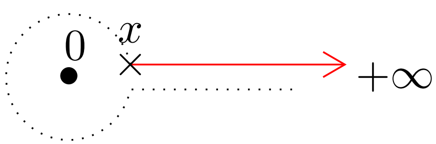


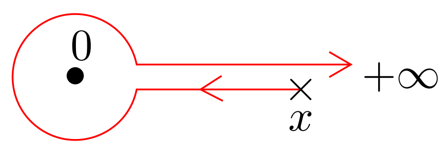


$$\tilde{I}_a^{\mu} u = \frac{1}{\Gamma(\mu)} \int_x^a u(t)(t-x)^{\mu-1} dt \;=\; e^{\mu\pi i} I_a^{\mu} u = \frac{e^{\mu\pi i}}{\Gamma(\mu)} \int_a^x u(t)(x-t)^{\mu-1} dt$$

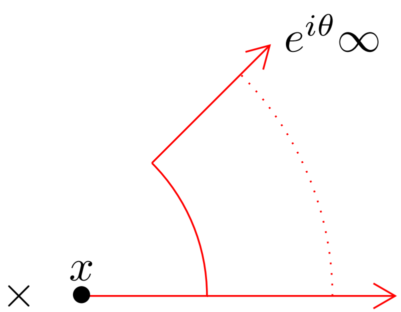
Singularity at ∞

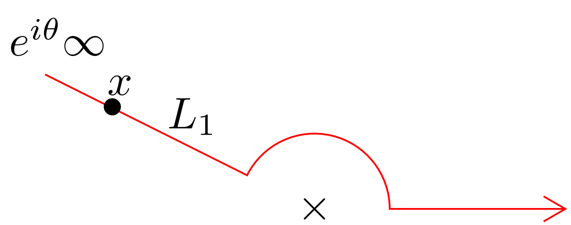
$$\boxed{(\tilde{I}_a^{\mu} u)(x) = x^{\mu-1} \cdot \left(I^{\mu}_{\frac{1}{a}} x^{-\mu-1} u(\tfrac{1}{x}) \right)(\tfrac{1}{x})} \qquad a = e^{i\theta}\infty \;\Rightarrow\; \tfrac{1}{a} = e^{-i\theta} 0$$





$(x \to +\infty)$





$$\frac{1}{\Gamma(\mu)} \int_x^{+\infty} t^{\lambda} e^{-t} (t-x)^{\mu-1} dt$$

Asymptotic at ∞ (Theorem)

(1) $u(x) \sim \Big(\sum_{n=0}^{\infty} a_n x^{-n}\Big) x^{\lambda} \quad (x \to e^{i\theta}\infty) \quad (\phi(x) := a_0 + a_1 x + \cdots \in \mathbb{C}[\![x]\!])$

$$\Rightarrow (\tilde{I}^{\mu}_{e^{i\theta}\infty} u)(x) \sim \frac{\Gamma(-\lambda-\mu)}{\Gamma(-\lambda)} \Big(\sum_{n=0}^{\infty} \frac{\Gamma(-\lambda-\mu)_n a_n}{\Gamma(-\lambda)_n} x^{-n}\Big) x^{\lambda+\mu}$$

(2) $u(x) \sim \phi(\tfrac{1}{x}) x^{\lambda} \exp\big(-C_0 x^{m_0} - C_1 x^{m_1} - \cdots\big) \quad (x \to e^{i\theta}\infty)$ and

$\mathrm{Re}\, C_0 e^{i m_0 \theta} > 0$, $\phi \in \mathbb{C}[\![x]\!]$

$$\Rightarrow (I^{\mu}_0 u)(x) \sim (m_0 C_0)^{-\mu} \psi(\tfrac{1}{x}) x^{\lambda+(1-m_0)\mu} \exp\big(-C_0 x^{m_0} - C_1 x^{m_1} - \cdots\big),$$

$\psi \in \mathbb{C}[\![x]\!]$, $\phi(0) = \psi(0)$, $m_0 > m_1 > \cdots > m_k > 0$ and $m_j \in \mathbb{Z}_{>0}$

(3) $u(x) \approx x^{\lambda'} \quad (x \to e^{i\theta'}\infty)$ and $\mathrm{Re}\, \lambda' > -1$

$$\Rightarrow (I^{\mu}_0 u)(x) \approx \frac{\Gamma(\lambda'+1)}{\Gamma(\lambda'+\mu+1)} x^{\lambda'+\mu}$$

(4) $u(x) \approx x^{\lambda'} \exp\big(C'_0 x^{m'_0} + C'_1 x^{m'_1} + \cdots\big) \quad (x \to e^{i\theta'}\infty)$ and $\mathrm{Re}\, C'_0 e^{i m'_0 \theta} > 0$

$$\Rightarrow (I^{\mu}_0 u)(x) \approx (m'_0 C'_0)^{-\mu} x^{\lambda+(1-m_0)\mu} \exp\big(C'_0 x^{m'_0} + C'_1 x^{m'_1} + \cdots\big)$$

§ Example (${}_{n-1}F_{n-1}$: a confluence of ${}_nF_{n-1}$)

$1^n|(n-1)1,1^n$ (n=2 ⇒ Kummer's Eq.)

$$G^{\pm} := \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu}\ (1\le\nu<n) & \lambda_{0,\nu}\ (1\le\nu\le n) \\ \pm x-\lambda_{1,n} & \end{matrix}\right\} \qquad \Bigl(\sum_{\nu=1}^{n}\lambda_{0,\nu}=\sum_{\nu=1}^{n}\lambda_{1,\nu}\Bigr)$$

$$\xrightarrow{\mathrm{Ad}(x^{-\lambda_{0,1}})} \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu}+\lambda_{0,1}\ (1\le\nu<n) & 0 \\ \pm x-\lambda_{1,n}+\lambda_{0,1} & \lambda_{0,\nu}-\lambda_{0,1}\ (2\le\nu\le n) \end{matrix}\right\}$$

$$\xrightarrow{\mathrm{mc}_{\lambda_{0,1}-\lambda_{1,1}}} \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu}+\lambda_{1,1}\ (2\le\nu<n) & \lambda_{0,\nu}-\lambda_{1,1}\ (2\le\nu\le n) \\ \pm x-\lambda_{1,n}+\lambda_{0,1} & \end{matrix}\right\}$$

$$\xrightarrow{\mathrm{Ad}(x^{\lambda_{1,1}-\lambda_{0,2}})} \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu}+\lambda_{0,2}\ (3\le\nu<n) & 0 \\ \pm x-\lambda_{1,n}+\lambda_{0,1}+\lambda_{0,2}-\lambda_{1,1} & \lambda_{0,\nu}-\lambda_{0,2}\ (3\le\nu\le n) \end{matrix}\right\}$$

$$\cdots \xrightarrow{\mathrm{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}})}$$

$$\left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0 \\ \pm x-\lambda_{1,n}+\lambda_{0,1}+\displaystyle\sum_{\nu=2}^{n-1}(\lambda_{0,\nu}-\lambda_{1,\nu-1}) & \lambda_{0,n}-\lambda_{0,n-1} \end{matrix}\right\}$$

$$= G_2^{\pm} := \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0 \\ \pm x+\lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{0,n-1} \end{matrix}\right\} \qquad \text{(Kummer)}$$

$$G_2^{\pm} = \begin{Bmatrix} x=\infty & x=0 \\ 1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0 \\ \pm x+\lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{0,n-1} \end{Bmatrix} \quad \text{(Kummer)}$$

$$\xrightarrow{\mathrm{mc}_{\lambda_{0,n-1}-\lambda_{1,n-1}}} \begin{Bmatrix} x=\infty & x=0 \\ \pm x+\lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{1,n-1} \end{Bmatrix} \ni x^{\lambda_{0,n}-\lambda_{1,n-1}} e^{\mp x}$$

$$G_2^{\pm} \xrightarrow{x\mapsto -x} G_2^{\mp} = \begin{Bmatrix} x=\infty & x=0 \\ 1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0 \\ \mp x+\lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{0,n-1} \end{Bmatrix}$$

$$\xrightarrow{\mathrm{Ad}(e^{\mp x})} \begin{Bmatrix} x=\infty & x=0 \\ \pm x+1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0 \\ \lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{0,n-1} \end{Bmatrix}$$

$$\xrightarrow{\mathrm{mc}_{\lambda_{1,n-1}-\lambda_{0,n-1}}} \begin{Bmatrix} x=\infty & x=0 \\ \pm x+1-\lambda_{1,n-1}+\lambda_{0,n-1} & \lambda_{1,n-1}-\lambda_{0,n-1}-1 \end{Bmatrix}$$

$$G^{\pm} := \begin{Bmatrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu} \ (1\le\nu<n) & \lambda_{0,\nu} \ (1\le\nu\le n) \\ \pm x-\lambda_{1,n} & \end{Bmatrix}$$

$$\begin{aligned} G^{+} : u_{\infty,x-\lambda_{1,n}}(x) &= \mathrm{Ad}(x^{\lambda_{0,1}})\tilde I_\infty^{\lambda_{1,1}-\lambda_{0,1}}\,\mathrm{Ad}(x^{\lambda_{0,2}-\lambda_{1,1}})\tilde I_\infty^{\lambda_{1,2}-\lambda_{0,2}}\cdots \\ &\quad \cdots\mathrm{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}})\tilde I_\infty^{\lambda_{1,n-1}-\lambda_{0,n-1}}(x^{\lambda_{0,n}-\lambda_{1,n-1}}e^{-x}) \\ &\approx x^{\lambda_{1,n}}e^{-x} \qquad (|x|\to\infty,\ |\arg x|<\tfrac{3\pi}{2}), \\ u^{\pm}_{\infty,1-\lambda_{1,\nu}}(x) &\approx (e^{\pm\pi i}x)^{\lambda_{1,\nu}-1} \qquad (|x|\to\infty,\ |\arg x\pm\pi|<\tfrac{3\pi}{2}), \\ u_{0,\lambda_{0,\nu}}(x) &\approx x^{\lambda_{0,\nu}} \qquad (x\to 0) \qquad (\Leftarrow \tilde I_\infty^{\mu}\mapsto I_0^{\mu}) \end{aligned}$$

$$G^+ : u_{\infty,x-\lambda_{1,n}}$$

$$u_1(x) := \tilde{I}_\infty^{\lambda_{1,n-1}-\lambda_{0,n-1}}(x^{\lambda_{0,n}-\lambda_{1,n-1}}e^{-x})$$

$$\sim \sum_{m_1=0}^{\infty} \frac{(\lambda_{1,n-1}-\lambda_{0,n-1})_{m_1}(\lambda_{1,n-1}-\lambda_{0,n})_{m_1}(-1)^{m_1}}{m_1!} x^{\lambda_{0,n}-\lambda_{1,n-1}-m_1}e^{-x}$$

$$= x^{\lambda_{0,n}-\lambda_{1,n-1}}e^{-x}\,{}_2F_0(\lambda_{1,n-1}-\lambda_{0,n-1},\lambda_{1,n-1}-\lambda_{0,n};-\tfrac{1}{x}) \qquad (x\to+\infty)$$

$$\approx \frac{\Gamma(\lambda_{0,n-1}-\lambda_{0,n})}{\Gamma(\lambda_{1,n-1}-\lambda_{0,n})} x^{\lambda_{0,n}-\lambda_{0,n-1}} \qquad (x\to+0,\ \mathrm{Re}\,\lambda_{0,n}<\mathrm{Re}\,\lambda_{0,n-1}),$$

$$u_2(x) := \tilde{I}_\infty^{\lambda_{1,n-2}-\lambda_{0,n-2}}\,\mathrm{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}})u_1(x)$$

$$\cdots\cdots\cdots\cdots$$

$$u_{n-1}(x) \sim \sum_{m_{n-1}=0}^{\infty}\cdots\sum_{m_1=0}^{\infty} \frac{\prod_{k=1}^{n-1}\bigl(m_1+\cdots+m_{k-1}+\sum_{\nu=1}^{k}(\lambda_{1,n-k}-\lambda_{0,n+1-k})\bigr)_{m_k}}{m_1!\cdots m_{n-1}!}$$

$$\times \prod_{k=1}^{n-1}(\lambda_{1,n-k}-\lambda_{0,n-k})_{m_k}(-\tfrac{1}{x})^{m_1+\cdots+m_{n-1}} \cdot x^{\sum_{k=1}^{n-1}(\lambda_{0,n+1-k}-\lambda_{1,n-k})}e^{-x}$$

$$\approx x^{\lambda_{1,n}-\lambda_{0,1}}e^{-x} \qquad (x\to+\infty)$$

$$\approx \prod_{k=1}^{n-1}\frac{\Gamma(\lambda_{0,k}-\lambda_{0,n})}{\Gamma(\lambda_{1,k}-\lambda_{0,n})}x^{\lambda_{0,n}-\lambda_{0,1}} \qquad (x\to+0,\ \mathrm{Re}\,\lambda_{0,n}<\mathrm{Re}\,\lambda_{0,k} \text{ for } k=1,\dots,n-1),$$

$$u_{\infty,x-\lambda_{1,n}} = x^{\lambda_{0,1}}u_{n-1} = \sum_{m=1}^{n}\frac{\prod_{k\in I_{n,m}}\Gamma(\lambda_{0,k}-\lambda_{0,m})}{\prod_{k\in J_{n-1}}\Gamma(\lambda_{1,k}-\lambda_{0,m})}u_{0,\lambda_{0,m}} \qquad \begin{cases} I_{n,m} := \{1,\dots,n\}\setminus\{m\} \\ J_{n-1} := \{1,\dots,n-1\}\end{cases}$$

$$G^{\pm}\ :\ u^{\pm}_{0,\lambda_{0,k}}$$

$$v_1^{\pm}(x) := I_0^{\lambda_{1,n-1}-\lambda_{0,n-1}}(x^{\lambda_{0,n}-\lambda_{1,n-1}}e^{\mp x})$$

$$= \frac{\Gamma(\lambda_{0,n}-\lambda_{1,n-1}+1)}{\Gamma(\lambda_{0,n}-\lambda_{0,n-1}+1)} x^{\lambda_{0,n}-\lambda_{0,n-1}}\,{}_1F_1(\lambda_{0,n}-\lambda_{1,n-1}+1;\lambda_{0,n}-\lambda_{0,n-1}+1;\mp x)$$

$$v_1^{-}(x) \approx x^{\lambda_{0,n}-\lambda_{1,n-1}}e^{x} \quad (x\to+\infty),$$

$$v_1^{+}(x) \in \mathbb{C}e^{-x} I_0^{\lambda_{0,n}-\lambda_{1,n-1}}(x^{\lambda_{1,n-1}-\lambda_{0,n-1}-1}e^{x}) \Rightarrow$$

$$v_1^{+}(x) \approx x^{\lambda_{1,n-1}-\lambda_{0,n-1}-1} \quad (x\to+\infty)$$

$${}_1F_1(\alpha;\gamma;x) = e^{x}\,{}_1F_1(\gamma-\alpha;\gamma;-x) \approx \frac{\Gamma(\gamma)}{\Gamma(\alpha)}x^{\alpha-\gamma}e^{x} \quad (x\to+\infty)$$

$$v_2^{\pm}(x) := I_0^{\lambda_{1,n-2}-\lambda_{0,n-2}}\,\mathrm{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}})v_1^{\pm}(x)$$

$$\cdots\cdots\cdots$$

$$v_{n-1}^{\pm}(x) = \prod_{\nu=1}^{n-1}\frac{\Gamma(\lambda_{0,n}-\lambda_{1,\nu}+1)}{\Gamma(\lambda_{0,n}-\lambda_{0,\nu}+1)}x^{\lambda_{0,n}-\lambda_{0,1}}$$

$$\times\ {}_{n-1}F_{n-1}\big(\lambda_{0,n}-\lambda_{1,\nu}+1\ \ (1\le\nu<n);\lambda_{0,n}-\lambda_{0,\nu}+1\ \ (1\le\nu<n);\mp x\big),$$

$$v_{n-1}^{-}(x) \approx x^{\lambda_{1,n}-\lambda_{0,1}}e^{x} \quad (x\to+\infty)$$

$$v_{n-1}^{+}(x) \approx \prod_{\nu=1}^{n-2}\frac{\Gamma(\lambda_{1,n-1}-\lambda_{1,\nu})}{\Gamma(\lambda_{1,n-1}-\lambda_{0,\nu})}x^{\lambda_{1,n-1}-\lambda_{0,1}-1} \quad (x\to+\infty,\ \mathrm{Re}\,\lambda_{1,n-1}>\mathrm{Re}\,\lambda_{1,m},\ m\in J_{n-2})$$

$$u_{0,\lambda_{0,k}}(x) = x^{\lambda_{0,k}}\,{}_{n-1}F_{n-1}\big(\{\lambda_{0,k}-\lambda_{1,\nu}+1\}_{\nu\in I_{n-1}};\{\lambda_{0,k}-\lambda_{0,\nu}+1\}_{\nu\in I_{n,k}};-x\big)$$

$$G^- \ : w_{\infty,1-\lambda_{1,m}},\ u^{\pm}_{\infty,1-\lambda_{1,m}}(x) = w_{\infty,1-\lambda_{1,m}}(e^{\pm\pi i}x)$$

$$\begin{aligned}
w_1(x) &= e^x \tilde{I}_\infty^{\lambda_{0,n}-\lambda_{1,n-1}+1}(x^{\lambda_{1,n-1}-\lambda_{0,n-1}-1}e^{-x}) \\
&= x^{\lambda_{1,n-1}-\lambda_{0,n-1}-1}{}_2F_0(\lambda_{0,n}-\lambda_{1,n-1}+1, \lambda_{0,n-1}-\lambda_{1,n-1}+1; -\tfrac{1}{x}) \\
&\approx \frac{\Gamma(\lambda_{0,n-1}-\lambda_{0,n}+1)}{\Gamma(\lambda_{0,n-1}-\lambda_{1,n-1}+1)} x^{\lambda_{0,n}-\lambda_{0,n-1}} \qquad (x \to 0,\ \operatorname{Re}\lambda_{0,n} < \operatorname{Re}\lambda_{0,n-1}), \\
w_2(x) &= \tilde{I}_\infty^{\lambda_{1,n-2}-\lambda_{0,n-2}} \operatorname{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}})w_1(x) \\
&\cdots\cdots
\end{aligned}$$

.

$$\begin{aligned}
w_{n-1}(x) &= \prod_{\nu=1}^{n-2} \frac{\Gamma(\lambda_{0,\nu}-\lambda_{1,n-1}+1)}{\Gamma(\lambda_{1,\nu}-\lambda_{1,n-1}+1)} \cdot x^{\lambda_{1,n-1}-\lambda_{0,1}-1} \\
&\quad \times {}_nF_{n-2}(\{\lambda_{0,\nu}-\lambda_{1,n-1}+1\}_{\nu\in I_n}; \{\lambda_{1,\nu}-\lambda_{1,n-1}+1\}_{\nu\in J_{n-2}}; -\tfrac{1}{x}) \\
&\approx \frac{\prod_{\nu=1}^{n-1}\Gamma(\lambda_{0,\nu}-\lambda_{0,n})}{\Gamma(\lambda_{0,n-1}-\lambda_{1,n-1}+1)\prod_{\nu=1}^{n-2}\Gamma(\lambda_{1,\nu}-\lambda_{0,n})} x^{\lambda_{0,n}-\lambda_{0,1}} \\
&\qquad (x \to +0,\ \operatorname{Re}\lambda_{0,n} < \operatorname{Re}\lambda_{0,\nu} \text{ for } 1 \le \nu < n)
\end{aligned}$$

$$\begin{aligned}
w_{\infty,1-\lambda_{1,n-1}}(x) &= x^{\lambda_{0,1}} \prod_{\nu=1}^{n-2} \frac{\Gamma(\lambda_{1,\nu}-\lambda_{1,n-1}+1)}{\Gamma(\lambda_{0,\nu}-\lambda_{1,n-1}+1)} w_{n-1}(x) \\
&= x^{\lambda_{1,n-1}-1} \cdot {}_nF_{n-2}(\{\lambda_{0,\nu}-\lambda_{1,n-1}+1\}_{\nu\in I_n}; \{\lambda_{1,\nu}-\lambda_{1,n-1}+1\}_{\nu\in J_{n-2}}; -\tfrac{1}{x}) \\
&\approx \frac{\prod_{\nu=1}^{n-2}\Gamma(\lambda_{1,\nu}-\lambda_{1,n-1}+1)\prod_{\nu=1}^{n-1}\Gamma(\lambda_{0,\nu}-\lambda_{0,n})}{\prod_{\nu=1}^{n-1}\Gamma(\lambda_{0,\nu}-\lambda_{1,n-1}+1)\prod_{\nu=1}^{n-2}\Gamma(\lambda_{1,\nu}-\lambda_{0,n})} x^{\lambda_{0,n}} \\
&\qquad (x \to +0,\ \operatorname{Re}\lambda_{0,n} < \operatorname{Re}\lambda_{0,\nu} \text{ for } 1 \le \nu < n).
\end{aligned}$$

Connection formula:

$$(u^{\pm}_{\infty,1-\lambda_{1,m}}(x) = w_{\infty,1-\lambda_{1,m}}(e^{\pm\pi i}x))$$

$$u^{\pm}_{\infty,1-\lambda_{1,m}}(x) = \sum_{k=1}^{n} c_\Gamma(\infty\!:\!1-\lambda_{1,m} \rightsquigarrow 0\!:\!\lambda_{0,k}) e^{\pm\lambda_{0,k}\pi i} u_{0,\lambda_{0,k}}(x)$$

$$\approx (e^{\pm\pi i}x)^{\lambda_{1,m}-1} \quad (|x|\to\infty,\ |\arg x \pm \pi| < \tfrac{3\pi}{2})$$

$$u_{\infty,x-\lambda_{1,n}}(x) = \sum_{k=1}^{n} c_\Gamma(\infty\!:\!x-\lambda_{1,n} \rightsquigarrow 0\!:\!\lambda_{0,k}) u_{0,\lambda_{0,k}}(x)$$

$$\approx x^{\lambda_{1,n}} e^{-x} \quad (|x|\to\infty,\ |\arg x| < \tfrac{3\pi}{2})$$

$$u_{0,\lambda_{0,k}}(x) = \sum_{m=1}^{n} c_\Gamma(0\!:\!\lambda_{0,k} \rightsquigarrow \infty\!:\!1-\lambda_{1,m}) e^{\pm(1-\lambda_{1,m})\pi i} u^{\pm}_{\infty,1-\lambda_{1,m}}(x)$$

$$+\, c_\Gamma(0\!:\!\lambda_{0,k} \rightsquigarrow \infty\!:\!x-\lambda_{1,n}) e^{\pm(\lambda_{1,n}-\lambda_{0,k})\pi i} u_{\infty,x-\lambda_{1,n}}(x)$$

$$\approx x^{\lambda_{0,k}} \quad (x \to +0)$$

Stokes relations:

$$e^{\mp(\lambda_{1,m}-1)\pi i}\cdot u^{\pm}_{\infty,1-\lambda_{1,m}}(e^{\mp 2\pi i}x) - e^{\pm(\lambda_{1,m}-1)\pi i}\cdot u^{\pm}_{\infty,1-\lambda_{1,m}}(x)$$

$$= \pm 2\pi i \cdot c_\Gamma(\infty\!:\!1-\lambda_{1.m} \rightsquigarrow \infty\!:\!x-\lambda_{1,n}) \cdot u_{\infty,x-\lambda_{1,n}}(x),$$

$$e^{-\lambda_{1,n}\pi i}\cdot u_{\infty,x-\lambda_{1,n}}(e^{\pi i}x) - e^{\lambda_{1,n}\pi i}\cdot u_{\infty,x-\lambda_{1,n}}(e^{-\pi i}x)$$

$$= -2\pi i \sum_{m=1}^{n-1} c_\Gamma(\infty\!:\!x-\lambda_{1,n} \rightsquigarrow \infty\!:\!1-\lambda_{1,m}) \cdot u^{\pm}_{\infty,1-\lambda_{1,m}}(e^{\mp\pi i}x)$$

$$n=2 \ \Rightarrow\ {}_2F_0(\alpha,\beta;-\tfrac{1}{e^{\pm\pi i}x}) - {}_2F_0(\alpha,\beta;-\tfrac{1}{e^{\mp\pi i}x}) = \frac{2\pi i x^{\alpha+\beta-1}e^{-x}}{\Gamma(1-\alpha)\Gamma(1-\beta)}\,{}_2F_0(1-\alpha,1-\beta;-\tfrac{1}{x})$$

Γ-factors:

$$c_\Gamma(\infty\!:\!1-\lambda_{1,m} \rightsquigarrow 0\!:\!\lambda_{0,k}) = \frac{\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,\nu}-\lambda_{0,k})\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{1,\nu}-\lambda_{1,m}+1)}{\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,\nu}-\lambda_{1,m}+1)\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{1,\nu}-\lambda_{0,k})}$$

$$c_\Gamma(\infty\!:\!x-\lambda_{1,n} \rightsquigarrow 0\!:\!\lambda_{0,k}) = \frac{\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,\nu}-\lambda_{0,k})}{\prod_{\nu\in J_{n-1}}\Gamma(\lambda_{1,\nu}-\lambda_{0,k})}$$

$$c_\Gamma(0\!:\!\lambda_{0,k} \rightsquigarrow \infty\!:\!1-\lambda_{1,m}) = \frac{\prod_{\nu\in I_{n-1,m}}\Gamma(\lambda_{1,m}-\lambda_{1,\nu})\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,k}-\lambda_{0,\nu}+1)}{\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{0,k}-\lambda_{1,\nu}+1)\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{1,m}-\lambda_{0,\nu})}$$

$$c_\Gamma(0\!:\!\lambda_{0,k} \rightsquigarrow \infty\!:\!x-\lambda_{1,n}) = \frac{\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,n}-\lambda_{0,\nu}+1)}{\prod_{\nu\in J_{n-1}}\Gamma(\lambda_{0,k}-\lambda_{1,\nu}+1)}$$

$$c_\Gamma(\infty\!:\!x-\lambda_{1,n} \rightsquigarrow \infty\!:\!1-\lambda_{1,m}) = \frac{\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{1,\nu}-\lambda_{1,m}+1)}{\prod_{\nu\in I_n}\Gamma(\lambda_{0,\nu}-\lambda_{1,m}+1)}$$

$$c_\Gamma(\infty\!:\!1-\lambda_{1,m} \rightsquigarrow \infty\!:\!x-\lambda_{1,n}) = \frac{\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{1,m}-\lambda_{1,\nu})}{\prod_{\nu\in I_n}\Gamma(\lambda_{1,m}-\lambda_{0,\nu})}$$

Irreducible $\quad\Leftrightarrow\quad \lambda_{0,k}-\lambda_{1,m}\notin\mathbb{Z}\quad (1\le k\le n,\ 1\le m<n)$

Kummer Whittaker

$$\begin{Bmatrix} x=\infty & 0 \\ 1-\lambda_{1,\nu} & \lambda_{0,\nu} \\ x-\lambda_{1,n} & \end{Bmatrix} \xrightarrow{\times x^{-\lambda_{0,1}}e^{x}} \begin{Bmatrix} x=\infty & 0 \\ -x+\gamma-\alpha & 0 \\ \alpha & 1-\gamma \end{Bmatrix} \xrightarrow{\times x^{\frac{\gamma}{2}}e^{-\frac{x}{2}}} \begin{Bmatrix} x=\infty & 0 \\ -\frac{x}{2}-k & \frac12-m \\ \frac{x}{2}+k & \frac12+m \end{Bmatrix}$$

§ Evaluation of connection coefficients $c(\lambda)$

$$\text{Gauss}: \left\{\begin{matrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & \alpha & ;\ x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\}$$

$$F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta-1;1-x) + O((1-x)^{\gamma-\alpha-\beta})$$

1. Evaluation of an integral expression of the corresponding solution
   $$F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}dt \quad (t \to 1-0)$$
2. Difference equation of $c(\lambda)$ ($\Leftarrow$ contiguous relation) and asymptotic of $c(\lambda)$
   $$c(\alpha,\beta,\gamma+1) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)} c(\alpha,\beta,\gamma),\ \lim_{\gamma\to+\infty} c(\alpha,\beta,\gamma) = 1$$
3. Possible poles of $c(\lambda)$ ($\Leftarrow$ difference of characteristic exponents) and known zeros of $c(\lambda)$ ($\Leftarrow$ reducibility) and asymptotic of $c(\lambda)$

   Poles: $1-\gamma = 1,2,\dots$ and $\gamma-\alpha-\beta = 0,1,\dots$ and zeros: $\gamma-\alpha,\ \gamma-\beta = 0,-1,\dots$
4. Change of asymptotic of solution under the Riemann-Liouville transform
   $I_0^\mu(x^\lambda(1-x)^{\lambda'}),\ x \to +0,\ 1-0$
5. Irregular singularities $\Rightarrow$ regular singularities (by versal unfolding)
   $$\lim_{\beta\to+\infty} F(\alpha,\beta,\gamma;\tfrac{x}{\beta}) = {}_1F_1(\alpha;\gamma;x), \quad \lim_{\gamma\to+\infty} F(\alpha,\beta,\gamma;\gamma x) = {}_2F_0(\alpha,\beta;x)$$
6. Extension to several variables
   Jordan-Pochhammer $\xrightarrow{\text{extension}}$ Lauricella's F_D $\xrightarrow{\text{restriction}}$ Gauss

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