Multiplicities of Representations on Homogeneous Spaces
of Real SemiSimple Lie Groups

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Let G be a connected real semisimple Lie group, K a maximal compact subgroup of G modulo the center of G and H a closed connected subgroup of G. Let $\mathcal{A}(G/H;\tau)$ denote the space of real analytic global sections of the associate bundle over G/H induced from a finite dimensional irreducible representation τ of H. By denoting $(\pi(g)f)(x) = f(g^{-1}x)$ for g, $x \in G$ and $f \in \mathcal{A}(G/H;\tau)$, the space $\mathcal{A}(G/H;\tau)$ is a G-module. If τ is trivial, we have naturally $\mathcal{A}(G/H) = \mathcal{A}(G/H;\tau)$.

Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H, respectively. We denote by $\hat{G}_{ad}$ the set of equivalence classes of the irreducible Harish-Chandra modules of G and by dV the infinitesimal character of $V \in \hat{G}_{ad}$, which belongs to $\tilde{\mathfrak{j}}_c^*/W(\tilde{\mathfrak{j}})$. Here $\tilde{\mathfrak{j}}$ is a Cartan subalgebra of $\mathfrak{g}$ and $W(\tilde{\mathfrak{j}})$ is a Weyl group of G corresponding to the complexification $\mathfrak{j}_c$ of $\mathfrak{j}$.

Let P be a minimal parabolic subgroup of G with the Levi decomposition $P = LN$, $\hat{L}$ the equivalence classes of the irreducible representations of L and E_ξ the Harish-Chandra module of the principal series of G induced from $\xi \in \hat{L}$.

Theorem 1 [Bien-Oshima]**.** i) *Suppose* $\mathfrak{g} = \mathfrak{h} + \mathrm{Lie}(P)$*. Let* ι *be the natural projection of* $\tilde{\mathfrak{j}}_c^*$ *onto* $\tilde{\mathfrak{j}}_c^*/W(\tilde{\mathfrak{j}})$*. Then for* $V \in \hat{G}_{ad}$

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathcal{A}(G/H;\tau)) \leq$$
$$\#\{\iota^{-1}(dV)\} \cdot \sum_{\substack{\xi \in \hat{L} \\ dE_\xi = dV}} \dim \mathrm{Hom}_{(\mathfrak{g},K)}(V, E_\xi^*) \cdot \dim \mathrm{Hom}_{P\cap H}(\tau, \xi).$$

ii) *If* (G,H) *is algebraic and there is no open* H*-orbit in* G/P*, there exists* $V \in \hat{G}_{ad}$ *with*

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathcal{A}(G/H)) = \infty.$$

iii) *If there exists a Borel subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}_c$ *with* $\mathfrak{g}_c = \mathfrak{h}_c + \mathfrak{b}$*, there exists a positive integer* m *with*

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathcal{A}(G/H;\tau)) \leq m \dim \tau \quad \text{for} \quad V \in \hat{G}_{ad}.$$

Remark. Suppose (G,H) is algebraic with the complexification (G_c, H_c). It is known that if there is no open H_c-orbit in the flag manifold G_c/B of G_c, then $\dim \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathcal{A}(G/H))$ is not bounded for $V \in \hat{G}_{ad}$.

Example. Let G be the direct product of $(1+n)$ copies of $SL(2,\mathbb{R})$ and $H = \{(b_{s,t}, k_{\lambda_1 t}, \ldots, k_{\lambda_n t}); (s,t) \in \mathbb{R}^2\}$ with $b_{s,t} = \begin{pmatrix} e^t & s \\ 0 & e^{-t} \end{pmatrix}$ and $k_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Then if $\lambda_1, \ldots \lambda_n$ are linearly independent over $\mathbb{Q}$, $\dim \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathcal{A}(G/H;\tau)) \leq 2$ for $V \in \hat{G}_{ad}$.

Hereafter we suppose H is the identity component of the group $G^\sigma = \{g \in G; \sigma(g) = g\}$ with an involutive automorphism σ of G satisfying $\sigma(K) = K$. Then H is a fundamental example satisfying the condition in the above 2).

Let $\hat{K}$ be the equivalence classes of the irreducible unitary representations of K. For $\delta \in \hat{K}$ we denote by $\mathcal{A}_\delta(G/H;\tau)$ the space spanned by $f \in \mathcal{A}(G/H;\tau)$ such that $\sum_{k\in K} \mathbb{C}\pi(k)f$ belong to δ under $\pi|_K$.

Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q} = \mathfrak{k} + \mathfrak{p}$ be the orthogonal decompositions of $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Let $\mathfrak{a}$, $\mathfrak{j}$ and $\tilde{\mathfrak{j}}$ be maximal abelian subspaces of $\mathfrak{p} \cap \mathfrak{q}$, $\mathfrak{q}$ and $\mathfrak{g}$, respectively, so that $\mathfrak{a} \subset \mathfrak{j} \subset \tilde{\mathfrak{j}}$.

We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}_c$ and by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. We identify $U(\mathfrak{g})$ with the ring of left invariant differential operators on G.

Suppose $\mathfrak{u}$ is $\mathfrak{a}$, $\mathfrak{j}$ or $\tilde{\mathfrak{j}}$. Then we denote by $S(\mathfrak{u})$ the symmetric algebra over the complexification $\mathfrak{u}_c$ of $\mathfrak{u}$, by $W(\mathfrak{u})$ the Weyl group of the root system corresponding to the pair $(\mathfrak{g}_c, \mathfrak{u}_c)$ and by $I(u)$ the set of the $W(\mathfrak{u})$-invariant elements of $S(\mathfrak{u})$. Let $Z_{K\cap H}(\mathfrak{a})$ (resp. $N_{K\cap H}(\mathfrak{a})$) be the centralizer (resp. normalizer) of $\mathfrak{a}$ in $K\cap H$. Then the quotient group $W(\mathfrak{a};H) = N_{K\cap H}(\mathfrak{a})/Z_{K\cap H}(\mathfrak{a})$ is identified with a subgroup of $W(\mathfrak{a})$. Here we note that $Z(\mathfrak{g})$ is isomorphic to $I(\tilde{\mathfrak{j}})$.

The projection maps of $\tilde{\mathfrak{j}}$ to $\mathfrak{a}$ and $\mathfrak{j}$ to $\mathfrak{a}$ with respect to the Killing form induce the maps

$$\begin{aligned} p &: Z(\mathfrak{g}) \simeq I(\tilde{\mathfrak{j}}) \to I(\mathfrak{a}) \\ p' &: I(\mathfrak{j}) \to I(\mathfrak{a}). \end{aligned}$$

Theorem 2 [Huang-Oshima-Wallach]**.** *Put* $\ell = \dim \mathfrak{a}$.

i) Suppose p *is surjective. Let* $D_1, \dots, D_\ell$ *be elements of* $Z(\mathfrak{g})$ *whose image under* p *generate* $I(\mathfrak{a})$ *as an algebra over* $\mathbb{C}$.

For any $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{C}^\ell$ *the dimension of the space*

$$\{u \in \mathcal{A}_\delta(G/H;\tau); D_j u = \lambda_j u \quad \text{for} \quad j = 1, \dots, \ell\}$$

equals

$$[W(\mathfrak{a}) : W(\mathfrak{a};H)] \cdot \dim \delta \cdot \dim \operatorname{Hom}_{Z_{K\cap H}(\mathfrak{a})}(\delta, \tau^*).$$

ii) Suppose p' *is surjective and* $\dim \tau = 1$. *Then replacing* $D_1, \dots, D_\ell$ *by* ℓ *elements of* $U(\mathfrak{g})^{\mathfrak{h}} = \{D \in U(\mathfrak{g}); [X, D] = 0$ *for any* $X \in \mathfrak{h}\}$ *which generate* $I(\mathfrak{a})$ *through the Harish-Chandra homomorphism, we have the same claim as in i).*

Note that the assumptions in the above theorems are satisfied if $\mathfrak{g}$ is of classical type.

Corollary. *For* δ, $\delta' \in \hat{K}$ *let denote by* $\mathcal{A}_{\delta,\delta'}(G)$ *the space of real analytic function on* G *whose right and left* K*-types are* δ *and* δ'*, respectively. If* $\mathfrak{g}$ *is a normal real form of* $\mathfrak{g}_c$*, the dimension of the simultaneous eigenspace of the* ℓ *generators of* $Z(\mathfrak{g})$ *in* $\mathcal{A}_{\delta,\delta'}(G)$ *equals* $\dim \delta \cdot \dim \delta' \cdot \dim \operatorname{Hom}_M(\delta^*, \delta')$. *Here* M *is the centralizer of a maximal abelian subspace of* $\mathfrak{p}$ *in* K.