A characterization of the monodromy group of Gauss hypergeometric equation

Toshio OSHIMA and Kouhei SHIMIZU

Abstract. We give a characterization of the monodromy group of the second order linear Fuchsian differential equation on the Riemann sphere which has three singular points.

1. Introduction

Hilbert’s twenty-first problem asks the existence of a linear differential equation of Fuchsian class with given singular points and monodromy group. Since the existence is not always true when the equation is single, the original problem was formulated as the problem of the existence of the first order Fuchsian system of Schlesinger canonical form with given singular points and monodromy group, which is called Riemann-Hilbert problem. Plemelj (cf. [Pl]) and independently Birkhoff gave affirmative answers. But their arguments were not sufficient and in 1990 Bolibrukh [Bo] gave a counterexample. Then the problem was affirmatively proved by Bolibrukh [Bo2] and Kostov [Ko] if the monodromy group is irreducible.

In this paper we consider single linear Fuchsian differential equations of higher order. When the spectral type of the monodromy group is not rigid, the problem is not true in general because the number of accessory parameters is not sufficient. But it is proved in [O1] that the problem is affirmative if the monodromy group is rigid and irreducible. Also it is shown in [O1, Example 2.2] that the problem is not affirmative if the monodromy group is ‘rigid’ but reducible.

In this paper we give a characterization of the monodromy group of a Fuchsian differential equation of the second order with three singular points. In this case the equation is essentially Gauss hypergeometric equation

\[ x(1-x)u'' + (\gamma - (\alpha + \beta + 1)x)u' - \alpha \beta u = 0 \]  

and hence the result is classically known (cf. [IKSY, Chapter 2, Corollary 4.3.4], [KS]). Here we give it by a simple argument based on a result in [O2], which

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studies Gauss hypergeometric equation only by an elementary calculus without any integration.

2. A characterization

Let

\[ u'' + a(x)u' + b(x)u = 0 \]  

be a Fuchsian differential equation of the second order with three singular points \( c_0, c_1 \) and \( c_2 \) in the Riemann sphere. To study the monodromy group of this equation we may assume \((c_0, c_1, c_2) = (0, 1, \infty)\) by a linear fractional transformation. Then \( x(1 - x)a(x) \) and \( x^2(1 - x)^2b(x) \) are polynomials of degree at most 1 and 2, respectively. Hence the equation (2) has 5 parameters.

Let

\[
\frac{x}{(1-x)}a(x) \text{ and } x^2(1-x)^2b(x)
\]

be polynomials of degree at most 1 and 2, respectively. Hence the equation (2) has 5 parameters.

Let

\[
\begin{cases}
x = 0 & 1 & \infty \\
\lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} ; x \\
\lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} 
\end{cases}
\]

be the Riemann scheme of the equation (2). Then we have the Fuchs relation

\[ \lambda_{0,1} + \lambda_{0,2} + \lambda_{1,1} + \lambda_{1,2} + \lambda_{\infty,1} + \lambda_{\infty,2} = 1. \]

Since Gauss hypergeometric equation is characterized by its Riemann scheme

\[
\begin{cases}
x = 0 & 1 & \infty \\
0 & 0 & \alpha ; x \\
1 - \gamma & \gamma - \alpha - \beta & \beta 
\end{cases}
\]

the equation (2) is obtained from (1) by the gauge transformation

\[ u \mapsto x^\lambda (1-x)^\mu u \]

with

\[
\begin{cases}
\lambda_{0,1} = \lambda, & \lambda_{1,1} = \mu, & \lambda_{\infty,1} = \alpha - \lambda - \mu, \\
\lambda_{0,2} = 1 - \gamma + \lambda, & \lambda_{1,2} = \gamma - \alpha - \beta + \mu, & \lambda_{\infty,2} = \beta - \lambda - \mu
\end{cases}
\]
and the explicit form of (2) is

\[ u'' - \frac{\lambda_{0,1} + \lambda_{0,2} - 1}{x} u' + \frac{\lambda_{1,1} + \lambda_{1,2} - 1}{1-x} u' + \frac{\lambda_{0,1}\lambda_{0,2}}{x^2} u + \frac{\lambda_{1,1}\lambda_{1,2}}{(1-x)^2} u + \frac{\lambda_{0,1}\lambda_{0,2} + \lambda_{1,1}\lambda_{1,2} - \lambda_{\infty,1}\lambda_{\infty,2}}{x(1-x)} u = 0. \]  

(6)

Let \((u_1, u_2)\) be a base of local solutions of the equation (6) at a generic point \(x_0\) as in the following figure. Let \(\gamma_p\) be closed paths starting from \(x_0\) and circling around the point \(x = p\) once in a counterclockwise direction for \(p = 0, 1\) and \(\infty\), respectively, as follows.

Let \(\gamma_p u_j\) be the local solutions in a neighborhood of \(x_0\) obtained by the analytic continuation of \(u_j\) along \(\gamma_p\), respectively. Then there exist \(M_p \in GL(2, \mathbb{C})\) satisfying \((\gamma_p u_1, \gamma_p u_2) = (u_1, u_2) M_p\). Here \(GL(2, \mathbb{C})\) is the group of invertible matrices of size 2 with entries in \(\mathbb{C}\). The matrices \(M_p\) are called the local generator matrices of monodromy of the equation (6) and the subgroup of \(GL(2, \mathbb{C})\) generated by \(M_0\), \(M_1\) and \(M_\infty\) is called the monodromy group. We note that the eigenvalues of \(M_p\) are \(e^{2\pi \sqrt{-1}\lambda_{p,1}}\) and \(e^{2\pi \sqrt{-1}\lambda_{p,2}}\) and moreover we have

\[ M_\infty M_1 M_0 = I_2 \] (the identity matrix)

and if we differently choose \(x_0\) and \((u_1, u_2)\), the set of local generator matrices of monodromy \((M_0, M_1, M_\infty)\) changes into \((gM_0 g^{-1}, gM_1 g^{-1}, gM_\infty g^{-1})\) with a certain \(g \in GL(2, \mathbb{C})\). If there exists a subspace \(V\) of \(\mathbb{C}^2\) such that \(\{0\} \subsetneq V \subsetneq \mathbb{C}^2\) and \(M_p V \subset V\) for \(p = 0, 1, \infty\), then we say that the monodromy of the equation (6) is reducible. If it is not reducible, it is called irreducible.

**Definition 2.1.** For \((A_0, A_1, A_2) \in GL(2, \mathbb{C})^3\) we put

\[ CA_{0,1,2} := \{(gA_0 g^{-1}, gA_1 g^{-1}, gA_2 g^{-1}) \mid g \in GL(2, \mathbb{C})\} \]
and
\[ \widetilde{M} := \left\{ C_{M_0,M_1,M_\infty} \mid M_0, M_1 \text{ and } M_\infty \text{ are local generator matrices of the equation } (6) \text{ with } \lambda_{i,j} \in \mathbb{C} \text{ satisfying } (4) \right\} \]
under the above notation.

Then the following theorem is our characterization of \( \widetilde{M} \).

**Theorem 2.2.** Under the above notation, \( \widetilde{M} \) is a subset of
\[ \tilde{X} := \{ C_{A_0,A_1,A_2} \mid A_i \in GL(2,\mathbb{C}) \text{ (} i = 0,1,2 \text{) and } A_2A_1A_0 = I_2 \} \]
characterized by
\[ \tilde{X} \setminus \tilde{M} = \left\{ C_{A_0,A_1,A_2} \mid A_i = \begin{pmatrix} a_i & 0 \\ 0 & a'_i \end{pmatrix}, \ a_i \neq a'_i \ (i = 0,1,2) \text{ and } A_2A_1A_0 = I_2 \right\} \]
\[ \bigcup \left\{ C_{A_0,A_1,A_2} \mid A_i = \begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix}, \ b_i \neq 0 \ (i = 0,1,2) \text{ and } A_2A_1A_0 = I_2 \right\}. \]

**Remark 2.3.** i) The above theorem is equivalent to [IKSY, Chapter 2, Corollary 4.3.4].

ii) The characterization of the monodromy group of the Gauss hypergeometric equation (1) is obtained by Theorem 2.2 imposing the condition
\[ (8) \quad \text{rank}(A_i - 1) \leq 1 \text{ for } i = 0 \text{ and } 1. \]

3. **Proof of the theorem**

We will show Theorem 2.2 by the following result in [O2].

**Theorem 3.1 ([O2, Theorem 8]).** Retain the notation in the previous section. Let \( M_0, M_1 \) and \( M_\infty \) be local monodromy matrices of the equation (6) with (4).

i) \((M_0,M_1,M_\infty)\) is irreducible if and only if
\[ (9) \quad \lambda_{0,1} + \lambda_{1,\nu} + \lambda_{\infty,\nu'} \notin \mathbb{Z} \quad (\forall \nu, \nu' \in \{1,2\}) \]

ii) Suppose
\[ (10) \quad \lambda_{0,2} + \lambda_{1,2} + \lambda_{\infty,\nu} \notin \{0,-1,-2,\ldots\} \quad (\nu = 1,2). \]
We may assume

\begin{equation}
\lambda_{p,1} - \lambda_{p,2} \notin \{1, 2, 3, \ldots\} \quad (p = 0, 1)
\end{equation}

by one or both of the permutations \(\lambda_{0,1} \leftrightarrow \lambda_{0,2}\) and \(\lambda_{1,1} \leftrightarrow \lambda_{1,2}\) if necessary.

When

\begin{equation}
\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,\nu} \notin \mathbb{Z} \quad (\nu = 1, 2),
\end{equation}

there exists \(g \in GL(2, \mathbb{C})\) such that the monodromy matrices satisfy

\begin{equation}
(gM_0g^{-1}, gM_1g^{-1}) = \left( \begin{pmatrix} e^{2\pi i \lambda_{0,2}} & a_0 \\ 0 & e^{2\pi i \lambda_{0,1}} \end{pmatrix}, \begin{pmatrix} e^{2\pi i \lambda_{1,1}} & 0 \\ a_1 & e^{2\pi i \lambda_{1,2}} \end{pmatrix} \right)
\end{equation}

with

\begin{equation}
a_0 = 2e^{-\pi i \lambda_{\infty,2}} \sin \pi(\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,2}),
a_1 = 2e^{-\pi i \lambda_{\infty,1}} \sin \pi(\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,1}).
\end{equation}

When (12) is not valid, we have (13) with a certain \(g \in GL(2, \mathbb{C})\) and

\begin{equation}
a_0 = \begin{cases} 1 & \text{if } \lambda_{0,1} + \lambda_{1,2} + \lambda_{\infty,\nu} \notin \{0, -1, -2, \ldots\} \quad (\nu = 1, 2), \\ 0 & \text{otherwise,} \end{cases}
\end{equation}

\begin{equation}
a_1 = \begin{cases} 1 & \text{if } \lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,\nu} \notin \{0, -1, -2, \ldots\} \quad (\nu = 1, 2), \\ 0 & \text{otherwise.} \end{cases}
\end{equation}

Note that \(a_0a_1 = 0\) in this case.

iii) Under a change of indices \(\lambda_{p,\nu} \rightarrow \lambda_{\sigma(p),\sigma(\nu)}\) with suitable permutations \((\sigma, \sigma_0, \sigma_1, \sigma_\infty) \in \mathcal{S}_3 \times \mathcal{S}_3\) we have (10) and (11). Here \(\mathcal{S}_3\) and \(\mathcal{S}_2\) are identified with the permutation groups of \(\{0, 1, \infty\}\) and \(\{1, 2\}\), respectively.

Proof of Theorem 2.2. We say \((A_0, A_1, A_2) \in GL(2, \mathbb{C})^3\) is irreducible if and only if there exists no subspace \(V\) of \(\mathbb{C}^2\) such that \(\{0\} \subsetneq V \subsetneq \mathbb{C}^2\) and \(A_pV \subset V\) for \(p = 0, 1, 2\).

Let \((A_0, A_1, A_2) \in GL(2, \mathbb{C})^3\) with \(A_\infty A_1 A_0 = I_2\). We can choose \(\lambda_{i,j} \in \mathbb{C}\) such that they satisfy (4) and the set of eigenvalues of \(A_i\) are \(\{e^{2\pi \sqrt{-1} \lambda_{i,1}}, e^{2\pi \sqrt{-1} \lambda_{i,2}}\}\) for \(i = 0, 1\) and \(\infty\), respectively.

First we suppose \((A_0, A_1, A_\infty)\) is irreducible. We will prove \(C_{A_0, A_1, A_\infty} \in \widetilde{\mathcal{M}}\).

Since the eigenvector \(v_0\) of \(A_0\) with the eigenvalue \(e^{2\pi \sqrt{-1} \lambda_{0,2}}\) and the eigenvector \(v_1\) of \(A_1\) with the eigenvalue \(e^{2\pi \sqrt{-1} \lambda_{1,2}}\) are linearly independent, there exists \(g \in \)
GL(2, \mathbb{C}) satisfying
\begin{equation}
(16) \quad (gA_0g^{-1}, gA_1g^{-1}) = \left( \begin{array}{cc}
2\pi \sqrt{-1} \lambda_{0.2} & a_0 \\
0 & e^{2\pi \sqrt{-1} \lambda_{1.1}}
\end{array} \right), \quad \left( \begin{array}{cc}
2\pi \sqrt{-1} \lambda_{1.1} & 0 \\
a_1 & e^{2\pi \sqrt{-1} \lambda_{1.2}}
\end{array} \right)
\end{equation}
with suitable complex numbers \(a_0\) and \(a_1\). Here the irreducibility of \((A_0, A_1, A_\infty)\) implies \(a_0a_1 \neq 0\). Since \(\text{trace} A_1A_0 = \text{trace} A_\infty^{-1}\), we have
\[
\begin{align*}
\text{trace} A_1A_0 &= e^{2\pi i (\lambda_{0.2} + \lambda_{1.1})} + a_0a_1 + e^{2\pi i (\lambda_{0.1} + \lambda_{1.2})} = e^{-2\pi i \lambda_{\infty,1}} + e^{-2\pi i \lambda_{\infty,2}}, \\
\text{trace} A_0a_1 &= e^{-2\pi i \lambda_{\infty,1}} + e^{-2\pi i \lambda_{\infty,2}} - e^{2\pi i (\lambda_{0.2} + \lambda_{1.1})} = e^{2\pi i (\lambda_{0.1} + \lambda_{1.2})} \\
&= e^{\pi i (\lambda_{0.1} + \lambda_{0.2} + \lambda_{1.1} + \lambda_{1.2})} (2i \sin \pi (\lambda_{0.2} + \lambda_{1.1} + \lambda_{\infty,2})) \\
&\quad \cdot \left(2i \sin \pi (\lambda_{0.1} + \lambda_{1.2} + \lambda_{\infty,2}) \right) \\
&= 4e^{-\pi i (\lambda_{\infty,1} + \lambda_{\infty,2})} \sin \pi (\lambda_{0.2} + \lambda_{1.1} + \lambda_{\infty,2}) \sin \pi (\lambda_{0.2} + \lambda_{1.1} + \lambda_{\infty,1}).
\end{align*}
\]
The condition \(a_0a_1 \neq 0\) implies that in (16) we may choose \(a_0\) and \(a_1\) by (14) for a suitable matrix \(g \in GL(2, \mathbb{C})\). Hence Theorem 3.1 ii) assures \(C_{A_0,A_1,A_\infty} \in \bar{M}\).

Now we suppose \((A_0, A_1, A_\infty)\) is reducible. Then there exists a simultaneous eigenvector \(v_0\) of \(A_0\) and \(A_1\), therefore
\[
\bar{X}': = \{ C_{A_0,A_1,A_2} \mid (A_0, A_1, A_2) \in GL(2, \mathbb{C})^3 \text{ is reducible and } A_2A_1A_0 = I_2 \}
\]
\[
= \left\{ C_{A_0,A_1,A_2} \mid A_i = \begin{pmatrix} a_i & b_i \\ 0 & a_i' \end{pmatrix} \quad (i = 0, 1, 2) \text{ and } A_2A_1A_0 = I_2 \right\}
\]
and \(\bar{X} \setminus \bar{M} \subset \bar{X}'\). Note that if we fix \(i \in \{0, 1, 2\}\) and there exists an eigenvector \(v_1\) of \(A_i\) with \(v_1 \notin \mathbb{C}v_0\), we may assume that \(A_i\) is a diagonal matrix in the above.

According to the number \(N = \# \{ i \in \{0, 1, 2\} \mid a_i = a_i' \}\), we can divide \(\bar{X}'\) into \(\bar{X}_N\) with \(N = 0, 1\) and 3:
\[
\bar{X}' = \bar{X}_0 \sqcup \bar{X}_1 \sqcup \bar{X}_3.
\]

Suppose \(C_{A_0, A_1, A_2} \in \bar{X}_0\). Then \(A_1\) is diagonalizable and according to the simultaneous diagonalizability of \((A_0, A_1, A_2)\), we divide \(\bar{X}_0\) as follows.
\[
\bar{X}_0 := \bar{X}_{0,0} \sqcup \bar{X}_{0,1},
\]
Theorem 3.1 ii) gives the above result. These 4 examples satisfy (10) and (11) but do not satisfy (12). Then the last claim of Theorem 3.1 ii) gives the above result.

\[ 
\tilde{X}_{0,\nu} := \begin{cases} 
C_{A_0, A_1, A_2} & A_i = \begin{pmatrix} a_i b_i \\ 0 \ a_i' \end{pmatrix}, \ a_i \neq a_i' \ (i = 0, 1, 2) \\
A_2 A_1 A_0 = I_2, \ b_0 = \nu \text{ and } b_1 = 0 
\end{cases}. 
\]

According to the existence of the scalar matrix in \( \{A_0, A_1, A_2\} \), we have

\[ \tilde{X}_1 = \tilde{X}_{1,0} \sqcup \tilde{X}_{0,1}, \]

\[ \tilde{X}_{1,\nu} := \begin{cases} 
C_{A_0, A_1, A_2}, \ C_{A_1, A_2, A_0}, \ C_{A_2, A_0, A_1} & A_0 = \begin{pmatrix} a_0 \nu \\ 0 \ a_0 \end{pmatrix}, \ A_1 = \begin{pmatrix} a_1 \ 0 \\ 0 \ a_1' \end{pmatrix}, \\
A_2 A_1 A_0 = I_2 \text{ and } a_1 \neq a_1'
\end{cases}. \]

Considering the number of scalar matrices in \( \{A_0, A_1, A_2\} \), we have

\[ \tilde{X}_3 = \tilde{X}_{3,3} \sqcup \tilde{X}_{3,1} \sqcup \tilde{X}_{3,0}, \]

\[ \tilde{X}_{3,3} := \{(a_0 I_2, a_1 I_2, a_2 I_2) \mid a_0 a_1 a_2 = 1\}, \]

\[ \tilde{X}_{3,1} := \begin{cases} 
C_{A_0, A_1, A_2}, \ C_{A_2, A_0, A_1}, \ C_{A_1, A_2, A_0} & A_0 = \begin{pmatrix} a_0 \ 1 \\ 0 \ a_0 \end{pmatrix}, \ A_1 = a_1 I_2, \\
A_2 A_1 A_0 = I_2
\end{cases}, \]

\[ \tilde{X}_{3,0} := \begin{cases} 
C_{A_0, A_1, A_2} & A_i = \begin{pmatrix} a_i \ b_i \\ 0 \ a_i' \end{pmatrix} \ (i = 0, 1, 2) \\
A_2 A_1 A_0 = I_2 \text{ and } b_0 b_1 b_2 \neq 0
\end{cases}. \]

We give examples of local monodromy matrices \( M_0 \) and \( M_1 \) of the equation (6) with the Riemann scheme (3). Here we assume \( \lambda \in \mathbb{C} \setminus \mathbb{Z} \) and denote \( (A, B) \sim (A', B') \) if there exists \( g \in \text{GL}(2, \mathbb{C}) \) satisfying \( A' = g A g^{-1} \) and \( B' = g B g^{-1} \).

<table>
<thead>
<tr>
<th></th>
<th>[ \lambda_{0,1} \lambda_{1,1} \lambda_{\infty,1} ]</th>
<th>( (M_0, M_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{1,0} )</td>
<td>\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; \lambda &amp; -\lambda \end{pmatrix}</td>
<td>\sim \begin{pmatrix} 10 &amp; 0 \ 01 &amp; 1 \end{pmatrix}, \begin{pmatrix} 10 &amp; 0 \ 01 &amp; e^{2\pi i \lambda} \end{pmatrix} \sim \begin{pmatrix} 10 &amp; 0 \ 01 &amp; e^{2\pi i \lambda} \end{pmatrix}</td>
</tr>
<tr>
<td>( X_{1,1} )</td>
<td>\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; \lambda &amp; 1 - \lambda \end{pmatrix}</td>
<td>\sim \begin{pmatrix} 11 &amp; 0 \ 01 &amp; 1 \end{pmatrix}, \begin{pmatrix} 10 &amp; 0 \ 01 &amp; e^{2\pi i \lambda} \end{pmatrix}</td>
</tr>
<tr>
<td>( X_{3,3} )</td>
<td>\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; 1 &amp; -1 \end{pmatrix}</td>
<td>\sim \begin{pmatrix} 10 &amp; 0 \ 01 &amp; 0 \end{pmatrix}, \begin{pmatrix} 10 &amp; 0 \ 01 &amp; 0 \end{pmatrix}</td>
</tr>
<tr>
<td>( X_{3,1} )</td>
<td>\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}</td>
<td>\sim \begin{pmatrix} 11 &amp; 0 \ 01 &amp; 0 \end{pmatrix}, \begin{pmatrix} 10 &amp; 0 \ 01 &amp; 0 \end{pmatrix}</td>
</tr>
</tbody>
</table>

These 4 examples satisfy (10) and (11) but do not satisfy (12). Then the last claim of Theorem 3.1 ii) gives the above result.
Now we note the following facts. If \( C_{A,B,(BA)^{-1}} \in \tilde{M} \), then \( C_{B,A,(AB)^{-1}} \in \tilde{M} \) and \( C_{A,(AB)^{-1},B} \in \tilde{M} \) because of the symmetry of the equation (6) determined by the Riemann scheme (3) and moreover \( C_{aA,bB,(abBA)^{-1}} \in \tilde{M} \) for \((a, b) \in (\mathbb{C} \setminus \{0\})^2\) because of a suitable gauge transformation \( u \mapsto x^\lambda (1 - x)^\mu u \).

It follows from these facts that the above \( X_{i,j} \) shows \( \tilde{X}_{i,j} \subset \tilde{M} \) for \((i, j) = (1, 0), (1, 1), (3, 3) \) and \((3, 1)\).

Suppose there exist complex numbers \( a_0 \) and \( a_1 \) satisfying \((M_0 - a_0)^2 = (M_1 - a_1)^2 = 0\) and moreover \( C_{M_0,M_1,(M_1M_0)^{-1}} \in \tilde{X}' \). Then (12) is not valid and Theorem 3.1 ii) and iii) imply that at least one of \( M_0, M_1 \) and \( M_1M_0 \) is a scalar matrix. Hence \( \tilde{X}_{3,0} \subset \tilde{X} \setminus \tilde{M} \).

Note that
\[
\{(a_0, a'_0, a_1, a'_1, a_2, a'_2) \mid \{a_i, a'_i\} \text{ are the sets of eigenvalues of } A_i \text{ for } C_{A_0,A_1,A_2} \in \tilde{M}\}
= \{(c_1, \ldots, c_6) \in \mathbb{C}^6 \mid c_1c_2c_3c_4c_5c_6 = 1\}.
\]

Suppose \( C_{M_0,M_1,M_\infty} \in \tilde{X}_0 \). We may assume \( \lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,1} \in \mathbb{Z} \). Owing to the Fuchs relation (4), we may moreover assume
\[
\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,1} \in \{0, -1, -2, \ldots\}
\]
and hence
\[
\lambda_{0,1} + \lambda_{1,2} + \lambda_{\infty,2} \in \{1, 2, 3, \ldots\}.
\]
Since \( M_1 \) is not a scalar matrix, we have
\[
\lambda_{0,2} + \lambda_{1,2} + \lambda_{\infty,1} \notin \mathbb{Z}.
\]
In this case (10) and (11) are valid and (12) is not valid. Hence Theorem 3.1 ii) assures \( a_0 = 1 \) and \( a_1 = 0 \) in (13). Thus we have \( \tilde{X}_0 \cap \tilde{M} = \tilde{X}_{0,1} \).

Combining the facts we have proved, we have the theorem. \(\square\)

**Remark 3.2.** Using our classification of elements of \( \tilde{X} \), it is easy to show
\[
\tilde{X} = \{C_{M_0,M_1,M_\infty} \mid M_0, M_1 \text{ and } M_\infty \text{ are local monodromy matrices of the equation } \frac{du}{dx} = \left(\frac{A_0}{x} + \frac{A_1}{1-x}\right)u \text{ with } A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad (a_i, b_i, c_i, d_i) \in \mathbb{C}^4 \text{ for } i = 0 \text{ and } 1\}.
\]
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