Confluence and versal unfolding Pfaffian systems

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Some corrections indicated by red.

p.128

$$D_{r,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & s_{3,1} & s_{3,12} & 0 & 0 & \cdots \\ 0 & 1 & s_{4,1} + s_{3,2} & s_{4,12} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.12}$$

P.133

Then we remark that

$$D'_{r,k,i,j}(\mathbf{a}') = \sum_{\nu=1}^{r} T'_{i,\nu}(\mathbf{a}')S'_{\nu,k}(\mathbf{a}')S'_{\nu,j}(\mathbf{a}') = D_{r,k,i+1,j+1}(\mathbf{a})\big|_{a_1=0} = D'_{r,\,j+1,i,\,k-1}(\mathbf{a}') \tag{5.9}$$

$$= \sum_{\nu=\max\{k,\,j+1\}}^{i} t_{\nu+1,\{k,k+1,\dots,i+1\}\setminus\{\nu+1\}}s'_{\nu+1,\{2,\dots,j\}},$$

$$T'_{i,\nu}(\mathbf{a}')S'_{\nu,j}(\mathbf{a}') = T_{i-1,\nu-1}(\mathbf{a}')S_{\nu-1,j-1}(\mathbf{a}') \tag{5.10}$$

and we get the following lemma by the same way as in the proof of the previous lemma.

Lemma 5.1. i) $D'_{r,k,i,j}(\mathbf{a}')$ are homogeneous polynomials of $\mathbf{a}'$ with degree $j+k-i-1$. In particular

$$D'_{r,k,i,j}(\mathbf{a}') = \begin{cases} 0 & (i+1<k \quad \text{or} \quad i<j \quad \text{or} \quad i>j+k-1), \\ 1 & (i=j+k-1). \end{cases}$$

The following propoision may be added at the end of §4 (p.131).

Proposition 4.4. The polynomials $D_{r,k,i,j}(\mathbf{a})$ $(1 \le i \le r,\ 1 \le j \le r,\ 1 \le k \le r)$ are given by

$$D_{r,k,i,j}(\mathbf{a}) = \begin{cases} 0 & (L<0 \quad \text{or} \quad i<j \quad \text{or} \quad i<k), \\ 1 & (L=0), \\ \displaystyle\sum_{0\le\nu_1\le\nu_2\le\cdots\le\nu_L\le i-k} \prod_{p=1}^{L}(a_{i-\nu_p} - a_{\nu_p+p}) & (L>0,\ i\ge j,\ i\ge k), \end{cases} \tag{4.19}$$

where $L = k+j-i-1$.

Proof. Lemma 4.2 implies the first two cases in (4.19). Notethat

$$\begin{aligned}\sum_{\nu=k}^{i} \frac{s_{\nu,\{1,\dots,j-1\}}}{s_{\nu,\{k,k+1,\dots,i\}\setminus\{\nu\}}} &= \frac{(a_k-a_{j-1})s_{k,\{1,\dots,j-2\}}}{s_{k,\{k+1,\dots,i\}}} + \sum_{\nu=k+1}^{i} \frac{(a_\nu - a_{j-1})s_{\nu,\{1,\dots,j-2\}}}{(a_\nu - a_k)s_{\nu,\{k+1,\dots,i\}\setminus\{\nu\}}} \\ &= \frac{(a_k-a_{j-1})s_{k,\{1,\dots,j-2\}}}{s_{k,\{k+1,\dots,i\}}} + \sum_{\nu=k+1}^{i}\Big(\frac{a_k-a_{j-1}}{a_\nu-a_k}+1\Big)\frac{s_{\nu,\{1,\dots,j-2\}}}{s_{\nu,\{k+1,\dots,i\}\setminus\{\nu\}}} \\ &= \sum_{\nu=k+1}^{i}\frac{s_{\nu,\{1,\dots,j-2\}}}{s_{\nu,\{k+1,\dots,i\}\setminus\{\nu\}}} + \sum_{\nu=k}^{i}(a_k-a_{j-1})\frac{s_{\nu,\{1,\dots,j-2\}}}{s_{\nu,\{k,k+1,\dots,i\}\setminus\{\nu\}}},\end{aligned}$$

which shows the recurcive relation

$$D_{r,k,i,j}(\mathbf{a}) = D_{r,k+1,i,j-1}(\mathbf{a}) + (a_k - a_{j-1})D_{r,k,i,j-1}(\mathbf{a}) \quad (k+j > i+1). \tag{4.20}$$

Then we have

$$\begin{aligned}
D_{r,k,i,j}(\mathbf{a}) &= (a_k - a_{j-1})D_{r,k,i,j-1} + (a_{k+1} - a_{j-2})D_{r,k+1,i,j-2} + (a_{k+2} - a_{j-3})D_{r,k+2,i,j-3} + \cdots \\
&= \sum_{\nu=0}^{\min\{i-k,j-2\}} (a_{k+\nu} - a_{j-1-\nu})D_{r,k+\nu,i,j-\nu-1} \\
&= \sum_{\mu=0}^{i-k} (a_{i-\mu} - a_{j+k-i-1+\mu})D_{r,i-\mu,i,j+k-i-1+\mu} \quad (\mu+\nu = i-k) \\
&= \sum_{\nu_L=0}^{i-k} (a_{i-\nu_L} - a_{L+\nu_L})D_{r,i-\nu_L,i,L+\nu_L} \quad (L := k+j-i-1,\ \nu_L = \mu) \\
&= \sum_{\nu_L=0}^{i-k} (a_{i-\nu_L} - a_{L+\nu_L}) \sum_{\nu_{L-1}}^{\nu_L} (a_{i-\nu_{L-1}} - a_{L-1+\nu_{L-1}})D_{r,i-\nu_{L-1},i,L-1+\nu_{L-1}} \\
&= \cdots \\
&= \sum_{0\le\nu_1\le\nu_2\le\cdots\le\nu_L\le i-k} (a_{i-\nu_L} - a_{L+\nu_L})(a_{i-\nu_{L-1}} - a_{L-1+\nu_{L-1}})\cdots(a_{i-\nu_1} - a_{1+\nu_1}),
\end{aligned}$$

which means (4.19) $\square$