Confluence and versal unfolding Pfaffian systems

Josai Mathematical Monograph **12**(2020), 117–151

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Some corrections indicated by red.

p.126

$$\frac{1}{x-a_i}=\sum_{j=1}^{i}\frac{\prod_{\nu=1}^{j-1}(a_i-a_\nu)}{(x-a_1)\cdots(x-a_j)} \tag{4.3}$$

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$$D_{r,3}=\begin{pmatrix}0&0&0&0&0&\cdots\\0&0&0&0&0&\cdots\\1&s_{3,1}&s_{3,12}&0&0&\cdots\\0&1&s_{4,1}+s_{3,2}&s_{4,12}&0&\cdots\\\vdots&\ddots&\ddots&\ddots&\ddots&\ddots\end{pmatrix} \tag{4.12}$$

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Then we remark that

$$D'_{r,k,i,j}(\mathbf{a}')=\sum_{\nu=1}^{r}T'_{i,\nu}(\mathbf{a}')S'_{\nu,k}(\mathbf{a}')S'_{\nu,j}(\mathbf{a}')=D_{r,k,i+1,j+1}(\mathbf{a})\big|_{a_1=0}=D'_{r,\,j+1,i,\,k-1}(\mathbf{a}') \tag{5.9}$$

$$=\sum_{\nu=\max\{k,\,j+1\}}^{i}t_{\nu+1,\{k,k+1,\dots,i+1\}\setminus\{\nu+1\}}s'_{\nu+1,\{2,\dots,j\}},$$

$$T'_{i,\nu}(\mathbf{a}')S'_{\nu,j}(\mathbf{a}')=T_{i-1,\nu-1}(\mathbf{a}')S_{\nu-1,j-1}(\mathbf{a}') \tag{5.10}$$

and we get the following lemma by the same way as in the proof of the previous lemma.

Lemma 5.1. i) $D'_{r,k,i,j}(\mathbf{a}')$ are homogeneous polynomials of $\mathbf{a}'$ with degree $j+k-i-1$. In particular

$$D'_{r,k,i,j}(\mathbf{a}')=\begin{cases}0 & (i+1<k \ \text{ or } \ i<j \ \text{ or } \ i>j+k-1),\\ 1 & (i=j+k-1).\end{cases}$$

The following propoision may be added at the end of §4 (p.131).

Proposition 4.4. The polynomials $D_{r,k,i,j}(\mathbf{a})\quad(1\le i\le r,\,1\le j\le r,\,1\le k\le r)$ are given by

$$D_{r,k,i,j}(\mathbf{a})=\begin{cases}0 & (L<0 \ \text{ or } \ i<j \ \text{ or } \ i<k),\\ 1 & (L=0),\\ \displaystyle\sum_{0\le\nu_1\le\nu_2\le\cdots\le\nu_L\le i-k}\ \prod_{p=1}^{L}(a_{i-\nu_p}-a_{\nu_p+p}) & (L>0,\ i\ge j,\ i\ge k),\end{cases} \tag{4.19}$$

where $L=k+j-i-1$.

Proof. Lemma 4.2 implies the first two cases in (4.19). Note that

$$\begin{aligned}\sum_{\nu=k}^{i}\frac{s_{\nu,\{1,\dots,j-1\}}}{s_{\nu,\{k,k+1,\dots,i\}\setminus\{\nu\}}} &= \frac{(a_k-a_{j-1})s_{k,\{1,\dots,j-2\}}}{s_{k,\{k+1,\dots,i\}}}+\sum_{\nu=k+1}^{i}\frac{(a_\nu-a_{j-1})s_{\nu,\{1,\dots,j-2\}}}{(a_\nu-a_k)s_{\nu,\{k+1,\dots,i\}\setminus\{\nu\}}}\\ &= \frac{(a_k-a_{j-1})s_{k,\{1,\dots,j-2\}}}{s_{k,\{k+1,\dots,i\}}}+\sum_{\nu=k+1}^{i}\Big(\frac{a_k-a_{j-1}}{a_\nu-a_k}+1\Big)\frac{s_{\nu,\{1,\dots,j-2\}}}{s_{\nu,\{k+1,\dots,i\}\setminus\{\nu\}}}\\ &= \sum_{\nu=k+1}^{i}\frac{s_{\nu,\{1,\dots,j-2\}}}{s_{\nu,\{k+1,\dots,i\}\setminus\{\nu\}}}+\sum_{\nu=k}^{i}(a_k-a_{j-1})\frac{s_{\nu,\{1,\dots,j-2\}}}{s_{\nu,\{k,k+1,\dots,i\}\setminus\{\nu\}}},\end{aligned}$$

which shows the recurcive relation

$$D_{r,k,i,j}(\mathbf{a}) = D_{r,k+1,i,j-1}(\mathbf{a})+(a_k-a_{j-1})D_{r,k,i,j-1}(\mathbf{a})\quad(k+j>i+1). \tag{4.20}$$

Then we have

$$\begin{aligned}D_{r,k,i,j}(\mathbf{a}) &= (a_k-a_{j-1})D_{r,k,i,j-1}+(a_{k+1}-a_{j-2})D_{r,k+1,i,j-2}+(a_{k+2}-a_{j-3})D_{r,k+2,i,j-3}+\cdots\\ &= \sum_{\nu=0}^{\min\{i-k,j-2\}}(a_{k+\nu}-a_{j-1-\nu})D_{r,k+\nu,i,j-\nu-1}\\ &= \sum_{\mu=0}^{i-k}(a_{i-\mu}-a_{j+k-i-1+\mu})D_{r,i-\mu,i,j+k-i-1+\mu}\quad(\mu+\nu=i-k)\\ &= \sum_{\nu_L=0}^{i-k}(a_{i-\nu_L}-a_{L+\nu_L})D_{r,i-\nu_L,i,L+\nu_L}\quad(L:=k+j-i-1,\ \nu_L=\mu)\\ &= \sum_{\nu_L=0}^{i-k}(a_{i-\nu_L}-a_{L+\nu_L})\sum_{\nu_{L-1}}^{\nu_L}(a_{i-\nu_{L-1}}-a_{L-1+\nu_{L-1}})D_{r,i-\nu_{L-1},i,L-1+\nu_{L-1}}\\ &= \cdots\\ &= \sum_{0\le\nu_1\le\nu_2\le\cdots\le\nu_L\le i-k}(a_{i-\nu_L}-a_{L+\nu_L})(a_{i-\nu_{L-1}}-a_{L-1+\nu_{L-1}})\cdots(a_{i-\nu_1}-a_{1+\nu_1}),\end{aligned}$$

which means (4.19) □