Fractional calculus of Weyl algebra and its applications to ordinary differential equations on the Riemann sphere

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§ **Introduction** global property of special functions

Gauss hypergeometric (HG) series $F(\alpha,\beta,\gamma;x)$

Gauss hypergeometric family: Bessel, Whittaker, Legendre, Hermite,...

$\longrightarrow$ generalization

1. ${}_nF_{n-1}(\alpha,\beta;x)$: Generalized Hypergeometric (GHG), Jordan-Pochhammer, Dotsenko-Fateev,...

2. Heun,... with accessory parameters ($=$ moduli : global/local) $\to$ Painlevé eq. ($\Leftarrow$ Isomonodromy deformation)

3. Several var. : Appell, Gelfand GHG, Heckman-Opdam HG,...

$\rightsquigarrow$ A unifying theory and understanding with computable results ($\Leftarrow$ Kac-Moody Weyl group) including
confluences, contiguity relations,
series expansions, integral representations of solutions,
monodromies and connection problems,...

$$F(\alpha,\beta,\gamma;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}$$

$$=1+\frac{\alpha\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\cdots$$

$$(\alpha)_k:=\prod_{\nu=0}^{k-1}(\alpha+\nu)=\alpha(\alpha+1)\cdots(\alpha+k-1)$$

Gauss summation formula: a connection coefficient

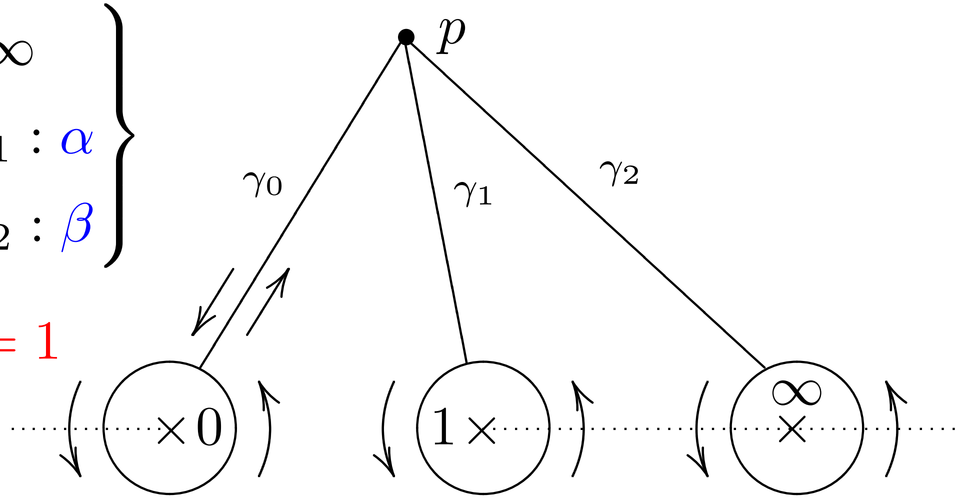
$$C_{\alpha,\beta,\gamma}:=F(\alpha,\beta,\gamma;1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)}$$

1. $\dfrac{C_{\alpha,\beta,\gamma+1}}{C_{\alpha,\beta,\gamma}}=\dfrac{\gamma(\gamma-\alpha-\beta)}{(\gamma-\beta)(\gamma-\alpha)}$ and $\lim\limits_{n\to\infty}C_{\alpha,\beta,\gamma+n}=1$

2. $F(\alpha,\beta,\gamma;x)=\dfrac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\displaystyle\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt$

$\to x=1$

$$P\left\{\begin{matrix} x=0 & 1 & \infty \\ \lambda_{0,1}: 1-\gamma & \lambda_{1,1}: \gamma-\alpha-\beta & \lambda_{2,1}: \alpha \\ \lambda_{0,2}: 0 & \lambda_{1,2}: 0 & \lambda_{2,2}: \beta \end{matrix}\right\}$$



$$\lambda_{0,1}+\lambda_{0,2}+\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,1}+\lambda_{2,2}=1$$

: Fuchs relation

$$x(1-x)u''+\big(\gamma-(\alpha+\beta+1)x\big)u'-\alpha\beta u=0$$

$u_{j,\nu}$: normalized local solution corresponds to the exponent $\lambda_{j,\nu}$

$(\gamma_j u_{0,2}, \gamma_j u_{1,2}) = (u_{0,2}, u_{1,2})M_j \quad (M_j \in GL(2,\mathbb{C}))$: monodromies

$$M_0=\begin{pmatrix} e^{2\pi i\lambda_{0,2}} & a_0 \\ 0 & e^{2\pi i\lambda_{0,1}} \end{pmatrix},\quad M_1=\begin{pmatrix} e^{2\pi i\lambda_{1,1}} & 0 \\ a_1 & e^{2\pi i\lambda_{1,2}} \end{pmatrix},\quad M_2M_1M_0=I_2$$

$$\operatorname{trace} M_1M_0 = e^{2\pi i(\lambda_{0,2}+\lambda_{1,1})}+a_0a_1+e^{2\pi i(\lambda_{0,1}+\lambda_{1,2})} = e^{-2\pi i\lambda_{2,1}}+e^{-2\pi i\lambda_{2,2}}$$

$$\begin{aligned} a_0a_1 &= e^{-2\pi i\lambda_{2,1}}+e^{-2\pi i\lambda_{2,2}}-e^{2\pi i(\lambda_{0,2}+\lambda_{1,1})}-e^{2\pi i(\lambda_{0,1}+\lambda_{1,2})} \\ &= e^{-2\pi i\lambda_{2,2}}\big(e^{2\pi i(\lambda_{0,2}+\lambda_{1,1}+\lambda_{2,2})}-1\big)\big(e^{2\pi i(\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,2})}-1\big) \end{aligned}$$

$\neq 0 \iff$ irreducible monodromy group

§ Fuchsian differential equations

$$Pu = 0, \quad P := a_n(x)\tfrac{d^n}{dx^n} + a_{n-1}(x)\tfrac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\tfrac{d}{dx} + a_0(x)$$

$x = 0$: singularity of P

Normalized at 0: $a_n(0) = \cdots = a_n^{(n-1)}(0) = 0$, $a_n^{(n)}(0) \neq 0$

$x = 0$ is regular singularity $\Leftrightarrow$ (order of zero of $a_j(x)$ at 0) $\geq j$

$$\boxed{P = \sum_{j=0}^{\infty} x^j p_j(\vartheta), \quad \vartheta := x\partial,\ \partial := \frac{d}{dx}}$$

$p_0(s) = 0$: indicial equation

the roots λ_j $(j = 1, \ldots, n)$: characteristic exponents

$\lambda_i - \lambda_j \notin \mathbb{Z} \quad (i \neq j) \ \Rightarrow$

$\exists 1$ solution $u_j(x) = x_j^{\lambda}\phi_j(x)$, $\phi_j(x)$ is analytic at 0 and $\phi_j(0) = 1$

${}_nF_{n-1}(\alpha, \beta; z)$: $\{$exponents at $1\} = \{0, 1, \ldots, n-1, -\beta_n\}$

the local monodromy is generically $(\Leftarrow\ \beta_n \notin \mathbb{Z})$ semisimple

$\Rightarrow$ generalize "characteristic exponents"

$$[\lambda]_{(m)} := \begin{pmatrix} \lambda \\ \lambda+1 \\ \vdots \\ \lambda+m-1 \end{pmatrix}, \quad m = 0, 1, \ldots$$

$$n = m_1 + \cdots + m_k, \quad \lambda_1, \ldots, \lambda_k \in \mathbb{C}$$

Def. P has generalized exponents $\{[\lambda_1]_{(m_1)}, \ldots, [\lambda_k]_{(m_k)}\}$ at 0 $\overset{\text{def}}{\iff}$

$\Lambda := \{\lambda_j + \nu\,;\, 0 \le \nu < m_j,\ j = 1, \ldots, k\}$: char. exponents at 0

- $\lambda_i - \lambda_j \notin \mathbb{Z}$ $(i \neq j)$ $\Rightarrow$

(Def. $\Leftrightarrow$ char. exp. are Λ and local monodromy is semisimple)

- $\lambda_1 = \cdots = \lambda_k$ $\Rightarrow$

(Def. $\Leftrightarrow$ char. exp. are Λ and Jordan normal form of the local monodromy type corresponds to the dual partition of $n = m_1 + \cdots + m_k$)

- $k = 1,\ \lambda_1 = 0$ $\Rightarrow$ (Def. $\Leftrightarrow$ $x = 0$: regular point)
- In general

$$\boxed{\prod_{j=1}^{k} \prod_{0 \le \nu < m_j - \ell} (s - \lambda_j - \nu) \,\Big|\, p_\ell(s) \quad (\ell = 0, \ldots, \max\{m_1, \ldots, m_k\} - 1)}$$

Def. P has the generalized Riemann scheme (GRS)

$$P\left\{\begin{matrix} x=c_0=\infty & c_1 & \cdots & c_p & \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} & \\ \vdots & \vdots & \vdots & \vdots & ;\ x \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} & \end{matrix}\right\}$$

$$\mathbf{m}=(\mathbf{m}_0,\ldots,\mathbf{m}_p)=\big((m_{0,1},\ldots,m_{0,n_0}),\ldots,(m_{p,1},\ldots,m_{p,n_p})\big)$$

: $(p+1)$-tuples of partitions of $n=\operatorname{ord}\mathbf{m}$ (spectral type)

Fuchs condition (FC): $|\{\lambda_{\mathbf{m}}\}|:=\sum m_{j,\nu}\lambda_{j,\nu}-\operatorname{ord}\mathbf{m}+\frac{1}{2}\operatorname{idx}\mathbf{m}=0$

$\operatorname{idx}\mathbf{m}:=\sum_{j,\nu}m_{j,\nu}^2-(p-1)(\operatorname{ord}\mathbf{m})^2$ (index of rigidity, Katz)

Normal form of P: $\partial=\frac{d}{dx}$

$$P=\Big(\prod_{j=1}^{p}(x-c_j)^n\Big)\partial^n+a_{n-1}(x)\partial^{n-1}+\cdots+a_1(x)\partial+a_0(x)$$

(order of zeros of $a_\nu(x)$ at c_j) $\geq\nu$ and $\deg a_\nu(x)\leq n(p-1)+\nu$

$\mathbf{m}$: realizable $\overset{\text{def}}{\Leftrightarrow}$ $\exists\, P$ with (GRS) for generic $\lambda_{j,\nu}$ under (FC)

$\mathbf{m}$: irreducibly realizable $\overset{\text{def}}{\Leftrightarrow}$ $\exists\, Pu = 0$ is irreducible for generic $\lambda_{j,\nu}$

Problem. Classify such $\mathbf{m}$! (Deligne-Simpson problem)

$\mathbf{m}$: monotone $\overset{\text{def}}{\Leftrightarrow}$ $m_{j,1} \ge m_{j,2} \ge m_{j,3} \ge \cdots$

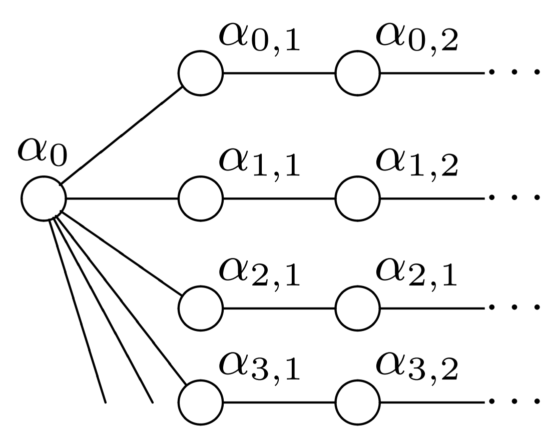
$\mathbf{m}$: indivisible $\overset{\text{def}}{\Leftrightarrow}$ $\gcd \mathbf{m} := \gcd\{m_{j,\nu}\} = 1$

$\mathbf{m}$: basic $\overset{\text{def}}{\Leftrightarrow}$ indivisible, monotone and $m_{j,1}+\cdots+m_{j,p} \le (p-1)\operatorname{ord}\mathbf{m}$

A Kac-Moody root system (Π, W)

$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi), \quad (\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1},$$

$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \ne j \quad \text{or} \quad |\mu - \nu| > 1) \\ -1 & (i = j \quad \text{and} \quad |\mu - \nu| = 1) \end{cases}$$



Δ_+^{re}: positive real roots $\qquad \Delta_+ = \Delta_+^{re} \cup \Delta_+^{im} \qquad (W\Delta_+^{re} = \Delta_+^{re} \cup \Delta_-^{re})$

Δ_+^{im}: positive imaginary roots $(k\Delta_+^{im} \subset \Delta_+^{im} = W\Delta_+^{im}, k = 2, 3, \ldots)$

$\boxed{\mathbf{m} \leftrightarrow \alpha_{\mathbf{m}} = (\operatorname{ord}\mathbf{m})\alpha_0 + \sum_{j\ge 0,\, k\ge 1} \sum_{\nu > k} m_{j,\nu}\alpha_{j,k} \text{ (Crawley-Boevey)}}$

Fact. $\operatorname{idx}\mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}})$

$$\Delta_+^{im} = \{kw\alpha_{\mathbf{m}}\,;\, w \in W,\ k = 1, 2, \ldots\ \ \mathbf{m} : \text{basic}\}$$

Thm. $\{\mathbf{m}$: realizable$\} \leftrightarrow \{k\alpha;\ \alpha \in \Delta_+\ \ \operatorname{supp}\alpha \ni \alpha_0,\ k = 1, 2, \ldots\}$

Suppose $\mathbf{m}$ is realizable.

★ $\mathbf{m}$: irreducibly realizable $\Leftrightarrow$ $\mathbf{m}$ is indivisible or $\operatorname{idx}\mathbf{m} < 0$

★ $\exists\ P_{\mathbf{m}}$: a universal model with (GRS) $\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$

★ $P_{\mathbf{m}}$ is of the normal form with the coefficients $a_\nu(x) \in \mathbb{C}[x, \lambda_{j,\nu}, g_i]$

★ $\forall\lambda_{j,\nu}$ under (FC), $\forall P$ with $\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$ are $P_{\mathbf{m}}$

★ $g_1, \ldots, g_N$: accessory parameters $\quad N = \begin{cases} 0 & (\operatorname{idx}\mathbf{m} > 0) \\ \gcd\mathbf{m} & (\operatorname{idx}\mathbf{m} = 0) \\ 1 - \frac{1}{2}\operatorname{idx}\mathbf{m} & (\operatorname{idx}\mathbf{m} < 0) \end{cases}$

$$\frac{\partial^2 P_{\mathbf{m}}}{\partial^2 g_i} = 0, \quad \operatorname{Top}(P_{\mathbf{m}}) = x^{L_i}\partial^{K_i}\operatorname{Top}\left(\frac{\partial P_{\mathbf{m}}}{\partial g_i}\right)$$

$\{(L_i, K_i)\,;\, i = 1, \ldots, N\}$ are explicitly given

$$Q = (c_k x_k + \cdots + c_0)\partial^m + a_{m-1}(x)\partial^{m-1} + \cdots + a_0(x),\ c_k \neq 0$$

$$\Rightarrow \operatorname{Top} Q = c_k x^k \partial^m$$

Def. $\mathbf{m}$ is rigid $\overset{\text{def}}{\Leftrightarrow}$ irreducibly realizable and $\operatorname{idx}\mathbf{m} = 2 \quad (\Rightarrow N = 0)$

(corresponds to $\alpha \in \Delta_+^{re}$ with $\operatorname{supp}\alpha \ni \alpha_0$)

Rigid tuples : 9 (ord ≤ 4), 306 (ord $= 10$), 19286 (ord $= 20$)

$\mathrm{ord} = 2$ $\quad 11, 11, 11$ (${}_2F_1$; Gauss)

$\mathrm{ord} = 3$ $\quad 111, 111, 21$ (${}_3F_2$) $\qquad 21, 21, 21, 21$(Pochhammer)

$\mathrm{ord} = 4$ $\quad 1^4, 1^4, 31$ (${}_4F_3$) $\qquad 1^4, 211, 22$ (Even family) $\quad 211, 211, 211$

$31, 31, 31, 31, 31$ (Pochhammer) $\qquad 211, 22, 31, 31 \qquad 22, 22, 22, 31$

Remark. The existence of $P_{\mathbf{m}}$ for fixed rigid $\mathbf{m}$ and generic $\{\lambda_{j,\nu}\}$ was an open problem by N. Katz (Rigid Local Systems, 1995).

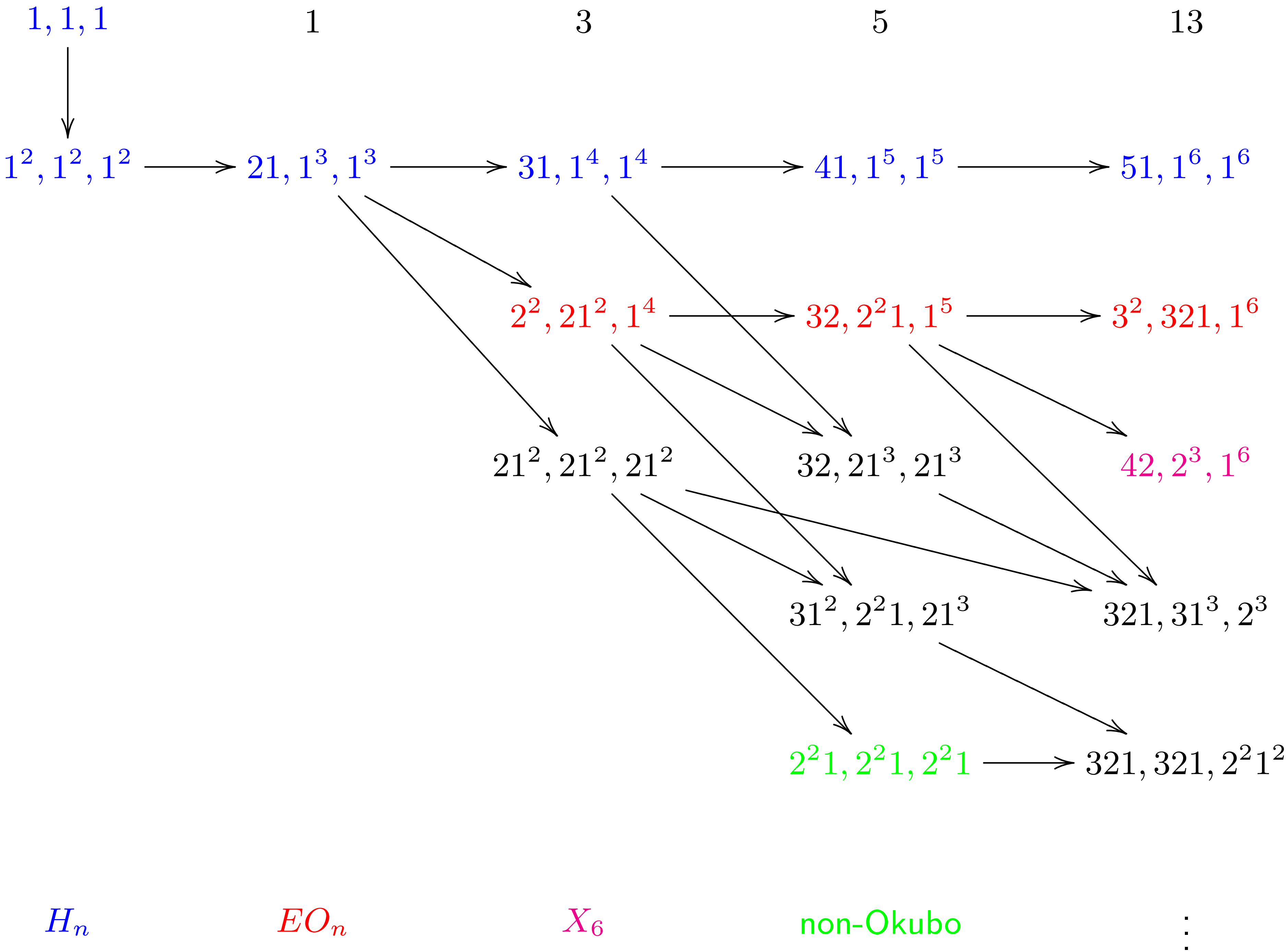
Reduction by "fractional calculus" $\Leftarrow W$ (Katz's middle convolution)

$\mathbf{m} \rightsquigarrow$ trivial ($\Leftarrow$ $\mathbf{m}$: rigid) or

fundamental $:= \{k\alpha\,;\mathbf{m}$: basic, $k = 1, 2, \ldots \ (k = 1 \Leftarrow \operatorname{idx}\mathbf{m} = 0)\}$

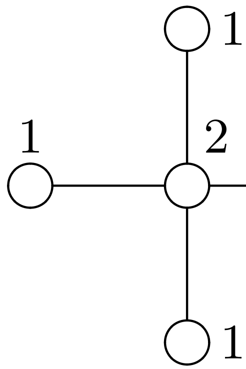
$\operatorname{idx}\mathbf{m} = 0 \ \rightarrow \ \tilde{D}_4$ ($\rightarrow$ Painlevé VI), $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (4 types)

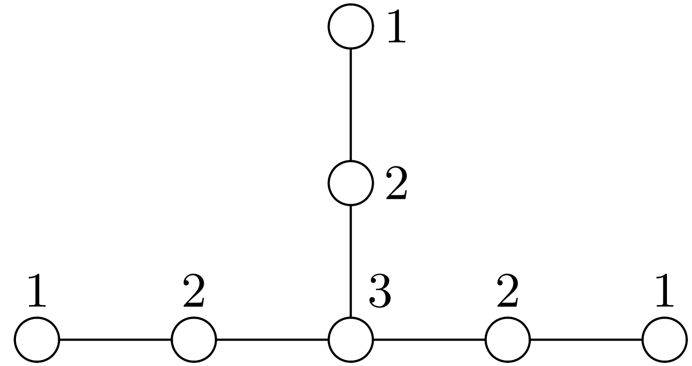
$\operatorname{idx}\mathbf{m} = -2 \ \rightarrow$ 13 types, etc...

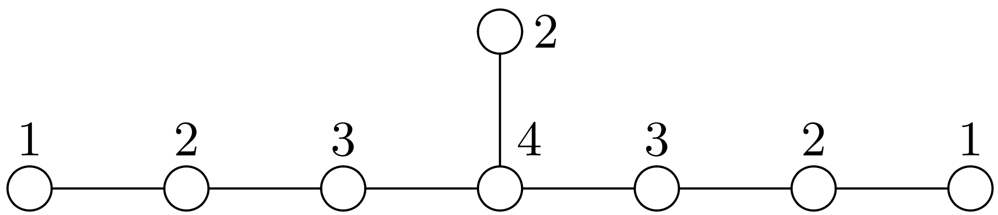

1, 1, 1
1
3
5
13
1^2, 1^2, 1^2
21, 1^3, 1^3
31, 1^4, 1^4
41, 1^5, 1^5
51, 1^6, 1^6
2^2, 21^2, 1^4
32, 2^2 1, 1^5
3^2, 321, 1^6
21^2, 21^2, 21^2
32, 21^3, 21^3
42, 2^3, 1^6
31^2, 2^2 1, 21^3
321, 31^3, 2^3
2^2 1, 2^2 1, 2^2 1
321, 321, 2^2 1^2
H_n
EO_n
X_6
non-Okubo
⋮

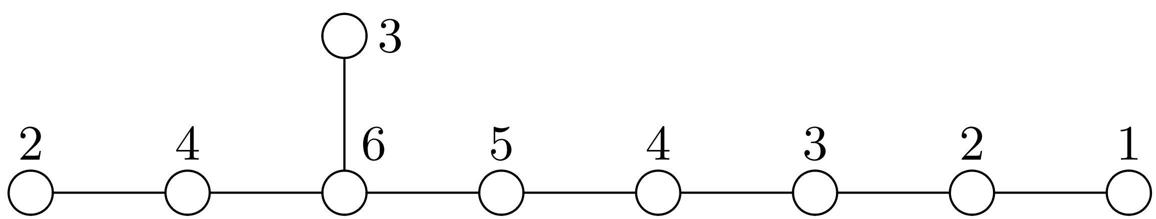
Basic tuples: $\mathrm{idx} = 0$ ($\Rightarrow$ affine)

Heun :

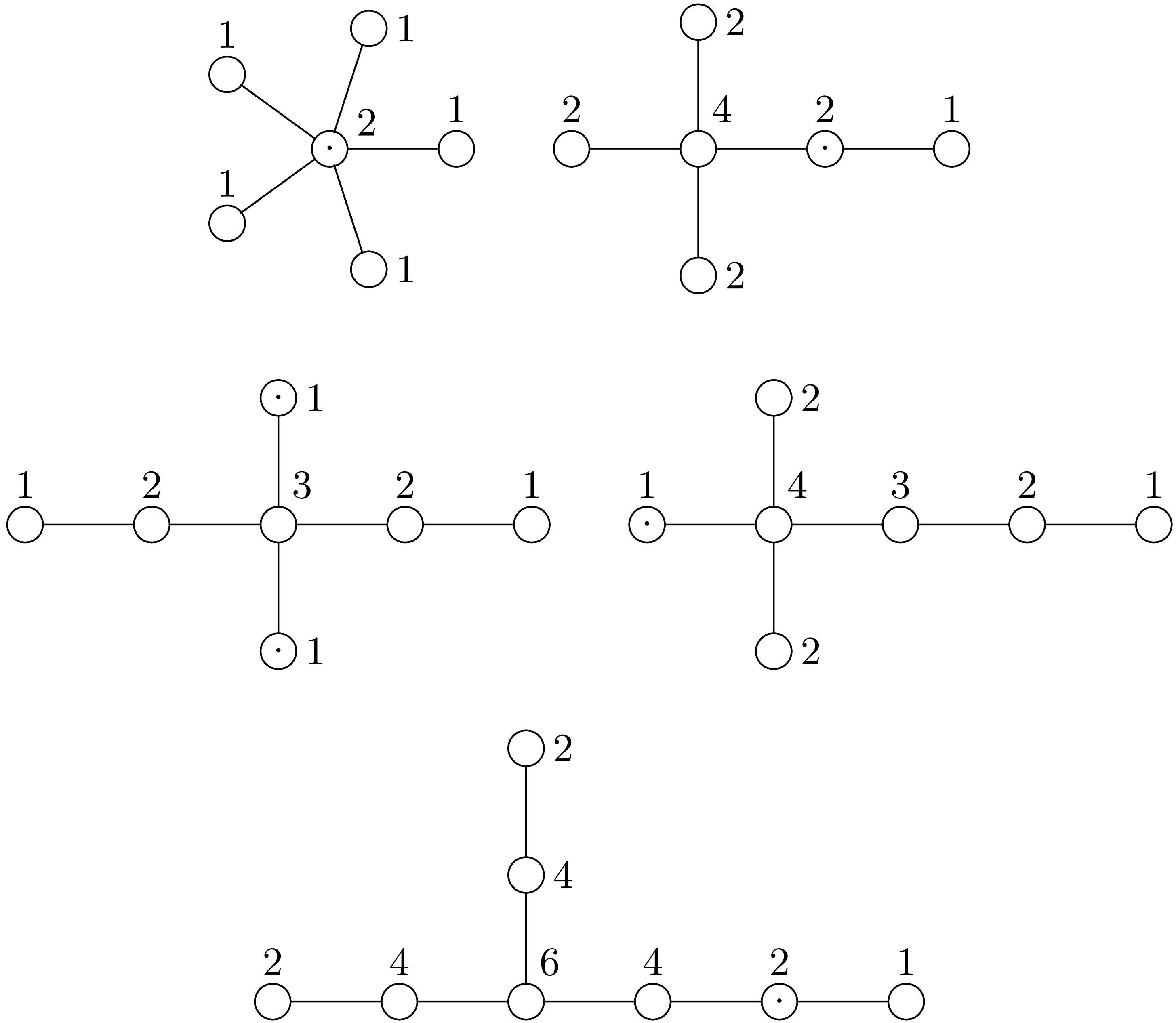


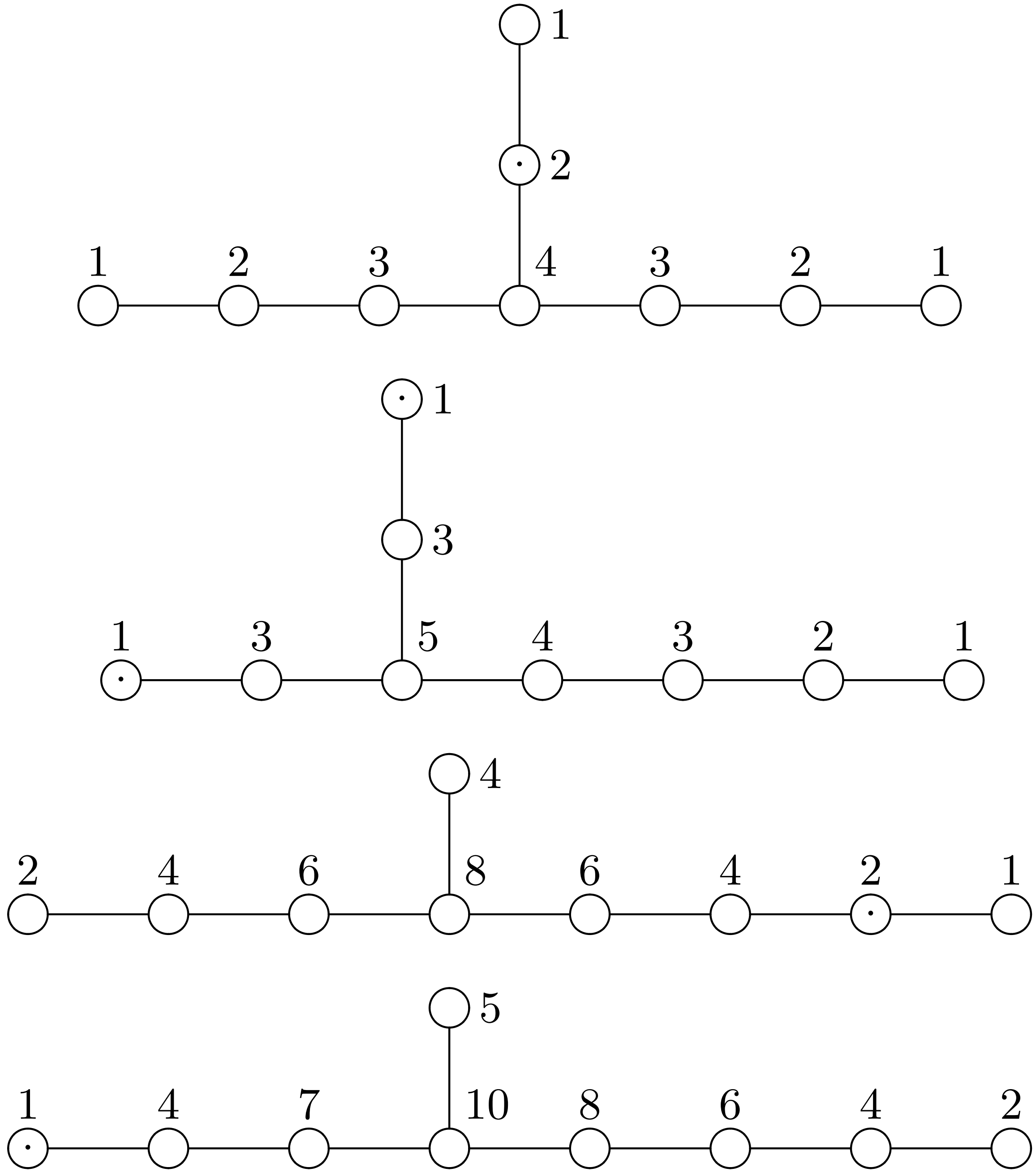




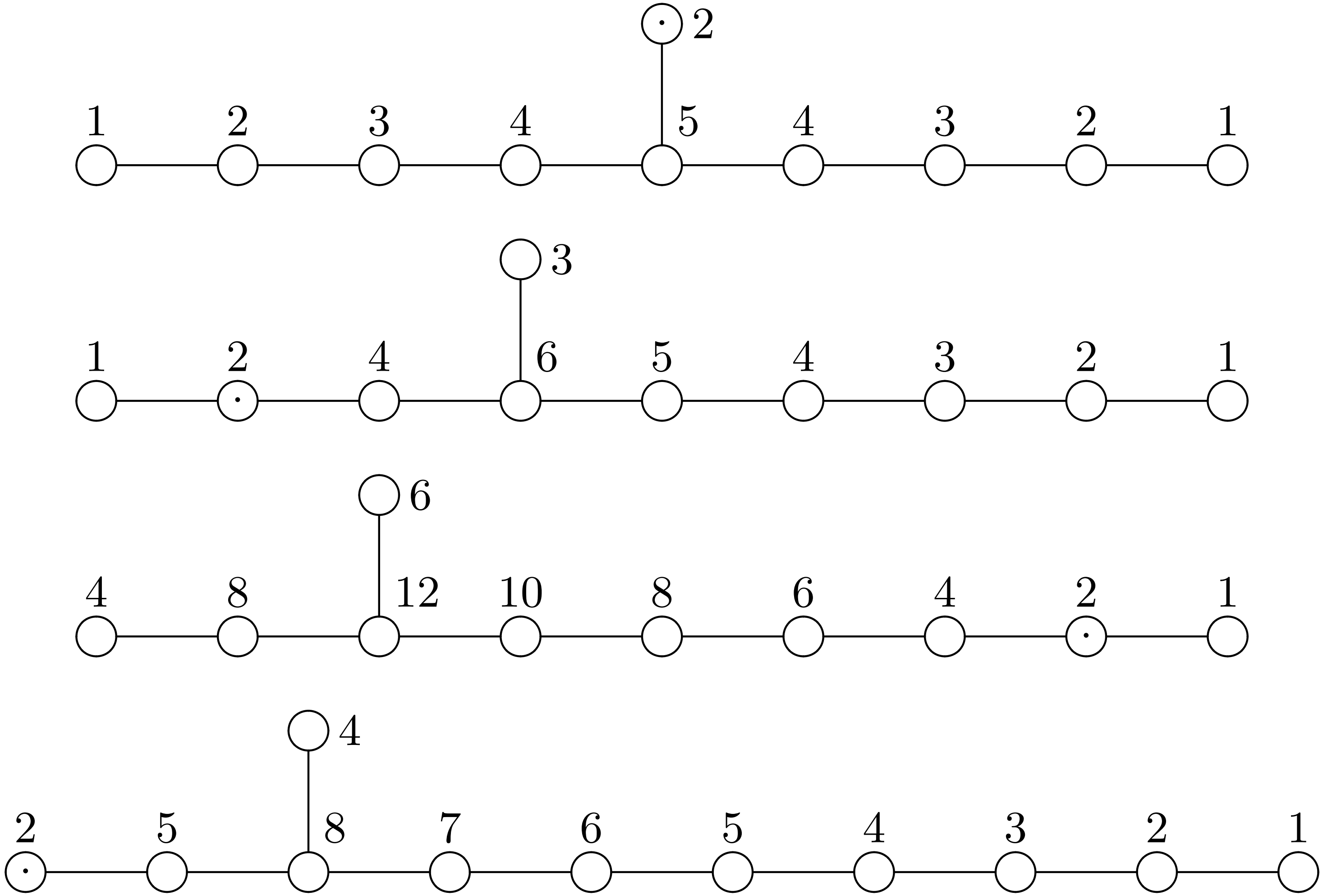


Basic tuples: $\mathrm{idx} = -2$ ($\Rightarrow$ Lorentzian)





1
2
1 2 3 4 3 2 1
1
3
1 3 5 4 3 2 1
4
2 4 6 8 6 4 2 1
5
1 4 7 10 8 6 4 2


2
1 2 3 4 5 4 3 2 1
3
1 2 4 6 5 4 3 2 1
6
4 8 12 10 8 6 4 2 1
4
2 5 8 7 6 5 4 3 2 1

§ Fractional calculus of Weyl algebra

Unified and computable interpretation (⇒ a computer program) of

Construction of equations

Integral representation of solutions

Congruences

Series expansion of solutions

Contiguity relations

Monodromy

Connection problem

Several variables (PDE)

$$W[x] := \langle x, \partial, \xi \rangle \otimes \mathbb{C}(\xi) \subset \overline{W}[x] := W[x] \otimes \mathbb{C}(x, \xi)$$
$$\simeq \overline{W}_L[x] := W[x] \otimes \mathbb{C}(\partial, \xi)$$

$\mathrm{R} : \overline{W}[x],\ \overline{W}_L[x] \to W[x]$ (reduced representative)

$\mathrm{L} : \partial_j \mapsto x_j,\ x_j \mapsto -\partial_j$

$$\mathrm{Ad}(f) \in \mathrm{Aut}(\overline{W}[x]),\ \partial_i \mapsto f(x,\xi) \circ \partial_i \circ f(x,\xi)^{-1} = \partial_i - \tfrac{f_i}{f}, \quad h_i = \tfrac{f_i}{f} \in \mathbb{C}(x,\xi)$$

$$\overline{\Delta}_+ := \{k\alpha\,;\, k=1,2,\ldots,\ \alpha\in\Delta_+,\ \operatorname{supp}\alpha \ni \alpha_0\}$$

$$\begin{array}{ccc}
\{P_{\mathbf{m}} : \text{Fuchsian differential operators}\} & \leftrightarrow & \overline{\Delta}_+ = \{\alpha_{\mathbf{m}}\} \\
\Big\downarrow \begin{array}{l}\text{Fractional operations} \\ \mathrm{RAd}(\partial^{-\mu})\circ\mathrm{RAd}(\prod_j (x-c_j)^{\lambda_j})\end{array} & & \Big\downarrow W\text{-action} \\
\{P_{\mathbf{m}} : \text{Fuchsian differential operators}\} & \leftrightarrow & \overline{\Delta}_+ = \{\alpha_{\mathbf{m}}\}
\end{array}$$

$$\mathrm{RAd}(\partial^{-\mu}) := \mathrm{L}^{-1}\circ\mathrm{R}\circ\mathrm{Ad}(x^{\mu})\circ\mathrm{L},\quad \mathrm{RAd}(f(x)) := \mathrm{R}\circ\mathrm{Ad}(f(x))$$

"W-action" for operators, series expansions and integral representations of solutions, contiguity relations, connection coefficients, monodromies,... are concretely determined.

Remark. On Fuchsian systems of Schlesinger canonical form

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x-c_j}u$$

the W-action is given by Katz + Detteweiler-Reiter + Crawley-Boevey.

Example: Jordan-Pochhammer Eq. ($p = 2 \Rightarrow$ Gauss)

$$p-11, p-11, \ldots, p-11 : \ (p+1)\text{-tuple of partitions of } p$$

$$P := \mathrm{RAd}\big(\partial^{-\mu}\big) \circ \mathrm{RAd}\Big(x^{\lambda_0} \prod_{j=1}^{p-1} (1 - c_j x)^{\lambda_j}\Big) \partial$$

$$= \mathrm{RAd}(\partial^{-\mu}) \circ \mathrm{R}\Big(\partial - \frac{\lambda_0}{x} + \sum_{j=2}^{p-1} \frac{c_j \lambda_j}{1 - c_j x}\Big)$$

$$= \partial^{-\mu+p-1}\Big(p_0(x)\partial + q(x)\Big)\partial^{\mu} = \sum_{k=0}^{p} p_k(x)\partial^{p-k}$$

$$p_0(x) = x \prod_{j=2}^{p-1} (1 - c_j x) \qquad q(x) = p_0(x)\Big(-\frac{\lambda_0}{x} + \sum_{j=2}^{p-1} \frac{c_j \lambda_j}{1 - c_j x}\Big)$$

$$p_k(x) = \binom{-\mu + p - 1}{k} p_0^{(k)}(x) + \binom{-\mu + p - 1}{k - 1} q^{(k-1)}(x)$$

$$u(x)=\frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\lambda_0+1)\Gamma(\mu)}\int_0^x\Bigl(t^{\lambda_0}\prod_{j=2}^{p-1}(1-c_jt)^{\lambda_j}\Bigr)(x-t)^{\mu-1}dt$$

$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_{p-1}=0}^{\infty}\frac{(\lambda_0+1)_{m_1+\cdots+m_{p-1}}(-\lambda_1)_{m_1}\cdots(-\lambda_{p-1})_{m_{p-1}}}{(\lambda_0+\mu+1)_{m_1+\cdots+m_{p-1}}m_1!\cdots m_{p-1}!}$$

$$c_2^{m_2}\cdots c_{p-1}^{m_{p-1}}x^{\lambda_0+\mu+m_1+\cdots+m_{p-1}}$$

$$P\left\{\begin{matrix} x=0 & 1=\frac{1}{c_1} & \cdots & \frac{1}{c_{p-1}} & \infty \\ [0]_{(p-1)} & {\color{blue}[0]}_{(p-1)} & \cdots & [0]_{(p-1)} & [1-\mu]_{(p-1)} \\ {\color{red}\lambda_0+\mu} & {\color{blue}\lambda_1+\mu} & \cdots & \lambda_{p-1}+\mu & -\lambda_1-\cdots-\lambda_{p-1}-\mu \end{matrix}\right\}$$

$$c({\color{red}\lambda_0+\mu}\rightsquigarrow{\color{blue}\lambda_1+\mu})=\frac{\Gamma(\lambda_0+\mu+1)\Gamma(-\lambda_1-\mu)}{\Gamma(\lambda_0+1)\Gamma(-\lambda_1)}\prod_{j=2}^{p-1}(1-c_j)^{\lambda_j}$$

$$c({\color{red}\lambda_0+\mu}\rightsquigarrow{\color{blue}0})=\frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\mu)\Gamma(\lambda_0+1)}\int_0^1 t^{\lambda_0}(1-t)^{\lambda_1+\mu-1}\prod_{j=2}^{p-1}(1-c_jt)^{\lambda_j}dt$$

Example: ${}_nF_{n-1}\xrightarrow{\mathrm{RAd}(\partial^{\mu_n-1})\circ\mathrm{RAd}(x^{\lambda_n})}{}_{n+1}F_n:\sum_{k\geq 0}\frac{(\alpha_1)_k\cdots(\alpha_{n+1})_k}{(\beta_1)_k\cdots(\beta_n)_k}\frac{x^k}{k!}$

Versal Pochhammer operator

$$p_0(x) = \prod_{j=1}^{p}(1 - c_j x), \quad q(x) = \sum_{k=1}^{p} \lambda_k x^{k-1} \prod_{j=k+1}^{p} (1 - c_j x)$$

$$P\left\{\begin{matrix} x = \frac{1}{c_j} \ (j = 1, \dots, p) & \infty \\ [0]_{(p-1)} & [1-\mu]_{(p-1)} \\ \displaystyle\sum_{k=j}^{p} \frac{\lambda_k}{c_j \prod_{\substack{1 \le \nu \le k \\ \nu \ne j}} (c_j - c_\nu)} + \mu & \displaystyle\sum_{k=1}^{p} \frac{(-1)^k \lambda_k}{c_1 \dots c_k} - \mu \end{matrix}\right\}$$

$$u_C(x) = \int_C \left(\exp \int_0^t \sum_{j=1}^{p} \frac{-\lambda_j s^{j-1}}{\prod_{1 \le \nu \le j}(1 - c_\nu s)} ds\right)(x - t)^{\mu-1} dt$$

$p = 2 \Rightarrow$ Unifying Gauss + Kummer + Hermite-Weber

$$c_1 = \cdots = c_p = 0 \Rightarrow u_C(x) = \int_{\infty}^{x} \exp\Big(-\sum_{j=1}^{p} \frac{\lambda_j t^j}{j!}\Big)(x - t)^{\mu-1} dt$$

Thm. $\mathbf{m}$: rigid monotone with $m_{0,n_0} = m_{1,n_1} = 1$, $\frac{1}{c_0} = 0$, $\frac{1}{c_1} = 1$

$$c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) = \frac{\displaystyle\prod_{\nu=1}^{n_0-1} \Gamma\bigl(\lambda_{0,n_0} - \lambda_{0,\nu} + 1\bigr) \cdot \prod_{\nu=1}^{n_1-1} \Gamma\bigl(\lambda_{1,\nu} - \lambda_{1,n_1}\bigr)}{\displaystyle\prod_{\substack{\mathbf{m}' \oplus \mathbf{m}'' = \mathbf{m} \\ m'_{0,n_0} = m''_{1,n_1} = 1}} \Gamma\bigl(|\{\lambda_{\mathbf{m}'}\}|\bigr) \cdot \prod_{j=2}^{p-1} (1 - c_j)^{L_j}}$$

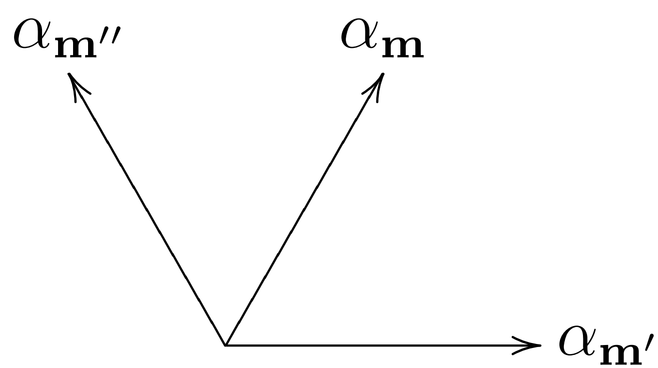
$$|\{\lambda_{\mathbf{m}'}\}| = \sum m'_{j,\nu}\lambda_{j,\nu} - \operatorname{ord}\mathbf{m}' + 1$$

$\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}'' \overset{\text{def}}{\iff}$ $\mathbf{m}$, $\mathbf{m}'$ realizable and $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$

$$\text{Gauss}: \begin{Bmatrix} x = \frac{1}{c_0} = 0 & \frac{1}{c_1} = 1 & \infty \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \overline{\lambda_{0,2}} & \underline{\lambda_{1,2}} & \lambda_{2,2} \end{Bmatrix} = \begin{Bmatrix} x = 0 & 1 & \infty \\ 1 - \gamma & \gamma - \alpha - \beta & \alpha \\ 0 & 0 & \beta \end{Bmatrix} \qquad \begin{aligned} &1\bar{1}, 1\underline{1}, 11 \\ &= 0\bar{1}, 10, 10 \\ &\oplus 10, 0\bar{1}, 01 \end{aligned}$$

$$c(\lambda_{0,2} \rightsquigarrow \lambda_{1,2}) = \frac{\Gamma(\lambda_{0,2} - \lambda_{0,1} + 1)\,\Gamma(\lambda_{1,1} - \lambda_{1,2})}{\Gamma(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,1})\,\Gamma(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,2})} \qquad \begin{matrix} \updownarrow \\ \Leftrightarrow \end{matrix}$$

$$P\left\{\begin{matrix} x=\frac{1}{c_0}=0 & \frac{1}{c_1}=1 & \cdots & \frac{1}{c_p}=\infty \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}$$



$\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$: rigid $\iff$ $\alpha_{\mathbf{m}}=\alpha_{\mathbf{m}'}+\alpha_{\mathbf{m}''}$: positive real roots

$\mathrm{ord}\le 40,\ p=2 \Rightarrow$ 4,111,704 independent cases by a computer

non-rigid case : $c(\lambda_{0,n_0}\rightsquigarrow\lambda_{1,n_1})=$

a gamma factor $\times$ a connection coefficient of a fundamental case

ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$
2	1	1	7	20	44	12	421	857	17	3276	6128
3	1	2	8	45	96	13	588	1177	18	5186	9790
4	3	6	9	74	157	14	1004	2032	19	6954	12595
5	5	11	10	142	306	15	1481	2841	20	10517	19269
6	13	28	11	212	441	16	2388	4644	21	14040	24748

2:11,11,11
3:111,111,21
3:21,21,21,21
4:211,211,211
4:1111,211,22
4:1111,1111,31
4:211,22,31,31
4:22,22,22,31
4:31,31,31,31,31
5:2111,221,311
5:2111,2111,32
5:221,221,221
5:11111,221,32
5:11111,11111,41
5:221,221,41,41
5:221,32,32,41
5:311,311,32,41
5:32,32,32,32
5:32,32,41,41,41
5:41,41,41,41,41,41
6:3111,3111,321
6:2211,2211,411
6:2211,321,321
6:222,3111,321
6:21111,222,411
6:21111,2211,42
6:21111,3111,33
6:2211,2211,33
6:222,222,321
6:21111,222,33
6:111111,321,33
6:111111,222,42
6:111111,111111,51
6:2211,222,51,51
6:2211,33,42,51
6:222,33,33,51
6:222,33,411,51
6:3111,33,411,51
6:321,321,42,51
6:321,42,42,42
6:33,33,33,42
6:33,33,411,42
6:33,411,411,42
6:411,411,411,42
6:33,42,42,51,51
6:321,33,51,51,51
6:411,42,42,51,51
6:51,51,51,51,51,51,51

Δ : {roots of the Kac-Moody root system (Π, W)} $= \Delta^{re} \cup \Delta^{im}$

Δ^{re} : {positive roots} $= W\Pi$

Δ^{im}_{+} : {positive imaginary roots} $= WK \subset Q_+ := \sum_{\alpha\in\Pi} \mathbb{Z}_{>0}\alpha$

$K := \{\beta \in Q_+\,;\ \mathrm{supp}\,\beta$ is connected and $(\beta,\alpha) \le 0 \quad (\forall\alpha\in\Pi)\}$

$\Delta^{re}_{+} = \Delta^{re} \cap Q_+$, $\Delta_+ := \Delta^{re}_{+} \cup \Delta^{im}_{+}$ and $\Delta = \Delta_+ \cup -\Delta_+$

$\mathcal{P}$ (spectral type)	Kac-Moody root system
$\mathrm{idx}(\mathbf{m},\mathbf{m}')$	$(\alpha_{\mathbf{m}}\vert\alpha_{\mathbf{m}'})$
middle convolution ∂_ℓ	the reflection with respect to α_ℓ
rigid	$\{\alpha \in \Delta^{re}_{+}\,;\ \mathrm{supp}\,\alpha \ni \alpha_0\}$
indivisible non-rigid realizable	indivisible roots $\in \Delta^{im}_{+}$
fundamental	$\alpha \in K$ and $((\alpha\vert\alpha) = 0 \ \Rightarrow\ \alpha$: indivisible)

idx of rigidity	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20
#fundamental	4	13	36	67	94	162	243	305	420	582	720
# triplets	3	9	24	44	60	97	144	163	223	303	342
# tuples of 4	1	3	9	17	25	45	68	95	128	173	239

Prop. $\mathbf{m}$: fund. $\Rightarrow$ $\mathrm{ord}\,\mathbf{m} \le 6 - 3\cdot\mathrm{idx}\,\mathbf{m}$ $(\le 2 - \mathrm{idx}\,\mathbf{m} \Leftarrow \mathbf{m} \notin \mathcal{P}_3)$

Thank you! End!

· *Classification of Fuchsian systems and their connection problem*, arXiv:0811.2916, 29 pages, 2008

· *Fractional calculus of Weyl algebra and Fuchsian differential equations*, preprint, 94 pages, 2009