Integral transformations of hypergeometric functions with several variables

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Abstract. As a generalization of Riemann-Liouville integral, we introduce integral transformations of convergent power series which can be applied to hypergeometric functions with several variables.

1. Introduction

Suppose a function $\phi(x)$ satisfies a linear ordinary differential equation on $\mathbb{P}^1$. Then the Riemann-Liouville integral (1) of $\phi(x)$ induces a middle convolution of the differential equation defined by Katz [**Ka**]. The multiplication of $\phi(x)$ by a simple function $(x-c)^\lambda$ induces an addition of the differential equation which is also important. For example, any rigid irreducible linear Fuchsian differential equation is constructed by successive applications of middle convolutions and additions from the trivial equation $u'=0$. Hence we have an integral representation of its solution, which is shown first by Katz [**Ka**] in the case of Fuchsian systems of the first order and by the author [**O1**] in the case of single differential equations of higher orders. Here the equation is called rigid if it is free from accessory parameters, namely, the equation is globally determined by the local structure at the singular points. Applying these transformations to linear ordinary differential equations on $\mathbb{P}^1$, we study many fundamental problems on their solutions in [**O1**].

The rigid Fuchsian ordinary differential equation on $\mathbb{P}^1$ can be extended to a Knizhnik-Zamolodchikov type equation (KZ equation in short, cf. [**KZ**]) regarding the singular points as new variables. Haraoka [**Ha**] shows this by extending middle convolutions on KZ equations and its generalization for equations with irregular singularities is given by the author [**O4, O5**]. Then these transformations are also useful to hypergeometric functions with several variables including Appell's hypergeometric functions (cf. [**O3**]).

These transformations do not give an integral representation of Appell's hypergeometric series F_4 but K. Aomoto gives an integral representation of F_4, which is written in [**O1**, §13.10.2]. We define integral transformations on the space of convergent power series of several variables in §2 and study hypergeometric functions with several variables, which extend a brief study of Appell's hypergeometric functions in [**O1**, §3.10]. The transformations are invertible and they are generalizations of Riemann-Liouville integrals of functions with a single variable. On the space of hypergeometric functions with several variables we have important transformations such as the multiplications of suitable functions and coordinate

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2010 *Mathematics Subject Classification.* Primary 34M35; Secondary 34A30, 33C70.

Key words and phrases. integral transformation, hypergeometric function.

This work was supported by Grant-in-Aid for Scientific Researches (C), No. 18K03341, Japan Society of Promotion of Science.

transformations. Note that a holonomic Fuchsian differential equation of several variables has solutions of convergent power series times simple functions $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ at its normally crossing singular points (cf. [**KO**]). We study a combination of the integral transformations and multiplications by these simple functions.

In §3 we show that the transformations defined in §2 give integral representations of Appell's or Lauricella's hypergeometric series and certain Horn's hypergeometric series with irregular singularities.

In §4 we show that our study gives a result related to the connection problem on the solutions, which will be discussed in another paper and related to the study by Matsubara [**Ma**, §3].

In §5 a combination of the integral transformations with coordinate transformations defined by products of powers of coordinate functions parametrized by $GL(n,\mathbb{Z})$. The transformations given in §5 are related to A-hypergeometric series introduced by Gel'fand, Kapranov and Zelevinsky [**GKZ**]. The transformation

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n}x^my^n \mapsto \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n}\frac{(\alpha)_{p_1m+q_1n}(\beta)_{p_2m+q_2n}}{(\gamma)_{(p_1+p_2)_m+(q_1+q_2)_n}}x^my^n$$

of convergent power series is an example. Here p_1, p_2, q_1 and q_2 are non-negative integers with $\left(\begin{smallmatrix} p_1 & p_2\\ q_1 & q_2\end{smallmatrix}\right)\in GL(2,\mathbb{Z})$ and we put $(a)_k = a(a+1)\cdots(a+k-1)$.

In §6 we study the transformation of differential equations corresponding to the transformations of their solutions.

In §7 we study the transformation given in §5 which keeps the space of KZ equations

$$\mathcal{M}:\begin{cases}\dfrac{\partial u}{\partial x} = \dfrac{A_{01}}{x-y}u + \dfrac{A_{02}}{x-1}u + \dfrac{A_{03}}{x}u,\\ \dfrac{\partial u}{\partial y} = \dfrac{A_{01}}{y-x}u + \dfrac{A_{12}}{y-1}u + \dfrac{A_{13}}{y}u\end{cases}$$

and give the induced transformations of the residue matrices $A_{i,j}$ defining the equations. The transformation is reduced to a coordinate transformation corresponding to the coordinate symmetries described in [**O3**, §6] and a middle convolution of the KZ equation. Hence we apply the result in [**Ha, O3**] to them. The hypergeometric series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_m\prod_{j=1}^{q}(\beta_j)_n\prod_{k=1}^{r}(\gamma_k)_{m+n}}{\prod_{i=1}^{p}(1-\alpha'_i)_m\prod_{j=1}^{q}(1-\beta'_j)_n\prod_{k=1}^{r}(1-\gamma'_k)_{m+n}}x^my^n$$

with $\alpha'_1=\beta'_1=0$ is a typical example satisfying a KZ equation, which is a generalization of Appell's F_1.

To the KZ equation we show Theorem 7.1 which gives an interesting correspondence between *simple* solutions along a line (cf. Definition 7.3) and *simple* solutions at a singular point where three singular lines meet.

In §8 we restrict our transformations in §7 to certain ordinary differential equations of Shlesinger canonical form. The transformation may be interesting since it may change the index of rigidity defined by [**Ka**].

Several applications of the results in this paper will be given in other papers.

2. Integral transformations

The Riemann-Liouville transform $I_c^\mu\phi$ of a function $\phi(x)$ is defined by

$$(1)\qquad (I_{x,c}^\mu\phi)(x) = (I_c^\mu\phi)(x) := \frac{1}{\Gamma(\mu)}\int_c^x \phi(t)(x-t)^{\mu-1}\mathrm{d}t.$$

Here c is usually a singular point of an integrable function $\phi(x)$. Since

$$\int_0^x \phi(t)(x-t)^{\mu-1}\mathrm{d}t = \int_0^1 \phi(xs)(x-xs)^{\mu-1}x\mathrm{d}s \qquad (t=xs)$$
$$= x^\mu \int_0^1 \phi(sx)(1-s)^{\mu-1}\mathrm{d}s,$$

the transformation

$$(K_x^\mu\phi)(x) := \frac{1}{\Gamma(\mu)}\int_0^1 \phi(tx)(1-t)^{\mu-1}\mathrm{d}t \tag{2}$$

of a function $\phi(x)$ satisfies

$$K_x^\mu = x^{-\mu}I_0^\mu, \tag{3}$$

$$K_x^\mu x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\mu+1)}x^\alpha. \tag{4}$$

Definition 2.1. We extend the integral transform K_x^μ to a function $\phi(x)$ of several variables $x=(x_1,\ldots,x_n)$ by

$$K_x^\mu\phi(x) := \frac{1}{\Gamma(\mu)}\int_{\substack{t_1>0,\ldots,t_n>0\\ t_1+\cdots+t_n<1}} (1-t_1-\cdots-t_n)^{\mu-1}\phi(t_1x_1,\ldots,t_nx_n)\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

Note that

$$\int_0^{1-s} t^\alpha(1-s-t)^{\mu-1}\mathrm{d}t = \int_0^{1-s} t^\alpha(1-s)^{\mu-1}(1-\tfrac{t}{1-s})^{\mu-1}\mathrm{d}t$$
$$= (1-s)^{\alpha+\mu}\int_0^1 t^\alpha(1-t)^{\mu-1}\mathrm{d}t$$
$$= \frac{\Gamma(\alpha+1)\Gamma(\mu)}{\Gamma(\alpha+\mu+1)}(1-s)^{\alpha+\mu}$$

and hence

$$\int_{\substack{t_1>0,\ldots,t_n>0\\ t_1+\cdots+t_n<1}} t^{\alpha_1}\cdots t^{\alpha_n}(1-t_1-\cdots-t_n)^{\mu-1}\mathrm{d}t$$
$$= \int_0^1 t_1^{\alpha_1}\mathrm{d}t_1\int_0^{1-t_1} t_2^{\alpha_2}\mathrm{d}t_2\cdots\int_0^{1-t_1-\cdots-t_{n-1}} t_n^{\alpha_n}(1-t_1-\cdots-t_n)^{\mu-1}\mathrm{d}t_n$$
$$= \frac{\Gamma(\mu)\Gamma(\alpha_n+1)}{\Gamma(\alpha_n+\mu+1)}\int_0^1 t_1^{\alpha_1}\mathrm{d}t_1\cdots\int_0^{1-t_1-\cdots-t_{n-2}} t_{n-1}^{\alpha_{n-1}}(1-t_1-\cdots-t_{n-1})^{\alpha_n+\mu}\mathrm{d}t_{n-1}$$
$$= \frac{\Gamma(\mu)\Gamma(\alpha_n+1)}{\Gamma(\alpha_n+\mu+1)}\times\frac{\Gamma(\alpha_n+\mu+1)\Gamma(\alpha_{n-1}+1)}{\Gamma(\alpha_{n-1}+\alpha_n+\mu+2)}\times\cdots$$
$$\cdots\times\frac{\Gamma(\alpha_2+\cdots+\alpha_n+\mu+n-1)\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+\cdots+\alpha_n+\mu+n)}$$
$$= \frac{\Gamma(\mu)\Gamma(\alpha_1+1)\cdots\Gamma(\alpha_n+1)}{\Gamma(\alpha_1+\cdots+\alpha_n+\mu+n)}.$$

Therefore we have

$$K_x^\mu x^{\boldsymbol{\alpha}} = \frac{\Gamma(\boldsymbol{\alpha}+1)}{\Gamma(|\boldsymbol{\alpha}+1|+\mu)}x^{\boldsymbol{\alpha}} \tag{5}$$

and

$$\begin{aligned} &K_x^\mu\phi(x_1)x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}\\ &\quad= \frac{\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_2+\cdots+\alpha_n+\mu)}(I_0^{\alpha_2+\cdots+\alpha_n+\mu}\phi)(x_1)\cdot x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}\end{aligned} \tag{6}$$

Here and hereafter we use the notation

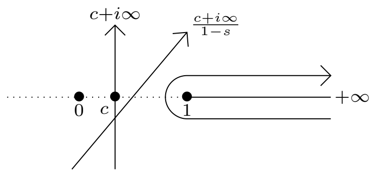
$$
\begin{aligned}
&\mathbb{N} = \{0, 1, 2, , \dots\},\\
&\mathbf{m} \ge 0 \ \Leftrightarrow\ m_1 \ge 0, \dots, m_n \ge 0,\\
&|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n,\ \boldsymbol{\alpha} + c = (\alpha_1 + c, \dots, \alpha_n + c),\ \mathbf{m}! = m_1! \cdots m_n!,\\
&x^{\boldsymbol{\alpha}} = \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ (c - \mathbf{x})^{\boldsymbol{\alpha}} = (c - x_1)^{\alpha_1} \cdots (c - x_n)^{\alpha_n},\\
&\Gamma(\boldsymbol{\alpha}) = \Gamma(\alpha_1) \cdots \Gamma(\alpha_n),\ (\boldsymbol{\alpha})_{\mathbf{m}} = \frac{\Gamma(\boldsymbol{\alpha} + \mathbf{m})}{\Gamma(\boldsymbol{\alpha})}
\end{aligned}
\tag{7}
$$

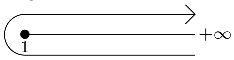
for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ and the variable $x = (x_1, \dots, x_n)$.

The arguments in this section are valid when $\operatorname{Re}\alpha_1 > 0, \dots, \operatorname{Re}\alpha_n > 0$ and $\operatorname{Re}\mu > 0$ but the right hand side of (5) is meromorphic for $\boldsymbol{\alpha}$ and μ and we define $K_x^\mu x^{\boldsymbol{\alpha}}$ by the analytic continuation with respect to these parameters.

We will define the inverse L_x^μ of K_x^μ. Suppose $0 \le \operatorname{Re} s < 1$ and $0 < c < 1 - \operatorname{Re} s$. Then

$$
\begin{aligned}
&\int_{c-i\infty}^{c+i\infty} t^{-\alpha}(1-s-t)^{-\tau}\tfrac{\mathrm{d}t}{t} = (1-s)^{-\tau}\int_{c-i\infty}^{c+i\infty} t^{-\alpha}(1-\tfrac{t}{1-s})^{-\tau}\tfrac{\mathrm{d}t}{t}\\
&\quad = (1-s)^{-\alpha-\tau}\int_{\frac{c-i\infty}{1-s}}^{\frac{c+i\infty}{1-s}} t^{-\alpha}(1-t)^{-\tau}\tfrac{\mathrm{d}t}{t}\\
&\quad = (1-s)^{-\alpha-\tau}\int_{c-i\infty}^{c+i\infty} t^{-\alpha}(1-t)^{-\tau}\tfrac{\mathrm{d}t}{t}\\
&\quad = (1-s)^{-\alpha-\tau}(-e^{-\tau\pi i} + e^{\tau\pi i})\int_1^\infty t^{-\alpha}(t-1)^{-\tau}\tfrac{\mathrm{d}t}{t}\\
&\quad = (1-s)^{-\alpha-\tau}\cdot 2i\sin\tau\pi\int_0^1 (\tfrac{1}{u})^{-\alpha}(\tfrac{1}{u}-1)^{-\tau}\tfrac{\mathrm{d}u}{u} \qquad (u = \tfrac{1}{t})\\
&\quad = \frac{2\pi i(1-s)^{-\alpha-\tau}}{\Gamma(\tau)\Gamma(1-\tau)}\int_0^1 u^{\alpha+\tau-1}(1-u)^{-\tau}\mathrm{d}u\\
&\quad = 2\pi i\frac{\Gamma(\alpha+\tau)}{\Gamma(\tau)\Gamma(\alpha+1)}(1-s)^{-\alpha-\tau}.
\end{aligned}
$$



Here the path of the integration of the above first line is $(-\infty, \infty) \ni s \mapsto c + is$ and we also use the path 

of the integration in the above.

Thus we have

$$
\begin{aligned}
&\int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty}\cdots\int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty} t^{-\boldsymbol{\alpha}}(1-t_1-\cdots-t_n)^{-\tau}\tfrac{\mathrm{d}t_1}{t_1}\cdots\tfrac{\mathrm{d}t_n}{t_n}\\
&= (2\pi i)^n\frac{\Gamma(\alpha_n+\tau)}{\Gamma(\tau)\Gamma(\alpha_n+1)}\frac{\Gamma(\alpha_{n-1}+\alpha_n+\tau)}{\Gamma(\alpha_n+\tau)\Gamma(\alpha_{n-1}+1)}\cdots\frac{\Gamma(|\boldsymbol{\alpha}|+\tau)}{\Gamma(\alpha_2+\cdots+\alpha_n+\tau)\Gamma(\alpha_1+1)}\\
&= (2\pi i)^n\frac{\Gamma(|\boldsymbol{\alpha}+1|+\tau-n)}{\Gamma(\boldsymbol{\alpha}+1)\Gamma(\tau)}.
\end{aligned}
$$

Definition 2.2. We define the transformation

$$
\begin{aligned}
(L_x^\mu\phi)(x) &:= \frac{\Gamma(\mu+n)}{(2\pi i)^n}\int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty}\cdots\int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty}\phi(\tfrac{x_1}{t_1},\dots,\tfrac{x_n}{t_n})(1-|\mathbf{t}|)^{-\mu-n}\tfrac{\mathrm{d}t_1}{t_1}\cdots\tfrac{\mathrm{d}t_n}{t_n}\\
&= \frac{\Gamma(\mu+n)}{(2\pi i)^n}\int_{\left|\frac{2s_1}{n+1}-1\right|=1}\cdots\int_{\left|\frac{2s_n}{n+1}-1\right|=1}\phi(s_1x_1,\dots,s_nx_n)\\
&\qquad\times(1-\tfrac{1}{s_1}-\cdots-\tfrac{1}{s_n})^{-\mu-n}\tfrac{\mathrm{d}s_1}{s_1}\cdots\tfrac{\mathrm{d}s_n}{s_n}.
\end{aligned}
$$

In the above, we mainly consider the case when $\phi(x) = x^{\boldsymbol{\lambda}}\varphi(x)$ with a convergent power series $\varphi(x)$. Then we have

$$L_x^\mu x^{\boldsymbol{\alpha}} = \frac{\Gamma(|\boldsymbol{\alpha}+1|+\mu)}{\Gamma(\boldsymbol{\alpha}+1)}x^{\boldsymbol{\alpha}}. \tag{8}$$

When $n=1$, we have

$$\begin{aligned}(L_x^\mu\phi)(x) &= \frac{\Gamma(\mu+1)}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}\phi(\tfrac{x}{t})(1-t)^{-\mu-1}\tfrac{\mathrm{d}t}{t}\\ &= \frac{\sin(\mu+1)\pi\cdot\Gamma(\mu+1)}{\pi}\int_1^\infty\phi(\tfrac{x}{t})(1-t)^{-\mu-1}\tfrac{\mathrm{d}t}{t}\\ &= \frac{1}{\Gamma(-\mu)}\int_0^1 s^\mu\phi(s)(x-s)^{-\mu-1}\mathrm{d}s\\ &= I_0^{-\mu}(x^\mu\phi).\end{aligned}$$

In general, we have

$$\begin{aligned}L_x^\mu(\phi(x_1)x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}) &= \frac{\Gamma(\alpha_2+\cdots+\alpha_n+\mu)}{\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}\\ &\times(I_{x_1,0}^{-\alpha_2-\cdots-\alpha_n-\mu}x_1^{\alpha_2+\cdots+\alpha_n+\mu}\phi(x_1))x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}.\end{aligned} \tag{9}$$

Definition 2.3. We define two transformations

$$K_x^{\mu,\boldsymbol{\lambda}} := x^{1-\boldsymbol{\lambda}}K_x^\mu x^{\boldsymbol{\lambda}-1} \quad\text{and}\quad L_x^{\mu,\boldsymbol{\lambda}} := x^{1-\boldsymbol{\lambda}}L_x^\mu x^{\boldsymbol{\lambda}-1} \tag{10}$$

which act on the ring $\mathcal{O}_0$ of convergent power series of $x=(x_1,\dots,x_n)$.

We have

$$K_x^{\mu,\boldsymbol{\lambda}}x^{\boldsymbol{\alpha}} = \frac{\Gamma(\boldsymbol{\lambda}+\boldsymbol{\alpha})}{\Gamma(|\boldsymbol{\lambda}+\boldsymbol{\alpha}|+\mu)}x^{\boldsymbol{\alpha}} \quad\text{and}\quad L_x^{\mu,\boldsymbol{\lambda}}x^{\boldsymbol{\alpha}} = \frac{\Gamma(|\boldsymbol{\lambda}+\boldsymbol{\alpha}|+\mu)}{\Gamma(\boldsymbol{\lambda}+\boldsymbol{\alpha})}x^{\boldsymbol{\alpha}}.$$

Theorem 2.1. *Putting*

$$u(x) = \sum_{\mathbf{m}\ge0}c_{\mathbf{m}}x^{\mathbf{m}} = \sum_{m_1=0}^\infty\cdots\sum_{m_n=0}^\infty c_{\mathbf{m}}x^{\mathbf{m}}\in\mathcal{O}_0 \quad (c_{\mathbf{m}}\in\mathbb{C}).$$

$$K_x^{\mu,\boldsymbol{\lambda}}u(x) = \sum_{\mathbf{m}}c_{\mathbf{m}}^Kx^{\mathbf{m}} \quad\textit{and}\quad L_x^{\mu,\boldsymbol{\lambda}}u(x) = \sum_{\mathbf{m}}c_{\mathbf{m}}^Lx^{\mathbf{m}} \quad (c_{\mathbf{m}}^K,\ c_{\mathbf{m}}^L\in\mathbb{C}),$$

we have

$$c_{\mathbf{m}}^K = \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(|\boldsymbol{\lambda}|+\mu)}\frac{(\boldsymbol{\lambda})_{\mathbf{m}}}{(|\boldsymbol{\lambda}|+\mu)_{|\mathbf{m}|}}c_{\mathbf{m}}, \tag{11}$$

$$c_{\mathbf{m}}^L = \frac{\Gamma(|\boldsymbol{\lambda}|+\mu)}{\Gamma(\boldsymbol{\lambda})}\frac{(|\boldsymbol{\lambda}|+\mu)_{|\mathbf{m}|}}{(\boldsymbol{\lambda})_{\mathbf{m}}}c_{\mathbf{m}}. \tag{12}$$

By the analytic continuation with respect to the parameters the transformations $K_x^{\mu,\boldsymbol{\lambda}}$ and $L_x^{\mu,\boldsymbol{\lambda}}$ are well-defined if

$$\lambda_\nu\notin\mathbb{Z}_{\le0} \quad (\nu=1,\dots,n) \tag{13}$$

and

$$|\boldsymbol{\lambda}|+\mu\notin\mathbb{Z}_{\le0}, \tag{14}$$

respectively. Namely, we may consider that $K_x^{\mu,\boldsymbol{\lambda}}$ and $L_x^{\mu,\boldsymbol{\lambda}}$ are defined by Theorem 2.1 by using (11) and (12). Hence if (13) and (14) are valid, $K_x^{\mu,\boldsymbol{\lambda}}$ and $L_x^{\mu,\boldsymbol{\lambda}}$ are bijective on $\mathcal{O}_0$ and the map $K_x^{\mu,\boldsymbol{\lambda}}\circ L_x^{\mu,\boldsymbol{\lambda}}$ is the identity map.

Proposition 2.4. *By the equations* (6) *and* (9) *we have*

$$K_x^{\mu,\boldsymbol{\lambda}}\phi(x_1) = \frac{\Gamma(\lambda_2)\cdots\Gamma(\lambda_n)}{\Gamma(\lambda_2+\cdots+\lambda_n+\mu)} x_1^{-|\boldsymbol{\lambda}|-\mu+1} I_{x_1,0}^{|\boldsymbol{\lambda}|-\lambda_1+\mu} x_1^{\lambda_1-1}\phi(x_1), \tag{15}$$

$$L_x^{\mu,\boldsymbol{\lambda}}\phi(x_1) = \frac{\Gamma(\lambda_2+\cdots+\lambda_n+\mu)}{\Gamma(\lambda_2)\cdots\Gamma(\lambda_n)} x_1^{1-\lambda_1} I_{x_1,0}^{-|\boldsymbol{\lambda}|+\lambda_1-\mu} x_1^{|\boldsymbol{\lambda}|+\mu-1}\phi(x_1). \tag{16}$$

3. Some hypergeometric functions

Under the notation (7) we note that

$$(1-|\mathbf{x}|)^{-\lambda} = \sum_{\mathbf{m}\ge 0} \frac{(\lambda)_{|\mathbf{m}|}}{\mathbf{m}!} x^{\mathbf{m}} \quad\text{and}\quad e^{|\mathbf{x}|} = \sum_{\mathbf{m}\ge 0}\frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!}.$$

Lauricella's hypergeometric series (cf. [**La, Er**]) and their integral representations are given as follows (cf. Theorem 2.1).

$$\begin{aligned} F_A(\lambda_0,\boldsymbol{\mu},\boldsymbol{\lambda};\mathbf{x}) &:= \sum_{\mathbf{m}\ge 0}\frac{(\lambda_0)_{|\mathbf{m}|}(\boldsymbol{\mu})_{\mathbf{m}}}{(\boldsymbol{\lambda})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \\ &= \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(\boldsymbol{\mu})} K_{x_1}^{\lambda_1-\mu_1,\mu_1}\cdots K_{x_n}^{\lambda_n-\mu_n,\mu_n}(1-|\mathbf{x}|)^{-\lambda_0} \\ &= \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(\lambda_0)} L_x^{\lambda_0-|\boldsymbol{\lambda}|,\boldsymbol{\lambda}}(1-\mathbf{x})^{-\boldsymbol{\mu}} \end{aligned} \tag{17}$$

$$F_B(\boldsymbol{\lambda},\boldsymbol{\lambda}',\mu;\mathbf{x}) := \sum_{\mathbf{m}\ge 0}\frac{(\boldsymbol{\lambda})_{\mathbf{m}}(\boldsymbol{\lambda}')_{\mathbf{m}}}{(\mu)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^m = \frac{\Gamma(\mu)}{\Gamma(\boldsymbol{\lambda})} K_x^{\mu-|\boldsymbol{\lambda}|,\boldsymbol{\lambda}}(1-\mathbf{x})^{-\boldsymbol{\lambda}'}, \tag{18}$$

$$F_C(\mu,\lambda_0,\boldsymbol{\lambda};\mathbf{x}) := \sum_{\mathbf{m}\ge 0}\frac{(\mu)_{|\mathbf{m}|}(\lambda_0)_{|\mathbf{m}|}}{(\boldsymbol{\lambda})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} = \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(\mu)} L_x^{\mu-|\boldsymbol{\lambda}|,\boldsymbol{\lambda}}(1-|\mathbf{x}|)^{-\lambda_0}, \tag{19}$$

$$F_D(\lambda_0,\boldsymbol{\lambda},\mu;\mathbf{x}) := \sum_{\mathbf{m}\ge 0}\frac{(\lambda_0)_{|\mathbf{m}|}(\boldsymbol{\lambda})_{\mathbf{m}}}{(\mu)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} = \frac{\Gamma(\mu)}{\Gamma(\boldsymbol{\lambda})} K_x^{\mu-|\boldsymbol{\lambda}|,\boldsymbol{\lambda}}(1-|\mathbf{x}|)^{-\lambda_0}. \tag{20}$$

When $n=2$, namely, the number of variables equals 2, the functions F_D, F_A, F_B and F_C are Appell's hypergeometric series F_1, F_2, F_3 and F_4 (cf. [**AK**]), respectively. Moreover we give examples of confluent Horn's series (cf. [**Ho, Er**]):

$$\begin{aligned} \Phi_2(\beta,\beta';\gamma;x,y) &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')} K_{x,y}^{\gamma-\beta-\beta',\beta,\beta'} e^{x+y}, \end{aligned} \tag{21}$$

$$\begin{aligned} \Psi_1(\alpha;\beta;\gamma,\gamma';x,y) &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n m!n!}x^m y^n \\ &= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)} L_{x,y}^{\alpha-\gamma-\gamma',\gamma,\gamma'}(1-x)^{-\beta}e^y, \end{aligned} \tag{22}$$

$$\begin{aligned} \Psi_2(\alpha;\gamma',\gamma';x,y) &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n m!n!}x^m y^n \\ &= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)} L_{x,y}^{\alpha-\gamma-\gamma',\gamma,\gamma'} e^{x+y}. \end{aligned} \tag{23}$$

When $n=1$, (17), (18), (19) and (20) are reduced to an integral representation of Gauss hypergeometric series. In fact, putting $(\lambda_0,\boldsymbol{\mu},\boldsymbol{\lambda}) = (\alpha,\beta,\gamma)$ in (17), we have

$$
\begin{aligned}
F(\alpha,\beta,\gamma;x) &= \sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m m!}x^m \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K_x^{\gamma-\alpha,\alpha}(1-x)^{-\beta} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}\mathrm{d}t
\end{aligned}
\tag{24}
$$

and Kummer function is

$$
\begin{aligned}
{}_1F_1(\alpha;\gamma;x) &:= \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{x^n}{n!} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K_x^{\gamma-\alpha,\alpha}e^x = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}e^{tx}\mathrm{d}t.
\end{aligned}
\tag{25}
$$

4. A connection problem

Integral representations of hypergeometric functions are useful for the study of global structure of the functions. Rigid linear ordinary differential equations on the Riemann sphere with regular or unramified irregular irregularities are reduced to the trivial equation by successive applications of middle convolutions and additions. These transformations correspond to the transformations of their solutions defined by Riemann-Liouville integrals and multiplications by elementary functions such as $(x-c)^\lambda$ or $e^{r(x)}$ with rational functions $r(x)$ of x.

In [**O1**, Chapter 12] and [**O6**], analyzing the asymptotic behavior of the Riemann Liouville integral when the variable x tends to a singular point of the function, we get the change of connection coefficients and Stokes coefficients under the integral transformations and finally such coefficients of the hypergeometric function we are interested in.

In this section a generalization of this way of study is shown in the case of several variables, which will be explained by using F_1. Since

$$
F_1(a,b,b',c;x,y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')}K_{x,y}^{c-b-b',b,b'}(1-x-y)^{-a},
$$

Proposition 2.4 implies

$$
F_1(a,b,b',c;x,0) = \frac{\Gamma(c)}{\Gamma(b)}x^{1-c}I_0^{c-b}x^{b-1}(1-x)^{-a}.
$$

The equalities (3) and (24) show $F_1(a,b,b',c;x,0) = F(b,a,c;x)$ but first we do not use this fact.

We pursue the changes of the Riemann scheme under the procedure given by Proposition 2.4 (cf. [**O1**, Chapter 5]). They are the change when we apply I_0^{c-b} to $x^{b-1}(1-x)^{-a}$ and the change when we multiply the resulting function by $\frac{\Gamma(c)}{\Gamma(b)}x^{1-c}$, which are

$$
\begin{aligned}
x^{b-1}(1-x)^{-a} : &\begin{Bmatrix} x=0 & 1 & \infty \\ \underline{b-1} & -a & a-b+1 \end{Bmatrix} \\
&\xrightarrow{I_0^{c-b}} \begin{Bmatrix} x=0 & 1 & \infty \\ 0 & 0 & b-c+1 \\ \underline{c-1} & c-a-b & a-c+1 \end{Bmatrix} \\
&\xrightarrow{\times\frac{\Gamma(c)}{\Gamma(b)}x^{1-c}} \begin{Bmatrix} x=0 & 1 & \infty \\ 1-c & 0 & b \\ \underline{0} & c-a-b & a \end{Bmatrix}.
\end{aligned}
\tag{26}
$$

Then $F_1(a,b,b',c;x,0)$ is characterized as the holomorphic function in a neighborhood of 0 with the Riemann scheme (26) and $F_1(a,b,b',c;0,0)=1$.

Since $F_1(a,b,b',c;x,y)$ satisfies a system of differential equations with singularities $x=0,1,\infty$, $y=0,1,\infty$ and $x=y$ in $\mathbb{P}^1\times\mathbb{P}^1$, we have a connection relation

$$F_1(a,b,b',c;x,y)=(-\tfrac{1}{x})^aC_af_1^a(x,y)+(-\tfrac{1}{x})^bC_bf_1^b(x,y) \tag{27}$$

in a neighborhood of $(-\infty,0)\times\{0\}$ in $\mathbb{P}^1\times\mathbb{P}^1$. Here $f_1^a(x,y)$ and $f_1^b(x,y)$ are holomorphic in a neighborhood of $[-\infty,0)\times\{0\}$ and $f_1^a(-\infty,0)=f_1^b(-\infty,0)=1$ and the connection coefficients C_a and C_b are given by those of $F_1(a,b,b',c;x,0)$, namely,

$$C_a=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\quad\text{and}\quad C_b=\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}. \tag{28}$$

Remark 4.1. We note that the general expression [**O1**, (0.25)] of the connection coefficient of Gauss hypergeometric function is simple and easy to be specialized (cf. [**O2**]). Moreover the connection formula

$$\begin{aligned}F(a,b,c;x)&=(-\tfrac{1}{x})^aC_aF(a,a-c+1,a-b+1;\tfrac{1}{x})\\&\quad+(-\tfrac{1}{x})^bC_bF(b,b-c+1,b-a+1;\tfrac{1}{x})\end{aligned}$$

follows from

$$\begin{aligned}\begin{Bmatrix}x=0 & 1 & \infty & \\ 0 & 0 & a; & x\\ 1-c & c-a-b & b & \end{Bmatrix}&=\begin{Bmatrix}x=0 & 1 & \infty & \\ a & 0 & 0; & \frac{1}{x}\\ b & c-a-b & 1-c & \end{Bmatrix}\\&=(-x)^a\begin{Bmatrix}x=0 & 1 & \infty & \\ 0 & 0 & a; & \frac{1}{x}\\ 1-(a-b+1) & c-a-b & a-c+1 & \end{Bmatrix}.\end{aligned}$$

We explicitly calculate (27) in the following way.

$$\begin{aligned}F_1(a,b,b',c;x,y)&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^my^n\\
&=\sum_{n=0}^{\infty}\frac{(a)_n(b')_n}{(c)_nn!}y^n\sum_{m=0}^{\infty}\frac{(a+n)_m(b)_m}{(c+n)_mm!}x^m=\sum_{n=0}^{\infty}\frac{(a)_n(b')_n}{(c)_nn!}y^nF(a+n,b,c+n;x)\\
&=\sum_{n=0}^{\infty}\frac{(a)_n(b')_n}{(c)_nn!}\Big(\\
&\qquad(-x)^{-a-n}\frac{\Gamma(c+n)\Gamma(b-a-n)}{\Gamma(b)\Gamma(c-a)}F(a+n,a-c+1,a-b+n+1;\tfrac{1}{x})\\
&\qquad+(-x)^{-b}\frac{\Gamma(c+n)\Gamma(a-b+n)}{\Gamma(a+n)\Gamma(c-b+n)}F(b,b-c-n+1,b-a-n+1;\tfrac{1}{x})\Big)\\
&=(-\tfrac{1}{x})^aF_1^a+(-\tfrac{1}{x})^bF_1^b,\\
F_1^a&=\sum\frac{\Gamma(c)(a)_n(b')_n\Gamma(b-a-n)(a+n)_m(a-c+1)_m}{\Gamma(b)\Gamma(c-a)(c)_n(a-b+n+1)_mm!n!}(\tfrac{1}{x})^m(-\tfrac{y}{x})^n\\
&=\sum\frac{\Gamma(c)(a)_{m+n}(b')_n\Gamma(a-b+1)\Gamma(b-a)(a-c+1)_m}{\Gamma(b)\Gamma(c-a)\Gamma(a-b+n+1)(a-b+n+1)_mm!n!}(\tfrac{1}{x})^m(\tfrac{y}{x})^n\\
&=\sum\frac{\Gamma(c)\Gamma(a-b+1)\Gamma(b-a)(a)_{m+n}(b')_n(a-c+1)_m}{\Gamma(b)\Gamma(c-a)\Gamma(a-b+1+m+n)m!n!}(\tfrac{1}{x})^m(\tfrac{y}{x})^n\\
&=\sum\frac{\Gamma(c)\Gamma(b-a)(a)_{m+n}(b')_n(a-c+1)_m}{\Gamma(b)\Gamma(c-a)(a-b+1)_{m+n}m!n!}(\tfrac{1}{x})^m(\tfrac{y}{x})^n\\
&=\sum\frac{\Gamma(c)\Gamma(b-a)(a)_{m+n}(a-c+1)_m(b')_n}{\Gamma(b)\Gamma(c-a)(a-b+1)_{m+n}m!n!}(\tfrac{1}{x})^m(\tfrac{y}{x})^n\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} F_1(a, a-c+1, b', a-b+1; \tfrac{1}{x}, \tfrac{y}{x}),\\
F_1^b &= \sum \frac{\Gamma(c)(b')_n\Gamma(a-b+n)(b)_m(b-c-n+1)_m}{\Gamma(a)\Gamma(c-b+n)(b-a-n+1)_m m!n!}(\tfrac{1}{x})^m y^n\\
&= \sum \frac{\Gamma(c)(b)_m(b')_n\Gamma(a-b+n)\Gamma(b-c-n+1)(b-c-n+1)_m(-\frac{1}{x})^m(-y)^n}{\Gamma(a)\Gamma(c-b)\Gamma(b-c+1)(a-b+n-1)\cdots(a-b+n-m)m!n!}\\
&= \sum \frac{\Gamma(c)(b)_m(b')_n\Gamma(a-b+n-m)\Gamma(b-c-n+1+m)(-\frac{1}{x})^m(-y)^n}{\Gamma(a)\Gamma(c-b)\Gamma(b-c+1)m!n!}\\
&= \sum \frac{\Gamma(c)(b)_m(b')_n\Gamma(a-b)(a-b)_{n-m}(b-c+1)_{m-n}}{\Gamma(a)\Gamma(c-b)m!n!}(-\tfrac{1}{x})^m(-y)^n\\
&= \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} G_2(b, b', a-b, b-c+1; -\tfrac{1}{x}, -y).
\end{aligned}
$$

Here

$$
G_2(\alpha, \beta, \gamma, \delta; x, y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\beta)_n(\gamma)_{n-m}(\delta)_{m-n}}{m!n!} x^m y^n \tag{29}
$$

and we have

$$
\begin{aligned}
f_1^a(x,y) &= F_1(a, a-c+1, b', a-b+1; \tfrac{1}{x}, \tfrac{y}{x}),\\
f_1^b(x,y) &= G_2(b, b', a-b, b-c+1; -\tfrac{1}{x}, -y)
\end{aligned} \tag{30}
$$

in (27).

Remark 4.2. The argument above is justified since $F_1(a; b, b'; c; x, y)$ satisfies a differential equation which has regular singularities along the hypersurface defined by $y = 0$ in $\{(x, y) \in \mathbb{C}^2 \mid \operatorname{Re} x < 0\}$ or Kummer's formula

$$
F(\alpha, \beta, \gamma; x) = (1-x)^{-\alpha} F(\alpha, \gamma-\beta, \gamma; \tfrac{x}{x-1}).
$$

We give an answer to a part of connection problem of Appell's F_1 which satisfies a KZ equation of rank 3 and the equation allows the coordinate transformations on $(\mathbb{P}^1)^5$ corresponding to the permutations of 5 coordinates. By the action of this transformation we get Kummer type formula for F_1 and solve the connection problem for F_1. Note that the singularity of the origin is not of the normally crossing type but for example, the map $(x, y) \mapsto (\frac{1}{x}, \frac{y}{x})$ is one of the coordinate transformations and the blowing up of the origin naturally corresponds to this coordinate transformation. This enables us to get all the analytic continuation of F_1 in $\mathbb{P}^1 \times \mathbb{P}^1$ in terms of F_1 and G_2 as is given in the above special case (cf. §7). Another independent solution f_1^c of the equation at $(\infty, 0)$ is characterized by the fact that the analytic continuation of f_1^c in a suitable neighborhood of $(-\infty, 0) \times \{0\}$ is a scalar multiple of f_1^c and then f_1^c is expressed by using F_1 as in the case of $(-\frac{1}{x})^a f_1^a$. This will be discussed in another paper with more general examples.

5. More transformations

In this section we examine transformations of power series obtained by a suitable class of coordinate transformations and the transformations $K_x^{\mu,\lambda}$ and $L_x^{\mu,\lambda}$. For a coordinate transformation $x \mapsto R(x)$ of $\mathbb{C}^n$ we put

$$
(T_{x \to R(x)}\phi) = \phi(R(x))
$$

for functions $\phi(x)$.

Definition 5.1. Choose a subset of indices $\{i_1,\dots,i_k\}\subset\{1,\dots,n\}$. Then we put $\mathbf{y}=(x_{i_1},\dots,x_{i_k})$. For $\mu\in\mathbb{C}$ and $\boldsymbol{\lambda}\in\mathbb{C}^k$ we define

$$K^{\mu,\boldsymbol{\lambda}}_{\mathbf{y},x\to R(x)} := T^{-1}_{x\to R(x)}\circ K^{\mu,\boldsymbol{\lambda}}_{\mathbf{y}}\circ T_{x\to R(x)},$$
$$L^{\mu,\boldsymbol{\lambda}}_{\mathbf{y},x\to R(x)} := T^{-1}_{x\to R(x)}\circ L^{\mu,\boldsymbol{\lambda}}_{\mathbf{y}}\circ T_{x\to R(x)}.$$

Let $\mathbf{p}=\Bigl(p_{i,j}\Bigr)_{\substack{1\le i\le n\\ 1\le j\le n}}\in GL(n,\mathbb{Z})$. We denote

$$\begin{aligned} x^{\mathbf{p}}=\mathbf{x}^{\mathbf{p}}&=(x^{p_{*,1}},\dots,x^{p_{*,n}})=\Bigl(\prod_{\nu=1}^n x_\nu^{p_{\nu,1}},\dots,\prod_{\nu=1}^n x_\nu^{p_{\nu,n}}\Bigr),\\ \mathbf{pm}&=(p_{1,*}\mathbf{m},\dots,p_{n,*}\mathbf{m})=\Bigl(\sum_{\nu=1}^n p_{1,\nu}m_\nu,\dots,\sum_{\nu=1}^n p_{n,\nu}m_\nu\Bigr)\end{aligned}\tag{31}$$

with $\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n$. Then $T^{-1}_{x\to x^{\mathbf{p}}}=T_{x\to x^{\mathbf{p}^{-1}}}$

We examine the transformations $K^{\mu,\boldsymbol{\lambda}}_{\mathbf{y},x\to x^{\mathbf{p}}}$ and $L^{\mu,\boldsymbol{\lambda}}_{\mathbf{y},x\to x^{\mathbf{p}}}$ under the assumption

$$p_{i_\nu,j}\ge 0\qquad (1\le\nu\le k,\ 1\le j\le n).\tag{32}$$

Since

$$\bigl(T^{-1}_{x\to x^{\mathbf{p}}}T_{x\to(t_1x_1,\dots,t_nx_n)}T_{x\to x^{\mathbf{p}}}\phi\bigr)(x)=\phi\Bigl(x_1\prod_{\nu=1}^n t_\nu^{p_{\nu,1}},\dots,x_n\prod_{\nu=1}^n t_\nu^{p_{\nu,n}}\Bigr),$$

we have

$$\begin{aligned}&\bigl(K^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}}\phi\bigr)(x)\\ &\quad=\frac{1}{\Gamma(\mu)}\int_{\substack{t_1>0,\dots t_k>0\\ t_1+\dots+t_k<1}}\mathbf{t}^{\boldsymbol{\lambda}-1}(1-|\mathbf{t}|)^{\mu-1}\phi\Bigl(x_1\prod_{\nu=1}^k t_\nu^{p_{i_\nu,1}},\dots,x_n\prod_{\nu=1}^k t_\nu^{p_{i_\nu,n}}\Bigr)\mathrm{d}t,\end{aligned}\tag{33}$$

$$\begin{aligned}\bigl(L^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}}\phi\bigr)(x)&=\frac{\Gamma(\mu+k)}{(2\pi i)^k}\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}\mathbf{t}^{\boldsymbol{\lambda}-1}(1-|\mathbf{t}|)^{-\mu-k}\\ \times\phi\Bigl(\frac{x_1}{\prod_{\nu=1}^k t_\nu^{p_{i_\nu,1}}},\dots,\frac{x_n}{\prod_{\nu=1}^k t_\nu^{p_{i_\nu,n}}}\Bigr)&\frac{\mathrm{d}t_1}{t_1}\cdots\frac{\mathrm{d}t_k}{t_k}\quad\text{with}\quad c=\tfrac{1}{k+1}.\end{aligned}\tag{34}$$

We note that (32) assures that these transformations are defined on $\mathcal{O}_0$.

Proposition 5.2. *Denoting*

$$(\mathbf{pm})_{i_1,\dots,i_k}=\Bigl(\sum_{\nu=1}^n p_{i_1,\nu}m_\nu,\dots,\sum_{\nu=1}^n p_{i_k,\nu}m_\nu\Bigr),$$

we have

$$K^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}}x^{\mathbf{m}}=\frac{\Gamma(\boldsymbol{\lambda}+(\mathbf{pm})_{i_1,\dots,i_k})}{\Gamma(|\boldsymbol{\lambda}+(\mathbf{pm})_{i_1,\dots,i_k}|+\mu)}x^{\mathbf{m}},\tag{35}$$

$$L^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}}x^{\mathbf{m}}=\frac{\Gamma(|\boldsymbol{\lambda}+(\mathbf{pm})_{i_1,\dots,i_k}|+\mu)}{\Gamma(\boldsymbol{\lambda}+(\mathbf{pm})_{i_1,\dots,i_k})}x^{\mathbf{m}}.\tag{36}$$

We give some examples hereafter in this section.

Let $(p_1,\dots,p_n)$ be a non-zero vector of non-negative integers. Suppose the greatest common divisor of $p_1,\dots,p_n$ equals 1. Then there exists $\mathbf{p}=\bigl(p_{i,j}\bigr)\in GL(n,\mathbb{Z})$ with $p_{1,j}=p_j$ and

$$K^{\mu,\lambda}_{x_1,x\to\mathbf{x}^{\mathbf{p}}}x^{\mathbf{m}}=\frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)}\frac{(\lambda)_{p_1m_1+\dots+p_nm_n}}{(\lambda+\mu)_{p_1m_1+\dots+p_nm_n}}x^{\mathbf{m}}.$$

In particular we have

$$K^{\mu,\lambda}_{x_1,x\to(x_1,\frac{x_1}{x_2},\dots,\frac{x_1}{x_n})}x^{\mathbf m}=\frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)}\frac{(\lambda)_{m_1+\cdots+m_n}}{(\lambda+\mu)_{m_1+\cdots+m_n}}x^{\mathbf m},\tag{37}$$

$$F_D(\lambda_0,\boldsymbol\lambda,\mu;\mathbf x)=\frac{\Gamma(\mu)}{\Gamma(\lambda_0)}K^{\mu-\lambda_0,\lambda_0}_{x_1,x\to(x_1,\frac{x_1}{x_2},\dots,\frac{x_1}{x_n})}(1-\mathbf x)^{-\boldsymbol\lambda}\quad\text{(cf. (20))}.\tag{38}$$

Here we note that the coordinate transformation $x\mapsto(x_1,\frac{x_1}{x_2},\dots,\frac{x_1}{x_n})$ gives a transformation of KZ equations of n variables (cf. [**O3**, §6]).

Let $\mathbf p=\begin{pmatrix}p_1&p_2\\q_1&q_2\end{pmatrix}\in GL(2,\mathbb Z)$ with p_1, p_2, q_1, $q_2\ge 0$. Put $\tilde{\mathbf p}=\mathbf p\otimes I_{n-2}\in GL(n,\mathbb Z)$. Then

$$K^{\mu,(\lambda_1,\lambda_2)}_{(x_1,x_2),x\to x^{\tilde{\mathbf p}}}x^{\mathbf m}=\frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2+\mu)}\frac{(\lambda_1)_{p_1m_1+p_2m_2}(\lambda_2)_{q_1m_1+q_2m_2}}{(\lambda_1+\lambda_2+\mu)_{(p_1+q_1)m_1+(p_2+q_2)m_2}}x^{\mathbf m},$$

$$L^{\mu,(\lambda_1,\lambda_2)}_{(x_1,x_2),x\to x^{\tilde{\mathbf p}}}x^{\mathbf m}=\frac{\Gamma(\lambda_1+\lambda_2+\mu)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\frac{(\lambda_1+\lambda_2+\mu)_{(p_1+q_1)m_1+(p_2+q_2)m_2}}{(\lambda_1)_{p_1m_1+p_2m_2}(\lambda_2)_{q_1m_1+q_2m_2}}x^{\mathbf m}.$$

Successive applications of these transformations to $(1-|\mathbf x|)^{-\lambda}$ or $(1-\mathbf x)^{-\boldsymbol\lambda}$ or $e^{|\mathbf x|},\dots$, we have many examples of integral representations of power series whose coefficients of $\frac{x^{\mathbf m}}{\mathbf m!}$ are expressed by the quotient of products of the form $(\lambda)_{p_1m_1+\cdots+p_nm_n}$.

The series

$$\phi(x,y)=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\prod_{\nu=1}^K(a_\nu)_{m+n}\prod_{\nu=1}^M(b_\nu)_m\prod_{\nu=1}^N(c_\nu)_n}{\prod_{\nu=1}^{K'}(a'_\nu)_{m+n}\prod_{\nu=1}^{M'}(b'_\nu)_m\prod_{\nu=1}^{N'}(c'_\nu)_n}\frac{x^m}{m!}\frac{y^n}{n!}\tag{39}$$

with the condition

$$(K+M)-(K'+M')=(K+N)-(K'+N')=1.$$

is an example. Then Appell's hypergeometric functions F_1, F_2, F_3 and F_4 correspond to $(K,M,N;K',N',N')=(1,1,1;1,0,0)$, $(1,1,1;0,1,1)$, $(0,2,2;1,0,0)$ and $(2,0,0;0,1,1)$, respectively. In general $\phi(x,y)$ may have several integral expressions as in the case of F_1 and F_2. The series (39) with $M=M'+1$, $N=N'+1$ and $K=K$ is a generalization of Appell's F_1, which will be given in §7 as an example.

The series

$$\begin{aligned}&K_x^{\gamma_2-\beta_2,\beta_2}\cdot(1-x)^{\alpha_2}\cdot K_x^{\gamma_1-\beta_1,\beta_1}(1-x)^{-\alpha_1}\\&=\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(\alpha_1)_m(\alpha_2)_m(\beta_1)_m(\beta_2)_{m+n}}{(\gamma_1)_m(\gamma_2)_{m+n}m!n!}x^{m+n}\end{aligned}\tag{40}$$

of $x\in\mathbb C$ satisfies a Fuchsian differential equation with the spectral type $211,211,211$ (cf. [**O1**, §13.7.5] and Remark 7.5) and the coefficients of x^k is not simple.

6. Differential equations

In this section we examine the differential equations satisfied by our invertible integral transformations of a function $u(x)$ in terms of the differential equation satisfied by $u(x)$. We denote by $W[x]$ the ring of differential operators with polynomial coefficients and put $W(x)=\mathbb C[x]\otimes W[x]$. Then $W[x]$ is called a Weyl algebra.

First we review the related results in [**O1**]. The integral transformation $u\mapsto I_c^\mu u$ given by (1) satisfies

$$I_c^{-\mu}\circ I_c^\mu=\mathrm{id}\tag{41}$$

$$I_c^\mu\circ\partial=\partial\circ I_c^\mu\quad\text{and}\quad I_c^\mu\circ\vartheta=(\vartheta-\mu)\circ I_c^\mu\tag{42}$$

under the notation

$$\partial=\tfrac{d}{dx},\quad \vartheta=x\partial.\tag{43}$$

Hence for an ordinary differential operator $P \in W[x]$, we define the middle convolution $\mathrm{mc}_\mu(P)$ of P by

$$\mathrm{mc}_\mu(P) := \partial^{-m} \sum_{i,j} a_{i,j} \partial^i (\vartheta - \mu)^j \in W[x]. \tag{44}$$

Here we first choose a positive integer k so that

$$\partial^k P = \sum_{i \geq 0,\ j \geq 0} a_{i,j} \partial^i \vartheta^j \quad ([a_{i,j}, x] = [a_{i,j}, \partial] = 0) \tag{45}$$

and then we choose the maximal positive integer m so that $\mathrm{mc}_\mu(P) \in W[x]$. The number k can be taken to be the degree of P with respect to x. Then we have

$$Pu = 0 \ \Rightarrow\ (\mathrm{mc}_\mu(P)) I_c^\mu u = 0. \tag{46}$$

The transformation $u \mapsto f(x)u$ of $u(x)$ defined by a suitable function $f(x)$ induces an automorphism $\mathrm{Ad}(f)$ of $W(x)$. Namely $\mathrm{Ad}(f)$ is called an addition and defined by

$$\mathrm{Ad}(f)\partial = \partial - \tfrac{\partial(f)}{f} \quad \text{and} \quad \mathrm{Ad}(f)x = x. \tag{47}$$

Hence $\frac{\partial(f)}{f}$ should be a rational function. Then $f(x)$ can be a function $(x-c)^\lambda$ or $f(x) = e^{r(x)}$ with a rational function $r(x)$.

There is another transformation $\mathrm{R}P$ of $P \in W(x) \setminus \{0\}$ where we define $\mathrm{R}P = r(x)P$ with $r(x) \in \mathbb{C}[x] \setminus \{0\}$ so that $r(x)P \in W[x]$ has the minimal degree with respect to x. Then $\mathrm{R}P$ is called the *reduced representative* of P. When we consider $\mathrm{mc}_\mu(P)$, we usually replace P by $\mathrm{R}P$.

Let $Pu = 0$ be a rigid Fuchsian differential equation on $\mathbb{P}^1$. Then it is proved in [**O1**] that P is obtained by successive applications of $\mathrm{Ad}(f)$ and $\mathrm{mc}_\mu \circ \mathrm{R}$ to ∂ and hence we have an integral representation of the solution to this equation and moreover its expansion into a power series.

In a similar way, the author [**O1**, §13.10] examines Appell's hypergeometric functions using the integral transformation

$$\begin{aligned} J_x^\mu(u)(x) &:= \int_\Delta (1 - s_1x_1 - \cdots - s_nx_n)^\mu u(s_1, \ldots, s_n)\mathrm{d}s \\ &= \frac{1}{x_1 \cdots x_n} \int_{\Delta'} (1 - t_1 - \cdots - t_n)^\mu u(\tfrac{t_1}{x_1}, \ldots, \tfrac{t_n}{x_n})\mathrm{d}t \quad (t_j = s_jx_j) \end{aligned} \tag{48}$$

with certain regions Δ and Δ' of integrations and get integral representations of Appell's hypergeometric functions. For example, we put $u(x) = x_1^{\beta-1}x_2^{\beta'-1}(1-x_1-x_2)^{\gamma-\beta-\beta'-1}$ and $\Delta = \{(s_1, s_2) \mid s_1 \geq 0,\ s_2 \geq 0,\ 1 - s_1 - s_2 \geq 0\}$ to get Appell's F_1 and put $u(x) = x_1^{\lambda_1-1}(1-x_1)^{\lambda_2-1}x_2^{\lambda'_1-1}(1-x_2)^{\lambda'_2-1}$ and $\Delta = \{(s_1, s_2) \mid 0 \leq s_1 \leq 1,\ 0 \leq s_2 \leq 1\}$ to get Appell's F_2 .

We show there the commuting relations

$$\begin{aligned} J_x^\mu \circ \vartheta_j &= (-1 - \vartheta_j) \circ J_x^\mu, \\ J_x^\mu \circ \partial_j &= x_j(\mu - \vartheta_1 - \cdots - \vartheta_n) \circ J_x^\mu, \end{aligned} \tag{49}$$

which correspond to (42) and imply the following proposition and then we get the differential equations satisfied by Appell's hypergeometric functions.

In general, we have the following proposition.

Proposition 6.1 ([**O1**, Proposition 13.2]). *For a differential operator*

$$P = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n} c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} \vartheta^{\boldsymbol{\beta}}$$

we have

$$J_x^\mu\big(Pu(x)\big) = J_x^\mu(P)J_x^\mu\big(u(x)\big),$$

$$J_x^\mu(P) := \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}^n} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}\Big(\prod_{k=1}^n \big(x_k(\mu-\vartheta_1-\cdots-\vartheta_n)\big)^{\alpha_k}\Big)(-\boldsymbol{\vartheta}-1)^{\boldsymbol{\beta}}.$$

Here the sums are finite and we use the notation.

$$\partial_x = \tfrac{\partial}{\partial x},\ \partial_y = \tfrac{\partial}{\partial y},\ \vartheta_x = x\partial_x,\ \vartheta_y = y\partial_y,\ \partial_i = \tfrac{\partial}{\partial x_i},\ \vartheta_i = x_i\partial_i. \tag{50}$$

Comparing the definition of integral transformations we have the following.

Proposition 6.2. *The integral transformations defined in §2 is expressed by* J_x^μ *as follows.*

$$\begin{aligned} K_x^\mu &= \frac{1}{\Gamma(\mu)}T_{x\to(\frac{1}{x_1},\dots,\frac{1}{x_n})}\circ x_1\cdots x_n\cdot J_x^{\mu-1} \\ &\textit{with}\quad \Delta' = \big\{(t_1,\dots,t_n)\mid t_1>0,\dots,t_n>0,\ t_1+\cdots+t_n<1\big\} \end{aligned} \tag{51}$$

and

$$\begin{aligned} L_x^\mu &= \frac{\Gamma(\mu+n)}{(2\pi i)^n}J_x^{-\mu-n}\circ T_{x\to(\frac{1}{x_1},\dots,\frac{1}{x_n})}\circ x_1\cdots x_n \\ &\textit{with}\quad \Delta' = \big\{(t_1,\dots,t_n)\mid \operatorname{Re}t_1=\cdots=\operatorname{Re}t_n=\tfrac{1}{n+1}\big\}. \end{aligned} \tag{52}$$

For $\mathbf{p}\in GL(n,\mathbb{Z})$ we put $\mathbf{q}=\mathbf{p}^{-1}$. Then

$$T_{x\to x^{\mathbf{p}}}(x_j) = x^{p_{*,j}} = \prod_{\nu=1}^n x_\nu^{p_{\nu,j}} \quad\text{and}\quad T_{x\to x^{\mathbf{p}}}(\partial_i) = \sum_{j=1}^n q_{i,j}\frac{x_j}{x^{p_{j,i}}}\partial_j.$$

In particular

$$T_{x\to(\frac{1}{x_1},\dots,\frac{1}{x_n})}(x_j) = \tfrac{1}{x_j},\quad T_{x\to(\frac{1}{x_1},\dots,\frac{1}{x_n})}(\partial_j) = -x_j^2\partial_j$$

and

$$T_{x\to(\frac{1}{x_1},\dots,\frac{1}{x_n})}\circ x_1\dots x_n = \frac{1}{x_1\cdots x_n}\circ T_{x\to(\frac{1}{x_1},\dots,\frac{1}{x_n})}$$

and thus we have the following lemma.

Lemma 6.3. *Defining*

$$\begin{aligned} \widetilde{u}(x_1,\dots,x_n) &:= \tfrac{1}{x_1\cdots x_n}u(\tfrac{1}{x_1},\dots,\tfrac{1}{x_n}), \\ \widetilde{P} = P^{\sim} &:= \sum a_{\boldsymbol{\alpha}}(\tfrac{1}{x_1},\dots,\tfrac{1}{x_n})\prod_\nu(-x_\nu^2\partial_\nu - x_\nu)^{\beta_\nu} \\ &\textit{for}\quad P = \sum a_{\boldsymbol{\alpha}}(x)\partial^{\boldsymbol{\alpha}}\in W(x), \end{aligned} \tag{53}$$

we have

$$\widetilde{Pu} = \widetilde{P}\widetilde{u}, \tag{54}$$

$$\widetilde{x_j} = \tfrac{1}{x_j},\ \widetilde{\partial_j} = -x_j^2\partial_j - x_j = -x_j(\vartheta_j+1),\ \widetilde{\vartheta_j} = -\vartheta_j - 1. \tag{55}$$

Hence Proposition 6.1, Proposition 6.2 and Lemma 6.3 show

$$K_x^\mu\circ\vartheta_j = (\widetilde{-1-\vartheta_j})\circ K_x^\mu = \vartheta_j\circ K_x^\mu, \tag{56}$$

$$\begin{aligned} K_x^\mu\circ\partial_j &= \big(x_j(\mu-1-\vartheta_1-\cdots-\vartheta_n)\big)^{\sim}\circ K_x^\mu \\ &= \tfrac{1}{x_j}(\vartheta_1+\cdots+\vartheta_n+\mu+n-1)\circ K_x^\mu. \end{aligned} \tag{57}$$

Similarly we have $L_x^\mu \circ \vartheta_i = \vartheta_i \circ L_x^\mu$ and

$$L_x^\mu \circ x_j(\vartheta_j + 1) = (x_j(\vartheta_1 + \cdots + \vartheta_n + \mu + n)) \circ L_x^\mu. \tag{58}$$

These relations can be checked by applying them to $x^{\boldsymbol{\alpha}}$. For example, it follows from (8) that

$$\begin{aligned}
L_x^\mu \circ x_j(\vartheta_j + 1)x^{\boldsymbol{\alpha}} &= L_x^\mu(\alpha_j + 1)x_j x^{\boldsymbol{\alpha}} \\
&= (\alpha_j + 1)\frac{\Gamma(|\boldsymbol{\alpha}| + \mu + n + 1)}{\Gamma(\alpha_1 + 1)\cdots\Gamma(\alpha_j + 2)\cdots\Gamma(\alpha_n + 1)} x_j x^{\boldsymbol{\alpha}} \\
&= \frac{(|\boldsymbol{\alpha}| + \mu + n)\Gamma(|\boldsymbol{\alpha}| + \mu + n)}{\Gamma(\boldsymbol{\alpha} + 1)} x_j x^{\boldsymbol{\alpha}} \\
&= x_j(\vartheta_1 + \cdots + \vartheta_n + \mu + n)L_x^\mu(x^{\boldsymbol{\alpha}}).
\end{aligned}$$

We also note that (56) is directly given by

$$(\vartheta_i K_x^\mu u)(x) = \frac{1}{\Gamma(\mu)}\int_0^1 (1 - |\mathbf{t}|)^{\mu-1} t_i x_i (\partial_i u)(tx)\mathrm{d}t = (K_x^\mu \vartheta_i u)(x)$$

and the equality

$$\tfrac{\partial}{\partial t_i}\big((1 - |\mathbf{t}|)^{\mu-1}u(tx)\big) = -(\mu - 1)(1 - |\mathbf{t}|)^{\mu-2}u(tx) + (1 - t)^{\mu-1}x_i(\partial_i u)(tx)$$

shows

$$x_i K_x^\mu \partial_i = (\mu - 1)K_x^{\mu-1}$$

and therefore

$$\begin{aligned}
\mu\int_0^1 (1 - |\mathbf{t}|)^{\mu-1}u(tx)\mathrm{d}t &= x_i\int_0^1 (1 - |\mathbf{t}|)^{\mu}(\partial_i u)(tx)\mathrm{d}t \\
&= x_i\int_0^1 (1 - |\mathbf{t}|)^{\mu-1}(1 - t_1 - \cdots - t_n)(\partial_i u)(tx)\mathrm{d}t, \\
x_i K_x^\mu \partial_i u &= \mu K_x^\mu u + \sum_{\nu=1}^n \frac{1}{\Gamma(\mu)}\frac{x_i}{x_\nu}\int_0^1 (1 - |\mathbf{t}|)^{\mu-1}(x_\nu \partial_i u)(tx)\mathrm{d}t \\
&= \mu K_x^\mu u + \sum_{\nu=1}^n \frac{x_i}{x_\nu} K_x^\mu \partial_i x_\nu u - K_x^\mu u \\
&= (\mu - 1)K_x^\mu u + \sum_{\nu=1}^n K_x^\mu \partial_\nu x_\nu u \\
&= (\mu + n - 1)K_x^\mu u + \sum_{\nu=1}^n \vartheta_\nu K_x^\mu u,
\end{aligned}$$

which implies (57).

Thus we have the following theorem.

THEOREM 6.1. *Suppose $u(x)$ satisfies $Pu(x) = 0$ with a certain $P \in W(x)$.*

i) *Putting*

$$Q = \mathrm{R}P = \sum_{\boldsymbol{\alpha},\,\boldsymbol{\beta}\in\mathbb{N}^n} a_{\boldsymbol{\alpha},\boldsymbol{\beta}} x^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}}. \tag{59}$$

we choose $\boldsymbol{\gamma} \in \mathbb{Z}^n$ so that

$$\partial^{\boldsymbol{\gamma}} Q = \sum_{\boldsymbol{\alpha},\,\boldsymbol{\beta}\in\mathbb{N}^n} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}\partial^{\boldsymbol{\alpha}}\vartheta^{\boldsymbol{\beta}} \quad (c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{C}). \tag{60}$$

Then we have $K_x^\mu(\partial^\gamma Q)K_x^\mu u(x)=0$ *with*

$$K_x^\mu\Big(\sum c_{\boldsymbol\alpha,\boldsymbol\beta}\partial^{\boldsymbol\alpha}\vartheta^{\boldsymbol\beta}\Big):=\mathrm{R}\sum c_{\boldsymbol\alpha,\boldsymbol\beta}\Big(\prod_{k=1}^n\big(\tfrac{1}{x_k}(\vartheta_1+\cdots+\vartheta_n+\mu+n-1)\big)^{\alpha_k}\Big)\vartheta^{\boldsymbol\beta}. \tag{61}$$

ii) *Putting*

$$Q=\mathrm{R}\tilde P, \tag{62}$$

we choose $\boldsymbol\gamma\in\mathbb{Z}^n$ *so that* (60) *holds. Then we have* $L_x^\mu(\partial^\gamma Q)L_x^\mu u(x)=0$ *with*

$$L_x^\mu\Big(\sum c_{\boldsymbol\alpha,\boldsymbol\beta}\partial^{\boldsymbol\alpha}\vartheta^{\boldsymbol\beta}\Big):=\mathrm{R}\sum c_{\boldsymbol\alpha,\boldsymbol\beta}\Big(\prod_{k=1}^n\big(x_k(\mu-\vartheta_1-\cdots-\vartheta_n)\big)^{\alpha_k}\Big)(-\boldsymbol\vartheta-1)^{\boldsymbol\beta}. \tag{63}$$

Remark 6.4. i) In Theorem 6.1 i), $\boldsymbol\gamma=(\gamma_1,\dots,\gamma_n)$ can be taken by

$$\gamma_j=\max\{0,\alpha_j-\beta_j|a_{(\alpha_1,\dots,\alpha_n),(\beta_1,\dots,\beta_n)}\neq 0\}\qquad(1\le j\le n).$$

ii) If $P\in W[x,y]$ in Theorem 6.1, it is clear that the theorem is valid under the assumption $c_{\boldsymbol\alpha,\boldsymbol\beta}\in W[y]$.

iii) Suppose $u(x)\in\mathcal{O}_0$ satisfies $P_1P_2u(x)=0$ with P_1, $P_2\in W[x]$ and $\{u\in\mathcal{O}_0\mid P_1u=0\}=\{0\}$. Then $P_2u(x)=0$.

iv) Without the assumption (32) we can define transformations $K^{\mu,\boldsymbol\lambda}_{x,x\to x^{\mathbf P}}$ and $L^{\mu,\boldsymbol\lambda}_{x,x\to x^{\mathbf P}}$ on $\mathcal{O}_0$ by (35) and (36). Even in this case the results in this section are clearly valid.

We will calculate some examples. By the integral expression

$$\begin{aligned}F_1(\lambda_0,\lambda_1,\lambda_2,\mu;x,y)&=\frac{\Gamma(\mu)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}K_{x,y}^{\mu-\lambda_1-\lambda_2,\lambda_1,\lambda_2}(1-x-y)^{-\lambda_0}\\&=C_1\int_{\substack{s>0,\,t>0\\s+t<1}}(1-s-t)^{\mu-\lambda_1-\lambda_2-1}s^{\lambda_1-1}t^{\lambda_2-1}(1-sx-ty)^{-\lambda_0}\mathrm{d}s\mathrm{d}t\end{aligned}$$

corresponding to (20) and (33), we calculate the system of differential equations satisfied by $F_1(\lambda_0,\lambda_1,\lambda_2,\mu;x,y)$ as follows. Putting

$$h:=x^{\lambda_1-1}y^{\lambda_2-1}(1-x-y)^{-\lambda_0},$$

we have

$$\begin{aligned}\mathrm{Ad}(h)\partial_x&=\partial_x-\tfrac{\lambda_1-1}{x}-\tfrac{\lambda_0}{1-x-y},\\\mathrm{Ad}(h)\partial_y&=\partial_y-\tfrac{\lambda_2-1}{y}-\tfrac{\lambda_0}{1-x-y},\\\mathrm{Ad}(h)(\vartheta_x+\vartheta_y)&=\vartheta_x+\vartheta_y-\tfrac{\lambda_0}{1-x-y}-(\lambda_1+\lambda_2-\lambda_0-2),\\\mathrm{Ad}(h)(\vartheta_x+\vartheta_y-\partial_x)&=\vartheta_x+\vartheta_y-\partial_x+\tfrac{\lambda_1-1}{x}-(\lambda_1+\lambda_2-\lambda_0-2).\end{aligned}$$

Hence we put

$$\begin{aligned}Q&:=\mathrm{R}\,\mathrm{Ad}(h)(\vartheta_x+\vartheta_y-\partial_x)\\&=(\vartheta_x+1)(\vartheta_x+\vartheta_y-\lambda_1-\lambda_2+\lambda_0+2)-\partial_x\vartheta_x+\lambda_1-1\end{aligned}$$

and we have

$$\begin{aligned}K_x^{\mu-\lambda_1-\lambda_2}(Q)&=x(\vartheta_x+1)(\vartheta_x+\vartheta_y-\lambda_1-\lambda_2+\lambda_0+2)\\&\quad-(\vartheta_x+\vartheta_y+\mu-\lambda_1-\lambda_2+1)(\vartheta_x-\lambda_1+1),\\\mathrm{Ad}(x^{1-\lambda_1}y^{1-\lambda_2})K_x^\mu(Q)&=x(\vartheta_x+\lambda_1)(\vartheta_x+\vartheta_y+\lambda_0)-(\vartheta_x+\vartheta_y+\mu-1)\vartheta_x\\&=x\big((\vartheta_x+\lambda_1)(\vartheta_x+\vartheta_y+\lambda_0)-\partial_x(\vartheta_x+\vartheta_y+\mu-1)\big).\end{aligned}$$

Hence $F_1(\lambda_0,\lambda_1,\lambda_2,\mu;x,y)$ is a solution of the system

$$\begin{cases}(\vartheta_x+\lambda_1)(\vartheta_x+\vartheta_y+\lambda_0)-\partial_x(\vartheta_x+\vartheta_y+\mu-1)u_1=0,\\(\vartheta_y+\lambda_2)(\vartheta_x+\vartheta_y+\lambda_0)-\partial_y(\vartheta_x+\vartheta_y+\mu-1)u_1=0.\end{cases}\tag{64}$$

Next we consider the integral representation

$$\begin{aligned}F_1(\lambda_0,\lambda_1,\lambda_2,\mu;x,y)&=\frac{\Gamma(\mu)}{\Gamma(\lambda_0)}K^{\mu-\lambda_0,\lambda_0}_{x,(x,y)\to(x,\frac{x}{y})}(1-x)^{-\lambda_1}(1-y)^{-\lambda_2}\\&=C_1'\int_0^1 t^{\lambda_0-1}(1-t)^{\mu-\lambda_0}(1-tx)^{-\lambda_1}(1-ty)^{-\lambda_2}\mathrm{d}t\end{aligned}$$

corresponding to (38). Since

$$T_{(x,y)\mapsto(x,\frac{x}{y})}:\partial_x\mapsto\tfrac{1}{x}(\vartheta_x+\vartheta_y),\ \partial_y\mapsto-\tfrac{y}{x}\vartheta_y,\ \vartheta_x\mapsto\vartheta_x+\vartheta_y,\ \vartheta_y\mapsto-\vartheta_y,$$

we have

$$\begin{aligned}&\partial_x\xrightarrow{\mathrm{Ad}((1-x)^{-\lambda_1}(1-y)^{-\lambda_2})}\partial_x-\tfrac{\lambda_1}{1-x}\xrightarrow{T_{(x,y)\to(x,\frac{x}{y})}}\tfrac{1}{x}(\vartheta_x+\vartheta_y)-\tfrac{\lambda_1}{1-x}\\&\xrightarrow{\mathrm{Ad}(x^{\lambda_0-1})}\tfrac{1}{x}(\vartheta_x+\vartheta_y)-\tfrac{\lambda_0-1}{x}-\tfrac{\lambda_1}{1-x}\\&\xrightarrow{\mathrm{R}}(1-x)(\vartheta_x+\vartheta_y-\lambda_0+1)-\lambda_1x\\&\xrightarrow{\partial_x}(\partial_x-\vartheta_x-1)(\vartheta_x+\vartheta_y-\lambda_0+1)-\lambda_1(\vartheta_x+1)\\&\xrightarrow{K_x^{\mu-\lambda_0}}(\partial_x+\tfrac{\mu-\lambda_0}{x}-\vartheta_x-1)(\vartheta_x+\vartheta_y-\lambda_0+1)-\lambda_1(\vartheta_x+1)\\&\xrightarrow{\mathrm{Ad}(x^{1-\lambda_0})}(\partial_x+\tfrac{\mu-1}{x}-\vartheta_x-\lambda_0)(\vartheta_x+\vartheta_y)-\lambda_1(\vartheta_x-\lambda_0)\\&\xrightarrow{T_{(x,y)\to(x,\frac{x}{y})}}(\tfrac{1}{x}(\vartheta_x+\vartheta_y)+\tfrac{\mu-1}{x}-\vartheta_x-\vartheta_y-\lambda_0)\vartheta_x-\lambda_1(\vartheta_x-\lambda_0)\\&=\partial_x(\vartheta_x+\vartheta_y+\mu-1)-(\vartheta_x+\lambda_1)(\vartheta_x+\vartheta_y+\lambda_0).\end{aligned}$$

Thus we also get the system (64) characterizing $F_1(\lambda_0,\lambda_1,\lambda_2,\mu;x,y)$.

We have similar calculations for other Appell's hypergeometric series as follows.

$$\begin{aligned}&F_2(\lambda_0;\mu_1,\mu_2;\lambda_1,\lambda_2;x,y)=\frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}K_x^{\lambda_1-\mu_1,\mu_1}K_y^{\lambda_2-\mu_2,\mu_2}(1-x-y)^{-\lambda_0}\\&=C_2\int_0^1\int_0^1 s^{\mu_1}t^{\mu_2}(1-s)^{\lambda_1-\mu_1-1}(1-t)^{\lambda_2-\mu_2-1}(1-sx-ty)^{-\lambda_0}\tfrac{\mathrm{d}s}{s}\tfrac{\mathrm{d}t}{t},\\&\partial_x\xrightarrow{\mathrm{Ad}(x^{\mu_1-1}y^{\mu_2-1}(1-x)^{\lambda_1-\mu_1-1}(1-y)^{\lambda_2-\mu_2-1})}\partial_x-\tfrac{\mu_1-1}{x}+\tfrac{\lambda_1-\mu_1-1}{1-x}\\&\xrightarrow{\mathrm{R}}x(1-x)\partial_x+(\lambda_1-2)x-(\mu_1-1)\\&\xrightarrow{\partial_x}\partial_xx(-\vartheta_x+\lambda_1-2)+\partial_x(\vartheta_x-\mu_1+1)\\&\xrightarrow{J_{x,y}^{-\lambda_0}}-\vartheta_x(\vartheta_x+1+\lambda_1-2)+x(-\lambda_0-\vartheta_x-\vartheta_y)(-1-\vartheta_x-\mu_1+1)\\&\quad=x\Big((\vartheta_x+\mu_1)(\vartheta_x+\vartheta_y+\lambda_0)-\partial_x(\vartheta_x+\lambda_1-1)\Big)\end{aligned}$$

and

$$F_3(\lambda_1,\lambda_2;\lambda'_1,\lambda'_2;\mu;x,y) = \frac{\Gamma(\mu)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} K^{\mu-\lambda_1-\lambda_2}_{x,y}(1-x)^{-\lambda'_1}(1-y)^{-\lambda'_2}$$
$$= C_3 \int_{\substack{s>0,t>0\\ s+t<1}} s^{\lambda_1}t^{\lambda_2}(1-s-t)^{\mu-1}(1-sx)^{-\lambda'_1}(1-ty)^{-\lambda'_2}\tfrac{ds}{s}\tfrac{dt}{t},$$
$$\partial_x \xrightarrow{\mathrm{R\,Ad}(x^{\lambda_1-1}y^{\lambda_2-1}(1-x)^{-\lambda'_1}(1-y)^{-\lambda'_2})} x(1-x)\partial_x + (\lambda_1-\lambda'_1-1)x - (\lambda_1-1)$$
$$\xrightarrow{\partial_x} \partial_x x(-\vartheta_x+\lambda_1-\lambda'_1-1) + \tfrac{1}{x}(\vartheta_x+\vartheta_y+\mu-\lambda'_1-\lambda_2-1)(\vartheta_x-\lambda_1+1)$$
$$\xrightarrow{\mathrm{Ad}(x^{1-\lambda_1}y^{1-\lambda_2})} -(\vartheta_x+\lambda_1)(\vartheta_x+\lambda'_1) + \partial_x(\vartheta_x+\vartheta_y+\mu-1)$$

and

$$F_4(\mu,\lambda_0;\lambda_1,\lambda_2;x,y) = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\mu)} L^{\mu-\lambda_1-\lambda_2,\lambda_1,\lambda_2}_{x,y}(1-x-y)^{-\lambda_0}$$
$$= C_4 \int_{\frac13-i\infty}^{\frac13+i\infty} s^{\lambda_1}t^{\lambda_2}(1-s-t)^{\lambda_1+\lambda_2-\mu-2}(1-sx-ty)^{-\lambda_0}\tfrac{ds}{s}\tfrac{dt}{t},$$
$$\partial_x \xrightarrow{\mathrm{Ad}((1-x-y)^{-\lambda_0})} \partial_x - \tfrac{\lambda_0}{1-x-y}$$
$$\partial_x - \vartheta_x - \vartheta_y - \lambda_0 \qquad \big((\partial_x - \vartheta_x - \vartheta_y - \lambda_0)(1-x-y)^{-\lambda_0} = 0\big)$$
$$\xrightarrow{\mathrm{Ad}(x^{\lambda_1-1}y^{\lambda_2-1})} \partial_x - \tfrac{\lambda_1-1}{x} - (\vartheta_x+\vartheta_y+\lambda_0-\lambda_1+1-\lambda_2+1)$$
$$\xrightarrow{\frac{1}{xy}T_{(x,y)\mapsto(\frac1x,\frac1y)}} -x(\vartheta_x+1) - (\lambda_1-1) - (\lambda_1-1)x + \vartheta_x+\vartheta_y-\lambda_0+\lambda_1+\lambda_2$$
$$\xrightarrow{\partial_x} -\partial_x x(\vartheta_x+\lambda_1) + \partial_x(\vartheta_x+\vartheta_y-\lambda_0+\lambda_1+\lambda_2)$$
$$\xrightarrow{\mathrm{Ad}(L^{\mu-\lambda_1-\lambda_2}_{x,y})} \vartheta_x(-\vartheta_x-1+\lambda_1)$$
$$+\, x(\lambda_1+\lambda_2-\mu-2-\vartheta_x-\vartheta_y)(-\vartheta_x-\vartheta_y-\lambda_0+\lambda_1+\lambda_2$$
$$\xrightarrow{\partial_x} -\partial_x x(\vartheta_x+\lambda_1) + \partial_x(\vartheta_x+\vartheta_y-\lambda_0+\lambda_1+\lambda_2)$$
$$\xrightarrow{L^{\mu-\lambda_1-\lambda_2}_{x,y}} \vartheta_x(-\vartheta_x-1-\lambda_1)$$
$$+\, x(\lambda_1+\lambda_2-\mu-2-\vartheta_x-\vartheta_y)(-\vartheta_x-\vartheta_y-\lambda_0+\lambda_1+\lambda_2-2)$$
$$\xrightarrow{\mathrm{Ad}(x^{1-\lambda_1}y^{1-\lambda_2})} x\Big((\vartheta_x+\vartheta_y+\mu)(\vartheta_x+\vartheta_y+\lambda_0) - \partial_x(\vartheta_x+\lambda_1-1)\Big).$$

Here C_1, C'_1, C_2, C_3 and C_4 are constants easily obtained from the integral formula in §2 with putting $x=y=0$. Hence

$$\begin{cases} u_2 = F_2(\lambda_0;\mu_1,\mu_2;\lambda_1,\lambda_2;x,y), \\ u_3 = F_3(\lambda_1,\lambda_2;\lambda'_1,\lambda'_2;\mu;x,y), \\ u_4 = F_4(\mu,\lambda_0;\lambda_1,\lambda_2;x,y) \end{cases} \tag{65}$$

are solutions of the system

$$\begin{cases}\big((\vartheta_x+\mu_1)(\vartheta_x+\vartheta_y+\lambda_0)-\partial_x(\vartheta_x+\lambda_1-1)\big)u_2=0,\\ \big((\vartheta_y+\mu_2)(\vartheta_x+\vartheta_y+\lambda_0)-\partial_y(\vartheta_y+\lambda_2-1)\big)u_2=0,\end{cases}\tag{66}$$

$$\begin{cases}\big((\vartheta_x+\lambda_1)(\vartheta_x+\lambda_1')-\partial_x(\vartheta_x+\vartheta_y+\mu-1)\big)u_3=0,\\ \big((\vartheta_y+\lambda_2)(\vartheta_x+\lambda_2')-\partial_y(\vartheta_x+\vartheta_y+\mu-1)\big)u_3=0,\end{cases}\tag{67}$$

$$\begin{cases}\big((\vartheta_x+\vartheta_y+\mu)(\vartheta_x+\vartheta_y+\lambda_0)-\partial_x(\vartheta_x+\lambda_1-1)\big)u_4=0,\\ \big((\vartheta_x+\vartheta_y+\mu)(\vartheta_x+\vartheta_y+\lambda_0)-\partial_y(\vartheta_y+\lambda_2-1)\big)u_4=0.\end{cases}\tag{68}$$

Remark 6.5. The above systems are directly obtained from the adjacent relations of the coefficients of Appell's hypergeometric series. Here we get them by the transformations of systems of differential equations corresponding to integral transformations of functions discussed in this paper so that it can be applied to general cases.

7. KZ equations

A Pfaffian system

$$\mathrm{d}u=\sum_{0\le i<j\le q}A_{i,j}\frac{\mathrm{d}(x_i-x_j)}{x_i-x_j}u\tag{69}$$

with an unknown N vector u and constant square matrices $A_{i,j}$ of size N is called a KZ (Knizhnik-Zamolodchikov type) equation of rank N (cf. [**KZ**]), which equals the system of the equations

$$\mathcal{M}\,:\,\frac{\partial u}{\partial x_i}=\sum_{\substack{0\le\nu\le q\\ \nu\ne i}}\frac{A_{i,\nu}}{x_i-x_\nu}u\qquad(i=0,\dots,q)\tag{70}$$

with denoting $A_{j,i}=A_{i,j}$. The matrix $A_{i,j}$ is called the *residue matrix* of $\mathcal{M}$ at $x_i=x_j$. Here we always assume the *integrability condition*

$$\begin{cases}[A_{i,j},A_{k,\ell}]=0 & (\forall\{i,j,k,\ell\}\subset\{0,\dots,q\}),\\ [A_{i,j},A_{i,k}+A_{j,k}]=0 & (\forall\{i,j,k\}\subset\{0,\dots,q\}),\end{cases}\tag{71}$$

which follows from the condition $\mathrm{dd}u=0$. Here i, j, k, ℓ are mutually different indices:

Definition 7.1. Using the notation

$$A_{i,i}=A_\emptyset=A_i=0,\quad A_{i,j}=A_{j,i}\quad(i,\,j\in\{0,1,\dots,q+1\}),$$
$$A_{i,q+1}:=-\sum_{\nu=0}^{n}A_{i,\nu},$$
$$A_{i_1,i_2,\dots,i_k}:=\sum_{1\le\nu<\nu'\le k}A_{i_\nu,i_{\nu'}}\quad(\{i_1,\dots,i_k\}\subset\{0,\dots,q+1\}),$$

we have

$$[A_I,A_J]=0\quad\text{if } I\cap J=\emptyset \text{ or } I\subset J \text{ with } I,\,J\subset\{0,\dots,q+1\}.\tag{72}$$

The matrix $A_{i,j}$ is called the residue matrix of $\mathcal{M}$ at $x_i=x_j$ and x_{q+1} corresponds to ∞ in $P^1_{\mathbb{C}}$.

We note that any rigid irreducible Fuchsian system

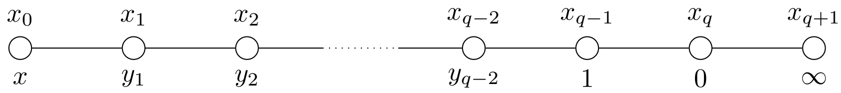
$$\mathcal{N}:\frac{\mathrm{d}u}{\mathrm{d}x}=\sum_{i=1}^{q}\frac{B_i}{x-x_i}u\tag{73}$$

can be extended to KZ equation $\mathcal{M}$ with $x = x_0$ and $B_i = A_{0,i}$, which follows from the result by Haraoka [**Ha**] extending a middle convolution on KZ equations.

We assume that $\mathcal{M}$ is irreducible at a generic value of the holomorphic parameter contained in $\mathcal{M}$. Then it is shown in [**O3**, §1] that $A_{0,\dots,q}$ is a scalar matrix κI_N with $\kappa \in \mathbb{C}$ and by the gage transformation $u \mapsto (x_{q-1} - x_q)^{-\kappa} u$ we may assume that $\mathcal{M}$ is *homogeneous*, which means

$$A_{i_0,\dots,i_q} = 0 \quad (0 \le i_0 < i_1 < \cdots < i_q \le q+1). \tag{74}$$

Then the symmetric group $\mathfrak{S}_{q+2}$ which is identified with the permutation group of the set of indices $\{0, 1, \dots, q+1\}$ naturally acts on the space of the homogeneous KZ equations (cf. [**O3**, §6]) :



$$\begin{array}{lll}
(0,1) & : & x \leftrightarrow y_1, \\
(i,i+1) & : & y_i \leftrightarrow y_{i+1} \qquad (1 \le i \le q-3), \\
(q-2,q-1) & : & (x, y_1, \dots, y_{q-1}, y_{q-2}) \leftrightarrow (\frac{x}{y_{q-2}}, \frac{y_1}{y_{q-2}}, \dots, \frac{y_{q-1}}{y_{q-2}}, \frac{1}{y_{q-2}}), \\
(q-1,q) & : & (x, y_1, \dots, y_{q-1}, y_{q-2}) \leftrightarrow (1-x, 1-y_1, \dots, 1-y_{q-1}, 1-y_{q-2}), \\
(q,q+1) & : & (x, y_1, \dots, y_{q-1}, y_{q-2}) \leftrightarrow (\frac{1}{x}, \frac{1}{y_1}, \dots, \frac{1}{y_{q-1}}, \frac{1}{y_{q-2}}).
\end{array}$$

Here we put $(x_0, \dots, x_{q+1}) = (x, y_1, \dots, y_{q-1}, 1, 0, \infty)$ by a transformation $\mathbb{P}^1 \ni x \mapsto ax + b$ which keeps the residue matrices $A_{i,j}$.

For simplicity we assume $n = q-1 = 2$ and put $(x_0, x_1, x_2, x_3, x_4) = (x, y, 1, 0, \infty)$. Then (74) means

$$A_{01} + A_{01} + A_{03} + A_{12} + A_{13} + A_{23} = 0 \tag{75}$$

and the five residue matrices A_{01}, A_{01}, A_{03}, A_{12} and A_{13} uniquely determine the other five residue matrices A_{23} and A_{i4} with $0 \le i \le 3$ and the action of $\mathfrak{S}_5$ is generated by the 4 involutions

$$\begin{array}{rcl}
(x_0, x_1, x_2, x_3, x_4) & \to & (x, y, 1, 0, \infty), \\
x_0 \leftrightarrow x_1 & \to & (x, y) \leftrightarrow (y, x), \\
x_1 \leftrightarrow x_2 & \to & (x, y) \leftrightarrow (\frac{x}{y}, \frac{1}{y}), \\
x_2 \leftrightarrow x_3 & \to & (x, y) \leftrightarrow (1-x, 1-y), \\
x_3 \leftrightarrow x_4 & \to & (x, y) \leftrightarrow (\frac{1}{x}, \frac{1}{y}).
\end{array}$$

In particular, the KZ system is determined by the equation

$$\mathcal{M} : \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{A_{01}}{x-y} u + \dfrac{A_{02}}{x-1} u + \dfrac{A_{03}}{x} u, \\ \dfrac{\partial u}{\partial y} = \dfrac{A_{01}}{y-x} u + \dfrac{A_{12}}{y-1} u + \dfrac{A_{13}}{y} u. \end{cases} \tag{76}$$

REMARK 7.2. The coordinate transformations corresponding to the involutions

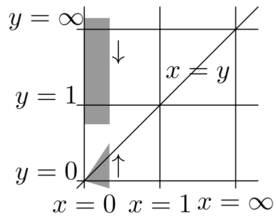
$$\begin{array}{rcl}
(x_0, x_1, x_2, x_3, x_4) \leftrightarrow (x_2, x_1, x_0, x_4, x_3) & \to & (x, y) \leftrightarrow (x, \frac{x}{y}) \\
(x_0, x_1, x_2, x_3, x_4) \leftrightarrow (x_0, x_2, x_1, x_4, x_3) & \to & (x, y) \leftrightarrow (\frac{y}{x}, y)
\end{array}$$

give the local coordinates of the blowing up of the singularities of the equation (76) at the origin:

$\mathfrak{S}_5 \ni x_0 \leftrightarrow x_1 \quad x_1 \leftrightarrow x_2 \quad x_2 \leftrightarrow x_3 \quad x_3 \leftrightarrow x_4$

$(x,y) \mapsto (y,x) \quad (\frac{x}{y}, \frac{1}{y}) \ (1-x, 1-y) \ (\frac{1}{x}, \frac{1}{y})$

$$\begin{aligned} (x,y) &\leftrightarrow (x, \tfrac{x}{y}) \\ \{|x| < \epsilon,\ |y| < C|x|\} &\leftrightarrow \{|x| < \epsilon,\ |y| > C^{-1}\} \\ x = y = 0 &\leftrightarrow x = 0 \end{aligned}$$



Now we review the result in [**DR, DR2, Ha**] by using the transformations defined in this paper. The convolution of the KZ equation (70) corresponds to the transformation defined by

$$(\tilde{\mathrm{mc}}_\mu u)(x,y) := \begin{pmatrix} I_{x,0}^{\mu+1} \frac{u(x,y)}{x-y} \\ I_{x,0}^{\mu+1} \frac{u(x,y)}{y} \\ I_{x,0}^{\mu+1} \frac{u(x,y)}{x} \end{pmatrix} = \begin{pmatrix} \frac{1}{\Gamma(\mu+1)} \int_0^x (1-t)^\mu \frac{u(t,y)}{t-y} \mathrm{d}t \\ \frac{1}{\Gamma(\mu+1)} \int_0^x (1-t)^\mu \frac{u(t,y)}{y} \mathrm{d}t \\ \frac{1}{\Gamma(\mu+1)} \int_0^x (1-t)^\mu \frac{u(t,y)}{t} \mathrm{d}t \end{pmatrix}.$$

We put $\tilde{K}_x^\mu = x^{-\mu} \circ \tilde{\mathrm{mc}}_\mu$ and $\tilde{K}_x^{\mu,\lambda} = x^{-\lambda} \circ \tilde{K}_x^\mu \circ x^\lambda$. Then

$$(\tilde{K}_x^{\mu,\lambda} u)(x,y) = \begin{pmatrix} K_x^{\mu+1,\lambda} \frac{xu(x,y)}{x-y} \\ K_x^{\mu+1,\lambda} \frac{xu(x,y)}{x-1} \\ K_x^{\mu+1,\lambda} u(x,y) \end{pmatrix}. \tag{77}$$

Putting $\tilde{u} = \tilde{K}_x^{\mu,\lambda} u$ for a solution u of the KZ equation (70), we have the KZ equation

$$\frac{\partial \tilde{u}}{\partial x_i} = \sum_{\substack{0 \le \nu \le 3 \\ \nu \ne i}} \frac{\tilde{A}_{i,\nu}}{x_i - x_\nu} \tilde{u} \tag{78}$$

satisfied by $\tilde{u}$.

Since this equation is reducible in general, we consider the reduced equation

$$\bar{\mathcal{M}} : \frac{\partial \bar{u}}{\partial x_i} = \sum_{\substack{0 \le \nu \le 3 \\ \nu \ne i}} \frac{\bar{A}_{i,\nu}}{x_i - x_\nu} \bar{u}. \tag{79}$$

The residue matrices $\tilde{A}_{i,j}$ and $\bar{A}_{i,j}$ are obtained from the results in [**DR, DR2, Ha**]:

(80)

$$\tilde{A}_{01} = \begin{pmatrix} \mu + A_{01} & A_{02} & A_{03} + \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{A}_{02} = \begin{pmatrix} 0 & 0 & 0 \\ A_{01} & \mu + A_{02} & A_{03} + \lambda \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{A}_{03} = \begin{pmatrix} -\mu - \lambda & 0 & 0 \\ 0 & -\mu - \lambda & 0 \\ A_{01} & A_{02} & A_{03} \end{pmatrix}, \qquad \tilde{A}_{04} = \begin{pmatrix} -A_{01} + \lambda & -A_{02} & -A_{03} - \lambda \\ -A_{01} & -A_{02} + \lambda & -A_{03} - \lambda \\ -A_{01} & -A_{02} & -A_{03} \end{pmatrix},$$

$$\tilde{A}_{12} = \begin{pmatrix} A_{12} + A_{02} & -A_{02} & 0 \\ -A_{01} & A_{12} + A_{01} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}, \quad \tilde{A}_{13} = \begin{pmatrix} A_{13} + A_{03} + \lambda & 0 & -A_{03} - \lambda \\ 0 & A_{13} & 0 \\ -A_{01} & 0 & A_{01} + A_{13} \end{pmatrix},$$

$$\tilde{A}_{14} = \begin{pmatrix} A_{23} - \mu - \lambda & 0 & 0 \\ A_{01} & A_{02} + A_{03} + A_{23} & 0 \\ A_{01} & 0 & A_{02} + A_{03} + A_{23} \end{pmatrix},$$

$$\tilde{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{03} + A_{23} + \lambda & -A_{03} - \lambda \\ 0 & -A_{02} & A_{02} + A_{23} \end{pmatrix},$$

$$\tilde{A}_{24} = \begin{pmatrix} A_{01} + A_{13} + A_{03} & A_{02} & 0 \\ 0 & A_{13} - \mu - \lambda & 0 \\ 0 & A_{02} & A_{01} + A_{13} + A_{03} \end{pmatrix},$$

$$\tilde{A}_{34} = \begin{pmatrix} A_{12} + A_{01} + A_{02} + \mu & 0 & A_{03} + \lambda \\ 0 & A_{12} + A_{01} + A_{02} + \mu & A_{03} + \lambda \\ 0 & 0 & A_{12} \end{pmatrix}.$$

Here we denote $A_{01} = A_{0,1}$ etc. for simplicity.

Then the subspace

$$\text{(81)} \quad \begin{aligned} \mathcal{L} &:= \begin{pmatrix} \ker A_{01} \\ \ker A_{02} \\ \ker A_{03} + \lambda \end{pmatrix} + \ker(\tilde{A}_{04} - \mu - \lambda) \\ &= \begin{pmatrix} \ker A_y \\ \ker A_1 \\ \ker A_0 + \lambda \end{pmatrix} + \ker \begin{pmatrix} A_y + \mu & A_1 & A_0 + \lambda \\ A_y & A_1 + \mu & A_0 + \lambda \\ A_y & A_1 & A_0 + \mu + \lambda \end{pmatrix} \end{aligned}$$

of $\mathbb{C}^{3N}$ satisfies $\tilde{A}_{i,j}\mathcal{L} \subset \mathcal{L}$. We define $\bar{A}_{i,j}$ the square matrices of size $3N - \dim \mathcal{L}$ which correspond to linear transformations induced by $\tilde{A}_{i,j}$, respectively, on the quotient space $\mathbb{C}^{3N}/\mathcal{L}$.

It is known that if the equation (73) is irreducible, then the corresponding ordinary differential equation defined by $\bar{A}_{0,1}$, $\bar{A}_{0,2}$ and $\bar{A}_{0,3}$ is irreducible (cf. [**DR**]) and so is the equation

$$\text{(82)} \quad \begin{cases} \dfrac{\partial \bar{u}}{\partial x} = \dfrac{\bar{A}_{01}}{x - y}\bar{u} + \dfrac{\bar{A}_{02}}{x - 1}\bar{u} + \dfrac{\bar{A}_{03}}{x}\bar{u}, \\ \dfrac{\partial \bar{u}}{\partial y} = \dfrac{\bar{A}_{01}}{y - x}\bar{u} + \dfrac{\bar{A}_{12}}{y - 1}\bar{u} + \dfrac{\bar{A}_{13}}{y}\bar{u}. \end{cases}$$

Note that if λ and μ are generic, we have

$$\mathcal{L} = \begin{pmatrix} \ker A_{01} \\ \ker A_{02} \\ 0 \end{pmatrix}.$$

Next we examine the transformations

$$\text{(83)} \quad \begin{aligned} \tilde{K}_y^{\mu,\lambda} &:= T_{(x,y)\mapsto(y,x)} \circ \tilde{K}_x^{\mu,\lambda} \circ T_{(x,y)\mapsto(y,x)}, \\ \tilde{K}_{x,y}^{\mu,\lambda} &:= T_{(x,y)\mapsto(x,\frac{x}{y})} \circ \tilde{K}_x^{\mu,\lambda} \circ T_{(x,y)\mapsto(x,\frac{x}{y})}. \end{aligned}$$

Note that $(x,y)\mapsto(y,x)$ and $(x,y)\mapsto(x,\frac{x}{y})$ correspond to $(x_0,x_1,x_2,x_3,x_4)\mapsto(x_1,x_0,x_2,x_3,x_4)$ and $(x_0,x_1,x_2,x_3,x_4)\mapsto(x_2,x_1,x_0,x_4,x_3)$, respectively. Hence the KZ equations satisfied by $\tilde K_y^{\mu,\lambda}u$ and $\tilde K_{x,y}^{\mu,\lambda}u$ are easily obtained from their definition and the equation satisfied by $\tilde K_x^{\mu,\lambda}u$. We consider the equation satisfied by $\tilde K_{x,y}^{\mu,\lambda}u$.

Putting

$$\tilde u(x,y)=(\tilde K_{x,y}^{\mu,\lambda}u)(x,y)=\begin{pmatrix} T_{(x,y)\mapsto(x,\frac{x}{y})}K_x^{\mu+1,\lambda}\frac{x}{x-y}u(x,\frac{x}{y})\\ T_{(x,y)\mapsto(x,\frac{x}{y})}K_x^{\mu+1,\lambda}\frac{x}{x-1}u(x,\frac{x}{y})\\ T_{(x,y)\mapsto(x,\frac{x}{y})}K_x^{\mu+1,\lambda}u(x,\frac{x}{y})\end{pmatrix},\tag{84}$$

the residue matrices of KZ equation satisfied by $\tilde u(x,y)$ are given by

(85)

$$\tilde A_{01}=\begin{pmatrix}A_{01}+A_{02} & -A_{02} & 0\\ -A_{12} & A_{01}+A_{12} & 0\\ 0 & 0 & A_{01}\end{pmatrix},\quad \tilde A_{02}=\begin{pmatrix}0&0&0\\ A_{12} & A_{02}+\mu & A_{24}+\lambda\\ 0&0&0\end{pmatrix},$$

$$\tilde A_{03}=\begin{pmatrix}A_{03} & A_{02} & 0\\ 0 & A_{14}-\mu-\lambda & 0\\ 0 & A_{02} & A_{03}\end{pmatrix},\quad \tilde A_{04}=\begin{pmatrix}A_{04}+A_{24} & 0 & 0\\ 0 & A_{04}+A_{24}+\lambda & -A_{24}-\lambda\\ 0 & -A_{02} & A_{02}+A_{04}\end{pmatrix},$$

$$\tilde A_{12}=\begin{pmatrix}A_{12}+\mu & A_{02} & A_{24}+\lambda\\ 0&0&0\\ 0&0&0\end{pmatrix},\quad \tilde A_{13}=\begin{pmatrix}A_{04}-\mu-\lambda & 0 & 0\\ A_{12} & A_{13} & 0\\ A_{12} & 0 & A_{13}\end{pmatrix},$$

$$\tilde A_{14}=\begin{pmatrix}A_{14}+A_{24}+\lambda & 0 & -A_{24}-\lambda\\ 0 & A_{14}+A_{24} & 0\\ -A_{12} & 0 & A_{12}+A_{14}\end{pmatrix},$$

$$\tilde A_{23}=\begin{pmatrix}-A_{12}+\lambda & -A_{02} & -A_{24}-\lambda\\ -A_{12} & -A_{02}+\lambda & -A_{24}-\lambda\\ -A_{12} & -A_{02} & -A_{24}\end{pmatrix},\quad \tilde A_{24}=\begin{pmatrix}-\mu-\lambda & 0 & 0\\ 0 & -\mu-\lambda & 0\\ A_{12} & A_{02} & A_{24}\end{pmatrix},$$

$$\tilde A_{34}=\begin{pmatrix}A_{01}+A_{02}+A_{12}+\mu & 0 & A_{24}+\lambda\\ 0 & A_{01}+A_{02}+A_{12}+\mu & A_{24}+\lambda\\ 0 & 0 & A_{01}\end{pmatrix}$$

and the invariant subspace to define the required residue matrices $\bar A_{i,j}$ is

$$\begin{aligned}\mathcal{L}&=\begin{pmatrix}\ker A_{12}\\ \ker A_{02}\\ \ker A_{24}+\lambda\end{pmatrix}+\ker(\tilde A_{23}-\mu-\lambda)\\ &=\begin{pmatrix}\ker B_1\\ \ker A_1\\ \ker A_{24}+\lambda\end{pmatrix}+\ker\begin{pmatrix}B_1+\mu & A_1 & A_{24}+\lambda\\ B_1 & A_1+\mu & A_{24}+\lambda\\ B_1 & A_1 & A_{24}+\mu+\lambda\end{pmatrix}\subset\mathbb{C}^{3N}.\end{aligned}\tag{86}$$

Lastly in this section we give an example of hypergeometric series characterized by a KZ equation. Namely, applying

$$\prod_{i=2}^{p}\tilde K_x^{-\alpha_i'-\alpha_i,\alpha_i}\prod_{j=2}^{q}\tilde K_y^{-\beta_j'-\beta_j,\beta_j}\prod_{r=1}^{r}\tilde K_{x,y}^{-\gamma_k'-\gamma_k,\gamma_k}\tag{87}$$

to a solution of the equation $\mathrm{d}u = \alpha_1 u\frac{\mathrm{d}x}{x-1} + \beta_1 u\frac{\mathrm{d}y}{y-1}$, we get a KZ equation (76) with the generalized Riemann scheme (see [**O3**, §4] for its definition)

$$\left\{\begin{matrix} A_{01} & A_{02} & A_{03} & A_{04} & A_{12} \\ [0]_{pq+(p+q-1)r} & [0]_{pr+(p+r-1)q} & [\alpha'_i]_{q+r} & [\alpha_i]_{q+r} & [0]_{qr+(q+r-1)p} \\ [-\alpha''-\beta'']_r & [-\alpha''-\gamma'']_q & \beta_j+\gamma'_k & \beta'_j+\gamma_k & [-\beta''-\gamma'']_p \end{matrix}\right.$$
$$\tag{88} \left.\begin{matrix} A_{13} & A_{23} & A_{14} & A_{24} & A_{34} \\ [\beta'_j]_{p+r} & [\gamma_k]_{p+q} & [\beta_j]_{p+r} & [\gamma'_k]_{p+q} & [0]_{pq+qr+rp-(p+q+r)+1} \\ \alpha_i+\gamma'_k & \alpha_i+\beta_j & \alpha'_i+\gamma_k & \alpha'_i+\beta'_j & [-\alpha''-\beta''-\gamma'']_2 \\ & & & & [-\alpha''-\beta'']_{r-1} \\ & & & & [-\beta''-\gamma'']_{p-1} \\ & & & & [-\alpha''-\gamma'']_{q-1} \end{matrix}\right\},$$

$$\tag{89} \begin{aligned} &\alpha''_i := \alpha_i+\alpha'_i,\ \beta''_j := \beta_j+\beta'_j,\ \gamma''_k := \gamma_k+\gamma'_k,\ \alpha'_1=\beta'_1=0, \\ &\alpha'' = \sum_{i=1}^p \alpha''_i,\ \beta'' = \sum_{j=1}^q \beta''_j,\ \gamma'' = \sum_{k=1}^r \gamma''_k, \\ &1\le i\le p,\ 1\le j\le q,\ 1\le k\le r \quad (p\ge 1,\ q\ge 1,\ r\ge 1). \end{aligned}$$

Here, for example, the eigenvalues of the square matrix A_{01} of size $R = pq+qr+rp$ are 0 with multiplicity $pq+(p+q-1)r$ and $-\alpha''-\beta''$ with multiplicity r. If the parameters α_i, β_j, γ_k, α'_i, β'_j, γ'_k are generic, the matrices $A_{i,j}$ are semisimple and the KZ equation is irreducible.

Note the hypergeometric series

$$\tag{90} \begin{aligned} \phi(x,y) = \sum_{m=0}^\infty\sum_{n=0}^\infty \frac{\prod_{i=1}^p(\alpha_i)_m \prod_{j=1}^q(\beta_j)_n \prod_{k=1}^r(\gamma_k)_{m+n}}{\prod_{i=1}^p(1-\alpha'_i)_m \prod_{j=1}^q(1-\beta'_j)_n \prod_{k=1}^r(1-\gamma'_k)_{m+n}} x^m y^n \\ \text{with}\quad \alpha'_1=\beta'_1=0 \end{aligned}$$

is a component of a solution of this KZ equation (cf. (77)).

The Riemann scheme (88) is obtained by [**O3**, Theorem 7.1] and (87). The precise argument and a further study of the hypergeometric series (90) will be given in another paper.

The index of the rigidity of this KZ equation with respect to x equals

$$\tag{91} \begin{aligned} \mathrm{Idx}_x\mathcal{M} &= (R-q)^2+q^2+(R-r)^2+r^2+2(p(q+r)^2+qr)-2R^2 \\ &= 2-2(q-1)(r-1)(q+r+1) \end{aligned}$$

and hence the ordinary differential equation with respect to the variable x is rigid if and only if $r=1$ or $q=1$.

If $p=q=r=1$, the corresponding KZ equation (78) is given by (85) with

$$\begin{aligned} &(A_{01},A_{02},A_{03},A_{04},A_{12},A_{13},A_{14},A_{23},A_{24},A_{34},\lambda,\mu) \\ &\quad = (0,\alpha_1,0,-\alpha_1,\beta_1,0,-\beta_1,-\alpha_1-\beta_1,\alpha_1+\beta_1,\gamma_1,-\gamma_1-\gamma'_1) \end{aligned}$$

and if follows from (84) that the equation has a solution with the last component

$$\phi(x,y) = F_1(\gamma_1,\alpha_1,\beta_1,1-\gamma'_1;x,y).$$

We define a simple local solution to (76) at the origin, which includes the solution we have just considered.

Definition 7.3. We define that a local solution to the equation (76) near the origin have a *simple monodromy* if the analytic continuation of the solution in a neighborhood of the origin spans one dimensional space. We simply call the solution a *simple* solution at the origin. We also define that a local solution of the equation to (76) near the line $x=0$ have a simple monodromy and call it a simple solution along $x=0$ if the analytic continuation of the solution in a neighborhood of $x=0$ spans one dimensional space.

By the correspondence between the equations (76) and (70) with $q=3$ and moreover a transformation by $\mathfrak{S}_5$ we define a local solution at $x_i=x_j=x_k$ and a local solution at $x_s=x_t$ to the equation (70) with $q=3$ when $\{i,j,k,s,t\}=\{0,1,2,3,4\}$.

Here, for example, the path of the analytic continuation in the latter case, namely along $x=0$, is in $\{(x,y)\in\mathbb{P}^1\times\mathbb{P}^1 \mid |x|<\epsilon,\ 0<|x|<\epsilon|y|\}$ with a small positive number ϵ.

Then we have the following theorem.

THEOREM 7.1. *Suppose* $\{i,j,k,s,t\}=\{0,1,2,3,4\}$ *as above. To the equation* (70) *with* $q=3$ *there is one to one correspondence between a simle solution at* $x_i=x_j=x_k$ *and a simple solution along* $x_s=x_t$.

PROOF. The coordinate $(x,\frac{x}{y})$ is a local coordinate of a blowing up of the singularities of the equation (70) around the origin. Then the origin corresponds to the line $x=0$. This coordinate transformation corresponds to the map $(x_0,x_1,x_2,x_3,x_4)\mapsto(x_2,x_1,x_0,x_4,x_3)$, which is explained in Remark 7.2. Since $x_2=x_4$ corresponds to $x_0=x_3$, we have the theorem when $(i,j,k,s,t)=(0,1,3,2,4)$. Note that the coordinate $(\frac{y}{x},y)$ gives the same conclusion. Then the symmetry $\mathfrak{S}_5$ proves the theorem. □

This theorem says that an eigenvalue of A_{24} with free multiplicity corresponds to a simple solution at $x_0=x_1=x_3$ which is the origin in (x,y) coordinate. Hence at the origin we have pq independent simple solutions of the KZ equation with the Riemann scheme (88) if the parameters are generic.

REMARK 7.4. The space of local solutions at a normally crossing singular point defined by $x_i=x_j$ and $x_s=x_t$ under the above notation is spanned by simple solutions if the parameters are generic (cf. [**KO**]).

REMARK 7.5. The transformation

$$\mathcal{O}_0\ni u=\sum c_{m,n}x^my^n\mapsto(1-x)^{-a}(1-y)^{-b}u=\sum(a)_i(b)_jc_{m,n}x^{i+m}y^{j+n}$$

induces the transformation $(A_{02},A_{12})\mapsto(A_{02}+a,A_{12}+b)$ of the equation (76). Then the coefficients of the resulting power series may be complicated (cf. (40)).

REMARK 7.6. The transformation of residue matrices induced by $\tilde{K}_x^\lambda$, $\tilde{K}_y^\lambda$ and the $\tilde{K}_{x,y}^\lambda$ and the calculation of Riemann scheme (88) for given p, q and r etc. are supported by the functions `m2mc` and `mc2grs` in the library [**O7**] of the computer algebra `Risa/Asir`. For example, by the commands

```
R=os_md.mc2grs(0,["K",[4,3,2]]);
os_md.mc2grs(R,"get"|dviout=1,div=5);
```

we get (88) on a display in the case $(p,q,r)=(4,3,2)$. This is enabled by using a `PDF` file output by functions in the library under the computer algebra. Here `div=5` indicates to divide the Riemann scheme by 5 columns into two parts as in (88) and `R` is a list of simultaneous eigenspace decomposition at 15 normally crossing singularities of the corresponding KZ equation. The algorithm for the calculation is given by [**O3**, Theorem 7.1]. Moreover by the command

```
os_md.mc2grs(R,"rest"|dviout=1);
```

we get the Riemann scheme of the induced equations on the 10 singular hypersurfaces corresponding to eigenvalues of 10 residue matrices. If `"spct"` is indicated in place of `"rest"`, a table of spectral types with respect to the variables x_i for

$i = 0, \ldots, 4$ and the indices of rigidities are displayed. If "`dviout=-1`" is indicated in place of `dviout=1`, the result is given by a TEX source in place of displaying the result on a screen. If "`|dviout=1`" is not indicated, the result is given in a format recognized by `Risa/Asir`.

8. Fuchsian ordinary differential equations

In this section we consider a Fuchsian differential equation

$$\mathcal{N} : \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{A_y}{x-y}u + \frac{A_1}{x-1}u + \frac{A_0}{x}u \tag{92}$$

with regular singularities at $x = 0$, 1, y and ∞. Here the residue matrices A_y, A_1 and A_0 are constant square matrices of size N. If (92) is irreducible and rigid or

$$\dim\bigl(Z(A_y) \cap Z(A_1) \cap Z(A_0)\bigr) = 1 \tag{93}$$

and

$$\dim Z(A_y) + \dim Z(A_1) + \dim Z(A_0) + \dim Z(A_y + A_1 + A_0) - 2N^2 = 2, \tag{94}$$

the equation (92) is constructed by successive applications of middle convolutions and additions to the trivial equation $u' = 0$. Here $Z(A)$ denotes the space of the centralizer in $M(N, \mathbb{C})$ for $A \in M(N, \mathbb{C})$ and the left hand side of (94) is the index of the rigidity of the equation. We note that middle convolutions can be replaced by the transformation of equations induced by $\tilde{K}^{\mu}_{x}$ with additions.

We assume that the equation (92) can be extended to the compatible equation

$$\frac{\partial u}{\partial y} = \frac{A_y}{x-y}u + \frac{B_1}{y-1}u + \frac{B_0}{y}u. \tag{95}$$

If the equation (92) is rigid, it extends to the compatible equation and moreover if the equation satisfies (93), the matrices B_0 and B_1 are uniquely determined by (92) up to the difference of scalar matrices. We can apply the transformations induced by $\tilde{K}^{\mu}_{y}$ and $\tilde{K}^{\mu}_{x,y}$ to (92). Note that these transformations may change the index of rigidity as was shown in the last section.

The transformations induced by $\tilde{K}^{\mu}_{x}$, $\tilde{K}^{\mu}_{y}$ and $\tilde{K}^{\mu}_{x,y}$ are given by the following (96), (97) and (98), respectively, with calculating the induced matrices of the residue matrices on $\mathbb{C}^{3N}/\mathcal{L}$.

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} = \frac{\begin{pmatrix} A_y{+}\mu & A_1 & A_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{x-y}\tilde{u} + \frac{\begin{pmatrix} 0 & 0 & 0 \\ A_y & A_1{+}\mu & A_0 \\ 0 & 0 & 0 \end{pmatrix}}{x-1}\tilde{u} + \frac{\begin{pmatrix} -\mu & 0 & 0 \\ 0 & -\mu & 0 \\ A_y & A_1 & A_0 \end{pmatrix}}{x}\tilde{u}, \\
&\mathcal{L} = \begin{pmatrix} \ker A_y \\ \ker A_1 \\ \ker A_0 \end{pmatrix} + \ker \begin{pmatrix} A_y{+}\mu & A_1 & A_0 \\ A_y & A_1{+}\mu & A_0 \\ A_y & A_1 & A_0{+}\mu \end{pmatrix},
\end{aligned} \tag{96}$$

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} = \frac{\begin{pmatrix} A_y{+}\mu & B_1 & B_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{x-y}\tilde{u} + \frac{\begin{pmatrix} A_1{+}B_1 & -B_1 & 0 \\ -A_y & A_1{+}A_y & 0 \\ 0 & 0 & A_1 \end{pmatrix}}{x-1}\tilde{u} + \frac{\begin{pmatrix} A_0{+}B_0 & 0 & -B_0 \\ 0 & A_0 & 0 \\ -A_y & 0 & A_0{+}A_y \end{pmatrix}}{x}\tilde{u}, \\
&\mathcal{L} = \begin{pmatrix} \ker A_y \\ \ker B_1 \\ \ker B_0 \end{pmatrix} + \ker \begin{pmatrix} A_y{+}\mu & B_1 & B_0 \\ B_y & A_1{+}\mu & B_0 \\ B_y & A_1 & B_0{+}\mu \end{pmatrix},
\end{aligned} \tag{97}$$

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} = \frac{\begin{pmatrix} A_y{+}A_1 & -A_1 & 0 \\ -B_1 & A_y{+}B_1 & 0 \\ 0 & 0 & A_y \end{pmatrix}}{x-y}\tilde{u} + \frac{\begin{pmatrix} 0 & 0 & 0 \\ B_1 & A_1{+}\mu & A_{24} \\ 0 & 0 & 0 \end{pmatrix}}{x-1}\tilde{u} + \frac{\begin{pmatrix} A_0 & A_1 & 0 \\ 0 & A_{14}{-}\mu & 0 \\ 0 & A_1 & A_0 \end{pmatrix}}{x}\tilde{u}, \\
&\mathcal{L} = \begin{pmatrix} \ker B_1 \\ \ker A_1 \\ \ker A_{24} \end{pmatrix} + \ker \begin{pmatrix} B_1{+}\mu & A_1 & A_{24} \\ B_1 & A_1{+}\mu & A_{24} \\ B_1 & A_1 & A_{24}{+}\mu \end{pmatrix}.
\end{aligned} \tag{98}$$

Here

$$\begin{aligned}A_{14} &= -A_y - B_0 - B_1,\\ A_{24} &= -(A_{02} + A_{12} + A_{23}) = (A_{01} + A_{02} + A_{03} + A_{12} + A_{13}) - A_{02} - A_{12}\\ &= A_{01} + A_{03} + A_{13} = A_y + A_0 + B_0.\end{aligned}$$

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