COMMUNTING FAMILIES OF DIFFERENTIAL OPERATORS INVARIANT UNDER THE ACTION OF A WEYL GROUP

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ABSTRACT. For a Weyl group W of a classical root system (Σ, E), we study W-invariant commuting differential operators on E whose highest order terms generate the W-invariant differential operators with constant coefficients. We show that the potential function for the Laplacian in this commuting family of differential operators is expressed by the Weierstrass elliptic functions. The commuting differential operators define a generalization of hypergeometric equations.

0. INTRODUCTION

Let (Σ, E) be an irreducible and reduced root system of rank n and let W be the corresponding Weyl group. We denote by $S(E)$ the symmetric algebra over the complexification E_c of the vector space E. Let ∂ denote the algebra homomorphism of $S(E)$ to the ring of differential operators on E such that

$$(\partial(X)\phi)(x) = \frac{d}{dt}\phi(x + tX)|_{t=0} \tag{0.1}$$

for functions ϕ on E and $X \in E$. We fix a system of homogeneous generators $p_1, \ldots, p_n$ of the algebra $S(E)^W$ of W-invariant elements of $S(E)$. Here we choose p_1 so that $\deg p_1 = 2$.

In this paper we shall study a system of differential operators

$$P_j = \partial(p_j) + R_j \quad \text{for} \quad j = 1, \ldots, n \tag{0.2}$$

satisfying

$$\begin{cases} P_j \text{ are } W\text{-invariant,} \\ \operatorname{ord} R_1 = 0, \\ \operatorname{ord} R_j \le \deg p_j - 1 \quad \text{for } 2 \le j \le n, \\ [P_i, P_j] = 0 \quad \text{for } 1 \le i < j \le n \end{cases} \tag{0.3}$$

in the case when the root system is of the classical type with $n > 1$.

We fix a W-invariant inner product $\langle\ ,\ \rangle$ on E and identify E and its dual by this inner product. We extend $\langle\ ,\ \rangle$ on $E_c \times E_c$ as a complex bilinear form. Since $R_1(x)$

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is a function and $\partial(p_1)$ is a Laplacian on E under a natural coordinate system of E, the operator

$$P_1 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + R(x) \tag{0.4}$$

is a Euclidean Laplacian with the potential $R(x)$ by putting $R = R_1$ for simplicity.

The radial parts of the generators of the ring of invariant differential operators on a Riemannian symmetric space give an example of the commuting family ([HC]). In this case

$$R(x) = \sum_{\alpha\in\Sigma^+} C_\alpha \sinh^{-2} \frac{\langle\alpha, x\rangle}{2}, \tag{0.5}$$

where Σ is a restricted root system corresponding to the Riemannian symmetric space, Σ^+ is its positive system,

$$C_\alpha = \frac{1}{4} m_\alpha(m_\alpha + 2m_{2\alpha} - 2)\langle\alpha, \alpha\rangle \tag{0.6}$$

and m_α is the dimension of the root space for $\alpha \in \Sigma$, which satisfies

$$m_\alpha = m_{w\alpha} \quad \text{for} \ \ w \in W \tag{0.7}$$

and only take special integers. Then J. Sekiguchi, Heckman-Opdam and Debiard ([Sj], [H1], [H2], [HO], [Op1], [Op2], [D]) studied the operator (0.4) with (0.5)–(0.7) and proved the existence of a commuting family.

On the other hand, the operator P_1 which allows such a commuting family is called a completely integrable quantum system and has been studied from the view point of mathematical physics (cf. [OP2]). The construction of such system is usually related to a root system and the most general potential function which have been proposed is written by elliptic functions. The similar fact is also true in the case of classical dynamical systems (cf. [OP1], [P], [IM], [I]).

The main purpose of this paper is to prove that the potential function $R(x)$ which allows the existence of a commuting family of differential operators with conditions (0.2) and (0.3) can be explicitly expressed by the Weierstrass elliptic function $\wp(t)$ and moreover to give certain uniqueness properties of the commuting family in terms of $R(x)$. We note that the results in this paper are also valid in the case of classical dynamical systems because the same but easier proof for them works.

In this paper we assume that the coefficients of the operators P_j can be extended to holomorphic functions on a W-invariant connected open subset Ω' of the complexification E_c of E. Here $\Omega' = \Omega \setminus V$ with a proper analytic subset V of an open neighborhood Ω of the origin of E_c.

In §2 we shall prove that the potential function $R(x)$ can be expressed by even functions $u_\alpha(t)$ of one variable:

$$R(x) = \sum_{\alpha\in\Sigma^+} u_\alpha(\langle\alpha, x\rangle) \tag{0.8}$$

with

$$u_\alpha(t) = u_{w\alpha}(t) \quad \text{for} \;\; \alpha \in \Sigma, \;\; w \in W. \tag{0.9}$$

Here Σ^+ is a positive system of Σ.

In §3 we shall prove a uniqueness for the commuting algebra $\mathbb{C}[P_1, \dots, P_n]$ in terms of two generators with small orders.

In §4 and §5 we shall study $R(x)$ when the root system is of type A_n and prove that

$$u_\alpha(t) = C_0 + C_1\wp(t) \quad \text{for } \alpha \in \Sigma^+ \tag{0.10}$$

with suitable C_0, $C_1 \in \mathbb{C}$. Moreover we shall construct the commuting operators $P_2, \dots, P_n$. These operators and their pairwise commutativity seem to be known. See [OP3] and references therein. But one of the proofs of the commutativity in [OP3] is insufficient (cf. Remark 3.7).

In §6 and §7 we shall study $R(x)$ when the root system is of type B_n or of type D_n. First we shall give a uniqueness theorem (cf. Theorem 6.5) and a functional differential equation (cf. Theorem 6.1) which is equivalent to the commutativity of P_1 and an operator of the fourth order. When $n > 2$, we shall solve the equation (cf. Theorem 7.10), which says that the potential function $R(x)$ is explicitly expressed by $\wp$ except for a trivial case.

When the root system is of type B_2, we shall only determine $R(x)$ when the coefficients of the differential operators have expansions of Harish-Chandra type (cf. Theorem 7.12). Moreover owing to this result we have a characterization of Sekiguchi-Heckman-Opdam's operators corresponding to classical Weyl groups (cf. Remark 7.14). The complete solutions for type B_2 and the explicit form of commuting differential operators for type B_n and D_n are given in successive papers [OOS], [OO] and [O].

For readers' convenience, in §8 we shall give some examples of commuting families we have constructed and write them in an algebraic form. We shall see that in general the ordinary differential equation corresponding to the potential of a higher rank equals the generic Fuchsian equation of the second order on $\mathbb{P}^1(\mathbb{C})$ which has four regular singular points. If we specialize parameters of the equation, it coincides with the equations of Lamé's functions, Mathieu's functions, Gauss' hypergeometric functions, Kummer's confluent hypergeometric functions or Bessel functions. Hence our commuting families are naturally considered as a generalization of these ordinary differential equations to systems of partial differential equations.

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The main result in this paper is announced in [Sh] and [OOS].

1. Notation

For a positive number m we fix an orthonormal basis $\{e_1, \dots, e_m\}$ of the Euclidean space $\mathbb{R}^m$ and use the coordinate system $(x_1, \dots, x_m)$ with $\mathbb{R}^m \ni x_1e_1 + \cdots + x_me_m$. Then the root system (Σ, E) of type A_n is naturally realized in

$$E = \{(x_1, \dots, x_m) \in \mathbb{R}^m \,;\, x_1 + \cdots + x_m = 0\} \tag{1.1}$$

with $m = n + 1$ and we may choose the positive system

$$\Sigma^+ = \{e_i - e_j\,;\, 1 \le i < j \le m\}. \tag{1.2}$$

Similarly in the case when (Σ, E) is of type D_n we have $E = \mathbb{R}^n$ and

$$\Sigma^+ = \{e_i - e_j,\ e_i + e_j\,;\, 1 \le i < j \le n\} \tag{1.3}$$

and in the case when (Σ, E) is of type B_n we have $E = \mathbb{R}^n$ and

$$\Sigma^+ = \{e_i\,;\, 1 \le i \le n\} \cup \{e_i - e_j,\ e_i + e_j\,;\, 1 \le i < j \le n\}. \tag{1.4}$$

We note that we need not to distinguish the root systems of type B_n and type C_n in our problem.

For the coordinate system $(x_1, \dots, x_m)$ of $\mathbb{R}^m$ we put

$$\begin{aligned}
\partial_i &= \frac{\partial}{\partial x_i},\\
\partial^\alpha &= \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m},\\
|\alpha| &= \alpha_1 + \cdots + \alpha_m,\\
\partial(i_1, \dots, i_k) &= \sum_{\substack{\nu \ne i_1, \dots, i_k\\ 1 \le \nu \le m}} \partial_\nu.
\end{aligned}$$

Here $\alpha = (\alpha_1, \dots, \alpha_m)$ with non-negative integers α_i.

Let $P = \sum p_\alpha(x)\partial^\alpha$ be a differential operator. Then we put

$${}^tP = \sum (-1)^{|\alpha|}\partial^\alpha p_\alpha(x).$$

In this paper we call the operator is *self-adjoint* (resp. *skew self-adjoint*) if ${}^tP = P$ (resp. ${}^tP = -P$).

For integers k and ℓ with $k < \ell$ we put $[k, \ell] = \{k, k+1, \dots, \ell\}$ and for a subset I of $[k, \ell]$ we denote by $|I|$ the number of elements of I.

For an element g of the permutation group $\mathfrak{S}_k$ of the set $[1, k]$ with $1 \le k \le m$ we denote by $g(P)$ the operator transformed from P by the coordinate transformation $(x_1, \dots, x_k, \dots, x_m) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(k)}, \dots, x_m)$. Then the operator P is said to be *symmetric* for the coordinate $(x_1, \dots, x_k)$ if $g(P) = P$ for all $g \in \mathfrak{S}_k$.

Moreover we denote by P^- the operator transformed from P by the coordinate transformation $(x_1, \dots, x_m) \mapsto (-x_1, \dots, -x_m)$. Then we say that P has an *even* (resp. *odd*) *parity* if $P^- = P$(resp. $P^- = -P$).

Lastly in this section we review on the Weierstrass elliptic function $\wp$ (cf. [WW]), which is a doubly periodic meromorphic function on $\mathbb{C}$ with the Laurent development

$$\wp(z|2\omega_1, 2\omega_2) = z^{-2} + a_2 z^2 + a_4 z^4 + a_6 z^6 + \cdots \tag{1.5}$$

at the origin. The complex numbers ω_1 and ω_2 are primitive half-periods:

$$\wp(z+2m_1\omega_1+2m_2\omega_2|2\omega_1,2\omega_2)=\wp(z|2\omega_1,2\omega_2) \quad \text{for} \quad m_1,m_2\in\mathbb{Z}. \tag{1.6}$$

It has the expansion

$$\wp(z|2\omega_1,2\omega_2)=\frac{1}{z^2}+\sum_{\omega\neq 0}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right) \tag{1.7}$$

where the sum ranges over all $\omega=2m_1\omega_1+2m_2\omega_2$ except 0 $(m_1,\ m_2\in\mathbb{Z})$. This $\wp$ is uniquely characterized by the differential equation

$$(\wp')^2=4\wp^3-g_2\wp-g_3 \tag{1.8}$$

with the condition

(1.9) $\wp$ has a pole of order 2 at the origin.

Here g_2 and g_3 are complex numbers, which have the relation

$$\begin{aligned} g_2&=60\sum_{\omega\neq 0}\omega^{-4}=20a_2,\\ g_3&=140\sum_{\omega\neq 0}\omega^{-6}=28a_4. \end{aligned} \tag{1.10}$$

The complex numbers ω_1 and ω_2 are linearly independent over $\mathbb{R}$ but we allow the period to be infinity. In other words, the numbers g_2 and g_3 are any complex numbers. For example we have

$$\begin{aligned} &\wp(z|\sqrt{-1}\pi,\infty)=\sinh^{-2}z+\frac{1}{3} \quad \text{when} \quad g_2=\frac{4}{3} \quad \text{and} \quad g_3=-\frac{8}{27},\\ &\wp(z|\infty,\infty)=z^{-2} \quad \text{when} \quad g_2=g_3=0. \end{aligned} \tag{1.11}$$

2. Reduction to one variable

Now we examine the potential function $R(x)$ of the operator P_1 in (0.4) which allows the commuting family (0.2) and (0.3) and we shall prove

Theorem 2.1. *Suppose the root system is of type A_n with $n>1$ or of type B_n with $n>1$ or of type D_n with $n>2$. Let $\{P_1,\dots,P_n\}$ be a system of differential operators of the form* (0.2) *which satisfies* (0.3). *Then there exist even functions $u(t)$ and $v(t)$ of one variable such that*

$$\begin{aligned} R(x)&=\sum_{1\le i<j\le n+1}u(x_i-x_j) && \textit{if } W \textit{ is of type } A_n,\\ R(x)&=\sum_{1\le i<j\le n}\Big(u(x_i-x_j)+u(x_i+x_j)\Big)+\sum_{1\le j\le n}v(x_j) && \textit{if } W \textit{ is of type } B_n,\\ R(x)&=\sum_{1\le i<j\le n}\Big(u(x_i-x_j)+u(x_i+x_j)\Big) && \textit{if } W \textit{ is of type } D_n. \end{aligned}$$

Note that ${}^tP_1 = P_1$ and ${}^t[P,Q] = -[{}^tP, {}^tQ]$ for differential operators P and Q. Hence in the following study to determine the potential function $R(x)$ we may assume

$$
{}^tP_j = (-1)^{\operatorname{ord} P_j} P_j \tag{2.1}
$$

by replacing P_j by $(P_j + (-1)^{\operatorname{ord} P_j}\,{}^tP_j)/2$.

First consider the case when the root system is of type A_n. Identifying E with a hyperplane of $\mathbb{R}^m$ with $m = n+1$ as in (1.1), we can assume the existence of the following system of commuting differential operators:

$$
\begin{aligned}
\Delta_1 &= \partial_1 + \cdots + \partial_m, \\
\Delta_2 &= \sum_{1 \le i < j \le m} \partial_i \partial_j + R(x), \\
\Delta_3 &= \sum_{1 \le i < j < k \le m} \partial_i \partial_j \partial_k + \sum_{1 \le i \le m} a_1^i \partial_i + a_0.
\end{aligned} \tag{2.2}
$$

Here $R(x), a_1^i$ and a_0 are functions of x and the function $-2R(x)$ corresponds to the original $R(x)$ in (0.4) because $P_1 = \Delta_1^2 - 2\Delta_2$. The commutativity $[\Delta_1, \Delta_j] = 0$ implies

$$
\Delta_1 R = \Delta_1 a_1^i = \Delta_1 a_0 = 0. \tag{2.3}
$$

Now consider the equation $[\Delta_2, \Delta_3] = 0$. Then the vanishing of the term ∂_i^2 implies $\partial(i) a_1^i = 0$ and by combining this with (2.3) we have

$$
\partial_i a_1^i = 0. \tag{2.4}
$$

The term $\partial_i \partial_j$ with $i < j$ implies $\partial(j) a_1^i + \partial(i) a_1^j = \partial(i,j) R$ and hence

$$
\partial_j a_1^i + \partial_i a_1^j = (\partial_i + \partial_j) R \quad \text{for} \ \ 1 \le i < j \le m. \tag{2.5}
$$

Therefore from (2.4) and (2.5) we have

$$
\partial_i \partial_j (\partial_i + \partial_j) R = 0 \quad \text{for} \ \ 1 \le i < j \le m. \tag{2.6}
$$

First we prepare

Lemma 2.2. *Let $u_1(x), \dots u_m(x)$ be functions satisfying*

$$
\partial_i u_j + \partial_j u_i = 0 \quad \textit{for } i \ne j.
$$

Then

$$
\partial_j \partial_k u_i = 0 \quad \textit{for different indices } i, j \textit{ and } k.
$$

Moreover if $\partial_i u_i = 0$ for any i, we have

$$
\partial_j \partial_k u_i = 0 \quad \textit{for } i,\ j,\ k = 1, \dots, m.
$$

Proof. When i, j and k are different indices, $\partial_j \partial_k u_i = -\partial_j \partial_i u_k = \partial_i \partial_k u_j = -\partial_k \partial_j u_i$ and we have the first claim. The last claim is also obtained by this equality for arbitrary indices i, j and k. □

Now we claim the following lemma which means that the potential function R is a sum of functions depends only on two coordinates in $(x_1, \dots, x_m)$.

Lemma 2.3 (type A_n). *Under the above notation*

$$\partial_i\partial_j\partial_k R = 0 \quad \text{for } \ 1 \le i < j < k \le m. \tag{2.7}$$

Proof. Let i, j and k are indices in $[1, m]$ which are mutually different. Then (2.5) implies $\partial_j(a_1^i - R) + \partial_i(a_1^j - R) = 0$ and we have $\partial_j\partial_k R = \partial_j\partial_k a_1^i$ by Lemma 2.2 and the lemma follows from (2.4). □

Now we shall continue the proof of Theorem 2.1. Put $R_{12} = \partial_1\partial_2 R$. Then it satisfies $(\partial_1 + \partial_2)R_{12} = \partial_3 R_{12} = \cdots = \partial_m R_{12} = 0$ and we have $R_{12} = r(x_1 - x_2)$ with a function $r(t)$. Note that $r(t)$ is an even holomorphic function for $0 < |t| \ll 1$ because of our assumption for Δ_2. Let $u(t)$ be a function with $u'' = -r$. Define a W-invariant function by

$$S(x) = R(x) - \sum_{1 \le i < j \le m} u(x_i - x_j).$$

Here $\partial_1\partial_k S(x) = 0$ for $k = 2, \ldots, m$ and we can choose a function $\phi(t)$ with $\partial_1\phi(x_1) = \partial_1 S(x)$. Then the function $T(x) = S(x) - \sum_{1 \le j \le m} \phi(x_j)$ satisfies $\partial_j T(x) = 0$ for $j = 1, \ldots, m$. Hence replacing $\phi(x_j)$ if necessary, we may assume

$$R(x) = \sum_{1 \le i < j \le m} u(x_i - x_j) + \sum_{1 \le j \le m} \phi(x_j).$$

Then by using (2.3) we have $\sum_j \phi'(x_j) = 0$ and therefore ϕ is constant. Modifying u by a constant, we may moreover assume $\phi = 0$.

Since $r(t)$ is an even function, we may assume $u(t) = w(t) + C\log t$ with an even holomorphic function $w(t)$ for $0 < |t| \ll 1$ and a complex number $C \in \mathbb{C}$. Then we have $C = 0$ because $R(x)$ is a single valued holomorphic function on Ω'. Thus we have Theorem 2.1 when the root system is of type A_n.

Remark 2.4 (H. Ochiai). Suppose the root system is of type A_n. Then it is clear from the above argument that we have

$$R(x) = \sum_{\alpha \in \Sigma^+} u_\alpha(\langle \alpha, x \rangle)$$

with suitable functions $u_\alpha(t)$ even if we omit the assumption of the W-invariance for P_j.

Next we consider the case when the root system is of type B_n with $n > 1$ or of type D_n with $n > 2$. Then we may put

(2.8)

$$\begin{aligned} P_1 &= \sum_{1 \le i \le n} \partial_i^2 + R(x), \\ P_2 &= \sum_{1 \le i < j \le n} \partial_i^2\partial_j^2 + \sum_{1 \le i \le n} a_2^i \partial_i^2 + \sum_{1 \le i < j \le n} a_{11}^{ij} \partial_i\partial_j + \sum_{1 \le i \le n} a_1^i \partial_i + a_0 \end{aligned}$$

as in the case of type A_n. We shall use the convention $a_{11}^{ij} = a_{11}^{ji}$ if $i > j$.

First we study the condition $[P_1, P_2] = 0$. The terms $\partial_i^3, \partial_i^2\partial_j$ and $\partial_i\partial_j\partial_k$ imply

$$\partial_i a_2^i = 0, \tag{2.9}$$

$$\partial_j a_2^i + \partial_i a_{11}^{ij} = \partial_j R \quad \text{for } 1 \le i, j \le n \text{ with } i \ne j, \tag{2.10}$$

$$\partial_i a_{11}^{jk} + \partial_j a_{11}^{ik} + \partial_k a_{11}^{ij} = 0 \quad \text{for } 1 \le i < j < k \le n, \tag{2.11}$$

respectively. Then we have $\partial_j^3 \partial_i R = \partial_j^2 \partial_i(\partial_j R) = \partial_j^2 \partial_i(\partial_j a_2^i + \partial_i a_{11}^{ij}) = \partial_i^2 \partial_j^2 a_{11}^{ij} = \partial_i^3 \partial_j R$ and hence

$$\partial_i \partial_j (\partial_i + \partial_j)(\partial_i - \partial_j) R = 0. \tag{2.12}$$

Now we prepare

Lemma 2.5. *Given functions $u_i(x)$ and $u_{jk}(x) = u_{kj}(x)$ of $(x_1, \ldots, x_n)$ for $1 \le i \le n$ and $1 \le j < k \ge n$. Suppose $n \ge 3$ and*

$$\begin{aligned} \partial_j u_i + \partial_i u_{ij} = 0 \quad &\textit{for } \ i \ne j, \\ \partial_i u_{jk} + \partial_j u_{ki} + \partial_k u_{ij} = 0 \quad &\textit{for } \ i \ne j \ne k. \end{aligned} \tag{2.13}$$

Then

$$\partial_j^2 \partial_k u_i = 0 \tag{2.14}$$

and

$$\partial_j \partial_k \partial_\ell u_i = 0. \tag{2.15}$$

Moreover if

$$\partial_i u_i = 0 \quad \textit{for } \ i = 1, \ldots, n, \tag{2.16}$$

then

$$\begin{aligned} &\partial_j^2 u_{ij} = 0, \\ &\partial^\alpha u_i = \partial^\alpha u_{ij} = 0 \quad \textit{if } \ |\alpha| \ge 3. \end{aligned} \tag{2.17}$$

Here i, j, k and ℓ are arbitrary indices in $[1, n]$ which are different to each other and if $n = 3$, we ignore (2.15).

Proof. It follows from (2.13) that $\partial_j^2 \partial_k u_i = -\partial_j^2 \partial_i u_{ik} = \partial_j \partial_i (\partial_i u_{kj} + \partial_k u_{ji}) = -\partial_i^2 \partial_k u_j - \partial_i \partial_i \partial_k u_j = -2\partial_i^2 \partial_k u_j$ and therefore this equals $-2(-2\partial_j^2 \partial_k u_i)$ and we have (2.14).

If $n > 3$, we have similarly $2\partial_j \partial_k \partial_\ell u_i = -\partial_j \partial_k \partial_i u_{i\ell} - \partial_j \partial_\ell \partial_i u_{ik} = -\partial_i^2 \partial_j u_{k\ell}$. Permuting the indices j, k and ℓ in this equation and summing up them, we get (2.15) because of (2.13).

Now suppose (2.16). Then $\partial_j^2 u_{ij} = -\partial_j \partial_i u_i = 0$ and $\partial_j^3 u_i = -\partial_j^2 \partial_i u_{ij} = 0$. Thus we have $\partial^\alpha u_i = 0$ if $|\alpha| \ge 3$. Hence if $|\alpha| \ge 2$, we have $\partial^\alpha \partial_i u_{ij} = -\partial^\alpha \partial_j u_i = 0$ and therefore we have $\partial_\ell \partial_k^2 u_{ij} = -\partial_\ell \partial_k (\partial_i u_{jk} + \partial_j u_{ki}) = 0$. Suppose $n > 4$. Then $\partial_\ell \partial_k u_{ij} = -\partial_\ell \partial_i u_{jk} - \partial_\ell \partial_j u_{ki} = \partial_i \partial_j u_{k\ell} + \partial_i \partial_k u_{\ell j} + \partial_j \partial_k u_{i\ell} + \partial_j \partial_i u_{\ell k} = 2\partial_i \partial_j u_{k\ell} - \partial_k \partial_\ell u_{ij}$ and so $\partial_\ell \partial_k u_{ij} = \partial_i \partial_j u_{k\ell}$. Hence $\partial_m \partial_i \partial_j u_{k\ell} = \partial_m \partial_\ell \partial_k u_{ij} = \partial_\ell \partial_i \partial_j u_{mk}$ and this also equals $\partial_k \partial_i \partial_j u_{\ell m}$. Since $\partial_k u_{\ell m} + \partial_\ell u_{mk} + \partial_m u_{k\ell} = 0$, we have $\partial_m \partial_i \partial_j u_{k\ell} = 0$. Thus we have completed the proof of the lemma. $\square$

Lemma 2.6 (type B_n or D_n).

$$\partial_i\partial_j\partial_k R = 0 \quad \textit{for} \ \ 1 \le i < j < k \le n. \tag{2.18}$$

Proof. Put $u_i = a_2^i - R$ and $u_{ij} = a_{11}^{ij}$. Then Lemma 2.5 and (2.10) and (2.11) imply

$$\begin{aligned} \partial_k^2\partial_j(R - a_2^i) = 0 &\quad \text{for } i \neq j \neq k \neq i,\\ \partial_j\partial_k\partial_\ell(R - a_2^i) = 0 &\quad \text{for different indices } i,\ j,\ k \text{ and } \ell. \end{aligned}$$

Applying ∂_i to these equations, we have

$$\begin{aligned} \partial_i\partial_k^2\partial_j R = 0 &\quad \text{for } i \neq j \neq k \neq i,\\ \partial_i\partial_j\partial_k\partial_\ell R = 0 &\quad \text{for different indices } i,\ j,\ k \text{ and } \ell \end{aligned}$$

because of (2.9).

Put $R_{12} = \partial_1\partial_2 R$. Then

$$R_{12} = \phi(x_1, x_2) + \sum_{i=3}^{n} C_i x_i \tag{2.19}$$

with a function ϕ of (x_1, x_2) and numbers $C_i \in \mathbb{C}$.

Here we note that C_i do not depend on i because of the W-invariance of R.

If the root space is of type B_n, R_{12} is invariant under the coordinate change $x_3 \mapsto -x_3$ and $C_i = 0$ in (2.19). Hence $\partial_1\partial_2\partial_3 R = 0$ and we have Lemma 2.6.

Suppose the root system is of type D_n. Since $D_3 \simeq A_3$, we may assume $n > 3$. Then by considering the coordinate change $(x_3, x_4) \mapsto (-x_3, -x_4)$ we have the same conclusion. $\square$

Now we shall continue the proof of Theorem 2.1 when the root system is of type B_n with $n > 1$ or type D_n with $n > 3$. Under the expression (2.19) it follows from (2.12) that

$$\phi(x_1, x_2) = u_1(x_1 + x_2) - u_2(x_1 - x_2)$$

with suitable holomorphic functions $u_1(t)$ and $u_2(t)$ with $0 < |t| \ll 1$. Here u_2 is an even function since $\phi(x_1, x_2)$ is symmetric for (x_1, x_2). Moreover we have $u_1 = u_2$ because the coordinate transformation $(x_1, x_2, \dots) \mapsto (x_1, -x_2, \dots)$ or $(x_1, x_2, x_3, x_4, \dots) \mapsto (x_1, -x_2, -x_3, x_4, \dots)$ transforms ϕ into $-\phi$.

Let $u(t)$ be the function with $u'' = u_2$. Then by the same argument as in the case of A_n, we have

$$R(x) = \sum_{1 \le i < j \le n} \big(u(x_i + x_j) + u(x_i - x_j)\big) + \sum_{1 \le i \le n} v(x_i) \tag{2.20}$$

with a suitable holomorphic function $v(t)$. Since $R(x)$ is a W-invariant holomorphic function, we can conclude that $u(t)$ and $v(t)$ are even holomorphic functions for $0 < |t| \ll 1$. Thus we have Theorem 2.1 when the root system is of type B_n.

The remaining part of the proof is to show that we may assume v equals 0 in the expression when the root system is of type D_n. Before we prove it, we express functions a_2^i and a_{11}^{ij} by the functions u and v for our later purpose:

Lemma 2.7. *Under the notation above we may assume*

$$
\begin{aligned}
a_2^i(x) &= \sum_{\substack{k,\ell\neq i\\ 1\le k<\ell\le n}} \bigl(u(x_k+x_\ell)+u(x_k-x_\ell)\bigr) + \sum_{\substack{k\neq i\\ 1\le k\le n}} v(x_k),\\
a_{11}^{ij} &= u(x_i+x_j)-u(x_i-x_j)
\end{aligned}
\tag{2.21}
$$

by replacing u and v if necessary.

Proof. Note that if we define a_2^i and a_{11}^{ij} by (2.21), the system of equations (2.11) holds. Hence if we denote the differences between the original functions and the above corresponding functions by the same notation with a bar, they satisfy

$$\partial_i \bar a_2^i = \partial_j \bar a_2^i + \partial_i \bar a_{11}^{ij} = \partial_i \bar a_{11}^{jk} + \partial_j \bar a_{11}^{ik} + \partial_k \bar a_{11}^{ij} = 0.$$

Owing to Lemma 2.5, we have

$$\bar a_{11}^{12} = 2Cx_1x_2 + (x_1+x_2)\phi_1(x') + \phi_2(x')$$

with a constant C and polynomial functions $\phi_j(x')$ of $x'=(x_3,\ldots,x_n)$ with degree at most j for $j=1$ and 2.

Since $\bar a_{11}^{12}$ is invariant or changes into $-\bar a_{11}^{12}$ under the coordinate transformation $(x_1,x_2,x_3)\mapsto(-x_1,-x_2,x_3)$ or $(x_1,x_2,x_3)\mapsto(-x_1,x_2,-x_3)$, respectively, we have $\phi_1=0$ and $\phi_2=C'x_3\cdots x_n$ with a constant C'. But since $\partial_3\bar a_{11}^{12}$ is symmetric for (x_1,x_2,x_3), we have $\partial_3\bar a_{11}^{12}=0$ by the relation $\partial_3\bar a_{11}^{12}+\partial_1\bar a_{11}^{23}+\partial_2\bar a_{11}^{13}=0$. Hence we can conclude $\bar a_{11}^{ij}=2Cx_ix_j$.

Replacing $u(t)$ and $v(t)$ by $u(t)+Ct^2$ and $v(t)-2C(n-1)t^2$, respectively, we may assume $\bar a_{11}^{ij}=0$. Then we have $\partial_i\bar a_2^i=\partial_j\bar a_2^i=0$ and therefore $\bar a_2^i$ are constant. Finally subtracting a constant multiple of P_1 from P_2, we have the Lemma. $\square$

Lastly we assume the root system is of type D_n to prove Theorem 2.1. We introduce the following operator which commutes with P_1.

$$P_n' = \partial_1\cdots\partial_n + \sum_{i_1+\cdots+i_n=n-2} a_{i_1\cdots i_n}\partial_1^{i_1}\cdots\partial_n^{i_n} + R'(x,\partial). \tag{2.22}$$

Here $R'(x,\partial)$ is a suitable W-invariant differential operator of order $<n-2$.

We put $a(j,k)=a_{i_1\cdots i_n}$ with the indices $i_1,\cdots,i_n$ given by

$$i_\nu = \begin{cases} 1 & \text{if } \nu\neq j \text{ and } \nu\neq k,\\ 0 & \text{if } \nu=j \text{ or } \nu=k.\end{cases}$$

Then by the term $\partial_2\partial_3\cdots\partial_n$ of $[P_1,P_n']$ we have

$$2\sum_{2\le j\le n}\partial_j a(1,j) = \partial_1 R. \tag{2.23}$$

Furthermore by the term $\partial_1^2\partial_2\cdots\partial_n$ of $[P_2,P_n']$ with Lemma 2.7 we have

$$
\begin{aligned}
2\sum_{2\le j\le n}\partial_j a(1,j) &= \partial_1 a_2^1 + \sum_{2\le j\le n}\partial_j a_{11}^{1j}\\
&= \sum_{2\le j\le n}\bigl(u'(x_1+x_j)+u'(x_1-x_j)\bigr).
\end{aligned}
\tag{2.24}
$$

Comparing this with (2.20) and (2.23), we have $v'=0$. Modifying u by a constant, we have Theorem 2.1 and subtracting a constant multiple of P_1 from P_2, we may assume Lemma 2.7 with $v=0$.

3. Uniqueness of the commuting family

In this section we shall prove that the generator P_1 and the generator, say P_2, having the lowest order among the remaining generators $\{P_2, \dots, P_n\}$ of the commuting family (0.3) uniquely determine the commuting algebra $\mathbb{C}[P_1, \dots, P_n]$.

In the subsequent sections we shall study the relation $[P_1, P_2] = 0$ and we shall get a more refined result on the dependence of the commuting algebra on the potential function $R(x)$. First we prepare

Lemma 3.1. *Let*

$$q(x,\xi) = \sum_{|\alpha|=K} q_\alpha(x)\xi^\alpha \tag{3.1}$$

be a homogeneous polynomial of $\xi = (\xi_1, \dots, \xi_m)$ of degree K whose coefficients are functions of $x = (x_1, \dots, x_m)$ and consider the conditions

$$\Big\{\sum_{i=1}^m \xi_i^\ell, q(x,\xi)\Big\} = 0 \tag{3.2.ℓ}$$

and

$$\sum_{|\alpha|=K} q_\alpha(x)\partial^\alpha \text{ is symmetric for } (x_1, \dots, x_m). \tag{3.3}$$

Here $\{\ ,\ \}$ is the Poisson bracket defined by

$$\{f,g\} = \sum_{i=1}^m \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \sum_{i=1}^m \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i}. \tag{3.4}$$

i) If (3.2.2) *holds, then $q_\alpha(x)$ are polynomials.*

ii) Fix a positive integer N with $N \geq 3$. Then the functions q_α are constants if one of the following conditions holds:

(3.5) $K \leq N-2$ *and condition* (3.2.N) *holds.*

(3.6) $K = N-1$ *and conditions* (3.2.2) *and* (3.2.N) *hold.*

(3.7) $K = N$ *and conditions* (3.2.2), (3.2.N) *and* (3.3) *hold.*

(3.8) $K = N+1,\ N \geq 4$ *and conditions* (3.2.2), (3.2.N) *and* (3.3) *hold.*

Proof. In this proof we always assume that the index $\alpha \in \mathbb{Z}^m$ satisfies $|\alpha| = K$. Put $\delta_\nu = (\delta_{1\nu}, \dots, \delta_{m\nu})$ with Kronecker's δ.

Note that for $\beta \in \mathbb{Z}^m$, the coefficients of the term ξ^β of (3.2.2) mean

$$\sum_{\nu=1}^m \partial_\nu q_{\beta-\delta_\nu} = 0 \tag{3.9}$$

and in general, the coefficients of the term ξ^β of (3.2.N) mean

$$\sum_{\nu=1}^{m} \partial_\nu q_{\beta-(N-1)\delta_\nu} = 0. \tag{3.10}$$

Here we use the convention that $q_\alpha = 0$ if α has a negative component.

Suppose (3.2.2) and fix an index j. Applying $\partial_j^{K-\alpha_j}$ to (3.9) with $\beta = \alpha + \delta_j$, we have

$$\partial_j^{K+1-\alpha_j} q_\alpha = -\sum_{\nu\neq j} \partial_j^{K-\alpha_j} \partial_\nu q_{\alpha+\delta_j-\delta_\nu}.$$

If $K-\alpha_j = 0$, then $\alpha_\nu = 0$ for $\nu \neq j$ and the above equation is reduced to $\partial_j q_\alpha = 0$. Then by the induction on the non-negative integer $K - \alpha_j$ we can prove

$$\partial_j^{K+1-\alpha_j} q_\alpha = 0. \tag{3.11}$$

Thus we have lemma 3.1 i).

Note that if $\alpha \in \mathbb{Z}^m$ satisfies $\alpha_\nu \leq N-2$ for $\nu = 1, \dots, m$, then equation (3.10) with $\beta = \alpha + (N-1)\delta_j$ equals

$$\partial_j q_\alpha = 0. \tag{3.12}$$

If (3.5) holds, (3.12) is valid for any j and α and therefore q_α are constant. To prove the remaining part of the Lemma, we may assume that $q_\alpha(x)$ are polynomials without constant terms because of the assumption (3.2.2).

Suppose (3.6). Then the above argument assures that we may assume $q_\alpha = 0$ if $\alpha_\nu \leq N-2$ for $\nu = 1, \dots, m$ and we have the expression

$$q(x,\xi) = \sum_{\nu=1}^{m} a_K^\nu(x)\xi_\nu^K. \tag{3.13}$$

Since $K > 1$, we have $\partial_i a_j^\nu = 0$ from equation (3.9) with $\beta = \delta_i + K\delta_j$ for any i and j and we have the lemma.

Suppose (3.7). Then by the same argument as above we can write

$$q(x,\xi) = \sum_{\nu=1}^{m} a_K^\nu(x)\xi_\nu^K + \sum_{\substack{\nu\neq\mu \\ 1\leq\nu,\mu\leq m}} a_{1k-1}^{\nu\mu}(x)\xi_\nu\xi_\mu^{K-1}. \tag{3.14}$$

In equation (3.10), putting $\beta = (2K-1)\delta_1$ and $\beta = (2K-2)\delta_1 + \delta_2$, we have

$$\partial_1 a_K^1 = \partial_1 a_{1K-1}^{21} = 0 \tag{3.15}$$

and putting $\beta = K\delta_1 + (K-1)\delta_2$ and $\beta = (K-1)\delta_1 + \delta_2 + (K-1)\delta_3$, we have

$$\partial_2 a_K^1 + \partial_1 a_{1K-1}^{12} = 0, \tag{3.16}$$

$$\partial_1 a_{1K-1}^{23} + \partial_3 a_{1K-1}^{21} = 0, \tag{3.17}$$

respectively. On the other hand, from equation (3.9) with $\beta = K\delta_1 + \delta_2$ we have

$$\partial_2 a^1_K + \partial_1 a^{21}_{1K-1} = 0. \tag{3.18}$$

It follows from (3.15), (3.16) and (3.18) that

$$\partial_2 a^1_K = \partial_1 a^{12}_{1K-1} = 0 \tag{3.19}$$

and it follows from (3.15), (3.17) and Lemma 2.2 that

$$\partial_1^2 a^{23}_{1K-1} = \partial_1 \partial_4 a^{23}_{1K-1} = 0. \tag{3.20}$$

Then from (3.7), (3.15), (3.19) and (3.20) we have $a^i_K = 0$ and $a^{ij}_{1K-1} = C \sum_{\nu \neq i,j} x_\nu$ with a constant number C. But equation (3.17) proves $C = 0$.

Suppose (3.8). Note that $K = N + 1 \geq 5$. We may assume

$$\begin{aligned} q(x,\xi) = &\sum_\nu a^\nu_K(x)\xi^K_\nu + \sum_{\nu \neq \mu} a^{\nu\mu}_{1K-1}(x)\xi_\nu \xi^{K-1}_\mu \\ &+ \sum_{\nu \neq \mu} a^{\nu\mu}_{2K-2}(x)\xi^2_\nu \xi^{K-2}_\mu + \sum_{\nu<\mu,\ \tau \neq \mu,\nu} a^{\nu\mu\tau}_{11K-2}(x)\xi_\nu\xi_\mu\xi^{K-2}_\tau. \end{aligned} \tag{3.21}$$

Putting $\beta = (2K-2)\delta_1$, $(2K-3)\delta_1 + \delta_2$, $(2K-4)\delta_1 + 2\delta_2$ and $(2K-4)\delta_1 + \delta_2 + \delta_3$, we obtain

$$\partial_1 a^1_K = \partial_1 a^{21}_{1K-1} = \partial_1 a^{21}_{2K-2} = \partial_1 a^{231}_{11K-2} = 0 \tag{3.22}$$

from (3.10). Similarly putting $\beta = K\delta_1 + (K-2)\delta_2$, $\beta = (K-1)\delta_1 + (K-2)\delta_2 + \delta_3$, $\beta = (K-2)\delta_1 + (K-2)\delta_2 + 2\delta_3$ and $\beta = (K-2)\delta_1 + (K-2)\delta_2 + \delta_3 + \delta_4$ in equation (3.10) we have

$$\partial_1 a^{12}_{2K-2} + \partial_2 a^1_K = 0, \tag{3.23}$$
$$\partial_1 a^{132}_{11K-2} + \partial_2 a^{31}_{1K-1} = 0, \tag{3.24}$$
$$\partial_1 a^{32}_{2K-2} + \partial_2 a^{31}_{2K-2} = 0, \tag{3.25}$$
$$\partial_1 a^{342}_{11K-2} + \partial_2 a^{341}_{11K-2} = 0, \tag{3.26}$$

respectively. On the other hand, putting $\beta = K\delta_1 + \delta_2$, $\beta = (K-1)\delta_1 + 2\delta_2$ and $\beta = (K-1)\delta_1 + \delta_2 + \delta_3$ in equation (3.9), we have

$$\partial_1 a^{21}_{1K-1} + \partial_2 a^1_K = 0, \tag{3.27}$$
$$\partial_1 a^{21}_{2K-2} + \partial_2 a^{21}_{1K-1} = 0, \tag{3.28}$$
$$\partial_1 a^{231}_{11K-2} + \partial_2 a^{31}_{1K-1} + \partial_3 a^{21}_{1K-1} = 0, \tag{3.29}$$

respectively.

By (3.22) and (3.27) we have $\partial_2 a^1_K = 0$ and in general we have $\partial_i a^j_K = 0$ for $i, j = 1, \ldots, m$ and hence $a^i_K = 0$.

Note that (3.22) and (3.29) means

$$\partial_2 a^{31}_{1K-1} + \partial_3 a^{21}_{1K-1} = 0. \tag{3.30}$$

Then from (3.28), (3.22), (3.30), (3.3) and Lemma 2.2 we obtain

$$\partial_2 a^{21}_{1K-1} = \partial_2^2 a^{31}_{1K-1} = \partial_2\partial_4 a^{31}_{1K-1} = 0 \tag{3.31}$$

and from (3.23), (3.22), (3.25), (3.3) and Lemma 2.2 we obtain

$$\partial_1 a^{12}_{2K-2} = \partial_1^2 a^{32}_{2K-2} = \partial_1\partial_4 a^{32}_{2K-2} = 0 \tag{3.32}$$

and from (3.22), (3.24), (3.31), (3.26), (3.3) and Lemma 2.2 we obtain

$$\partial_1^2 a^{132}_{11K-2} = \partial_1\partial_3 a^{132}_{11K-2} = \partial_1^2 a^{342}_{11K-2} = \partial_1\partial_5 a^{342}_{11K-2} = 0. \tag{3.33}$$

Thus from (3.3), (3.31), (3.32), (3.32) we have the expression

$$\begin{aligned} a^{21}_{1K-1} &= C_1(x_3 + \cdots + x_n),\\ a^{21}_{2K-2} &= C_2(x_3 + \cdots + x_n),\\ a^{231}_{11K-2} &= C_3(x_2 + x_3) + C_4(x_4 + \cdots + x_n) \end{aligned} \tag{3.34}$$

with suitable constant numbers C_i. Then from (3.24), (3.25), (3.26) and (3.30) we can conclude $C_3 + C_1 = 0$, $2C_2 = 0$, $2C_4 = 0$ and $2C_1 = 0$, respectively, which completes the proof of the Lemma. $\square$

Now we give the theorem in this section:

Theorem 3.2. *Without loss of generality we suppose that the order of the generator P_2 of our commuting family* (0.2) *equals* 3 (*resp.* 4) *in the case when the root system is of type A_n* (*resp. B_n or D_n*). *Then P_1 and P_2 uniquely determine the commuting algebra $\mathbb{C}[P_1, \ldots, P_n]$.*

Proof. First consider the case when the root system is of type A_n. We may assume that

$$\Delta'_k = \sum_{i=1}^{m} \partial_i^k + R'_k(x, D) \quad \text{for} \quad k = 1, \ldots, m \tag{3.35}$$

generate our commuting algebra with the identification (1.1). Here $\operatorname{ord} R'_k(x,\partial) < k$, $R'_1(x,\partial) = 0$ and $\operatorname{ord} R'_2(x,\partial) = 0$. We shall prove that Δ'_{N+1} is uniquely determined modulo $\mathbb{C}[\Delta'_1, \ldots, \Delta'_N]$ for $N = 3, \ldots, m-1$, which implies the theorem.

Suppose this is not true for some N. Then there exist W-invariant differential operators $\Delta'_{N+1}(1)$ and $\Delta'_{N+1}(2)$ with the same principal symbol $\sum_i \xi_i^{N+1}$ which commute with $\Delta'_1, \ldots, \Delta'_N$.

Put $Q = \Delta'_{N+1}(1) - \Delta'_{N+1}(2)$ and $K = \operatorname{ord} Q$. We may assume the principal symbol $q(x,\xi) = \sigma(Q)$ of Q really depends on x because otherwise we can reduce the order of Q by subtracting an element of $\mathbb{C}[\Delta'_1, \ldots, \Delta'_N]$.

Then the condition $[\Delta_2', Q] = [\Delta_N', Q] = 0$ implies one of the conditions (3.5), (3.6) and (3.7) and therefore Lemma 3.1 proves that $q(x, \xi)$ does not depend on x. This contradicts our assumption and hence we have the theorem.

In the case when the root system is of type B_n we may assume $\operatorname{ord} P_j = 2j$ and $\sigma(P_j) = \sum_i \xi_i^{2j}$. Then the proof proceeds in the same way as in the case when the root system is of type A_n. In the case when the root system is of type D_n we can prove the theorem in the same way if we define the operators P_j from lower order ones. □

Since the condition

$$\{\sum_{i=1}^{n} \xi_i^{2k}, q(x_1, \dots, x_n, \xi_1, \dots, \xi_n)\} = 0 \quad \text{for any } k \geq 1 \tag{3.36}$$

implies that q does not depend on x, we have the following as in the proof of Theorem 3.2.

Proposition 3.3. *Let P be a W-invariant differential operator which commutes with any element of the commuting algebra $\mathbb{C}[P_1, \dots, P_n]$. Then $P \in \mathbb{C}[P_1, \dots, P_n]$.*

Now we give the lemmas which shall be used later.

Lemma 3.4. *Let $Q_0(x, \partial)$, $Q_1(x, \partial)$ and $Q_2(x, \partial)$ be differential operators of the form*

$$Q_0(x, \partial) = \sum_{i=0}^{n} \partial_i^2 + q_0(x),$$
$$Q_1(x, \partial) = \sum q_1^\alpha(x) \partial^\alpha, \quad Q_2(x, \partial) = \sum q_2^\alpha(x) \partial^\alpha.$$

Suppose $q_0(x)$ and $q_2^\alpha(x)$ are polynomial functions of x and furthermore suppose $[Q_0(x, \partial), Q_1(x, \partial)] = Q_2(x, \partial) + r(x) Q_1(x, \partial)$ with a polynomial function $r(x)$. Then $q_1^\alpha(x)$ are also polynomial functions of x.

Proof. We shall prove that $q_1^\alpha(x)$ are polynomial functions of x by the induction on the number $|\alpha|$.

If $|\alpha| > \operatorname{ord} Q_1(x, \partial)$, the claim is clear. Let k be a nonnegative integers and suppose $q_1^\alpha(x)$ are polynomial functions of x if $|\alpha| > k$. Then the $(k+1)$-th order term of $[Q_0(x, \partial), Q_1(x, \partial)] = Q_2(x, \partial)$ shows

$$\{\sum_{i=1}^{n} \xi_i^2, \sum_{|\alpha|=k} q_1^\alpha(x) \xi^\alpha\} = \sum_{|\beta|=k+1} \big(a_\beta(x) + r(x) q_1^\beta(x)\big) \xi^\beta$$

with some polynomial functions $a_\beta(x)$.

Choosing a positive integer N so that $\deg \big(a_\beta(x) + r(x) q_1^\beta(x)\big) < N$, we have

$$\{\sum_{i=1}^{n} \xi_i^2, \sum_{|\alpha|=K} \partial_\ell^N q_1^\alpha(x) \xi^\alpha\} = 0$$

for $\ell = 1, \dots, n$. Then Lemma 3.1 proves that $\partial_\ell^N q_1^\alpha(x)$ are polynomial functions of x for $|\alpha| = k$ and $\ell = 1, \dots, n$ and so are $q_1^\alpha(x)$. □

Lemma 3.5. *Let $Q_0(x,\partial)$, $Q_1(x,\partial)$ and $Q_2(x,\partial)$ be holomorphic differential operators defined on a connected open dense subset of the n-dimensional complex vector space E_c such that Q_0 is of the form*

$$Q_0(x,\partial) = \sum_{i=0}^{n} \partial_i^2 + q_0(x)$$

and they satisfy

$$[Q_0, Q_1] = [Q_0, Q_2] = 0,\ q_0(-x) = q_0(x).$$

Suppose there exist linearly independent vectors $\tau_1, \dots, \tau_n$ in E_c such that the operators Q_0, Q_1 and Q_2 are invariant under the parallel translations on E_c by the vectors τ_j for $j = 1, \dots, n$. Then ${}^tQ_1 = Q_1^-$, ${}^tQ_2 = Q_2^-$ and $[Q_1, Q_2] = 0$

Proof. First note that ${}^t(P^-) = ({}^tP)^-$ for any differential operator P. Put $S = Q_1 - {}^tQ_1^-$. Then $[Q_0, S] = -[Q_0, {}^tQ_1^-] = [Q_0, Q_1]^- = 0$ and Lemma 3.1 proves that $\sigma(S)$ is a polynomial function of (x,ξ) and hence the invariance by the parallel translations shows that $\sigma(S)$ does not depend on x. Combining this with ${}^tS^- = -S$, we can conclude $S = 0$ and therefore ${}^tQ_1 = Q_1^-$.

Put $R = [Q_1, Q_2]$. Since $[Q_0, R] = [Q_1, [Q_0, Q_2]] - [Q_2, [Q_0, Q_1]] = 0$, we have similarly ${}^tR = R^-$ and ${}^tQ_2 = Q_2^-$. Then

$$R = {}^t[Q_1, Q_2]^- = {}^t[Q_1^-, Q_2^-] = [{}^tQ_2^-, {}^tQ_1^-] = [Q_2, Q_1] = -R,$$

which proves the Lemma. □

The following proposition also gives a uniqueness for the commuting algebra.

Proposition 3.6. *Let $P_1, \dots, P_n$ be the commuting differential operators corresponding to the Weyl group of type A_n, B_n or D_n. Suppose the coefficients of P_j are holomorphic on a connected open dense subset of E_c and moreover suppose there exist linearly independent vectors $\tau_1, \dots, \tau_n$ of E_c such that P_i are invariant under the parallel translations by $\tau_j (i,\ j = 1, \dots, n)$. Let Q be a W-invariant differential operator with the same invariant property under the parallel translations. Then the condition $[P_1, Q] = 0$ implies $Q \in \mathbb{C}[P_1, \dots, P_n]$.*

Proof. This is a direct consequence of Lemma 3.5 and Proposition 3.3. □

Remark 3.7. In [OP3, §5 Proposition 1] and [OP2] it is claimed that W-invariant differential operators which commute with P_1 are completely determined by their terms of highest degree. But it is incorrect, which is clear by example (8.18). Note that if $\delta = -2\beta$ in (8.18), it corresponds to type I ($v(q) = q^{-2}$) for the root system B_2 under the notation in [OP3]. The same incorrect argument is used to prove the pairwise commutativity of $P_2, \dots, P_n$ (cf. [OP3, §5 Proposition 2 and Appendix E]).

The following lemma will be used in the proof of Theorem 6.5.

Lemma 3.8. *Let $p(x_1, \dots, x_n, \xi_1, \dots, \xi_n) = \sum_{|\alpha|=3} p_\alpha(x)\xi^\alpha$ be a homogeneous polynomial of ξ such that $p(x,\partial)$ is symmetric and invariant under the coordinate transformation $(x_1, x_2) \mapsto (-x_1, -x_2)$. Suppose $\{\sum \xi_i^2, p(x,\xi)\} = 0$.*

i) If $n > 4$, then $p(x,\xi) = 0$.

ii) If $n \geq 2$ and $p(x,\partial)$ is invariant under the coordinate change $x_1 \mapsto -x_1$, then $p(x,\xi) = 0$.

iii) If $n = 4$, then

(3.37)
$$p(x,\xi) = C \sum_{g \in \mathfrak{S}_4} g\big(\frac{1}{3!} x_2 x_3 x_4 \xi_1^3 - \frac{1}{2!} x_1 x_3 x_4 \xi_1^2 \xi_2 - \frac{1}{3!}(x_1^2 + x_2^2 + x_3^2 - x_4^2) x_4 \xi_1 \xi_2 \xi_3\big)$$

with a suitable constant C, where g naturally acts on suffices.

Proof. Since $p_\alpha(x)$ are polynomials by Lemma 3.1 i) and the assumption implies $2 \sum \xi_i \frac{\partial p}{\partial x_i} = 0$, we have

$$p(x,\xi) = h(x_2 - \frac{\xi_2}{\xi_1} x_1, x_3 - \frac{\xi_3}{\xi_1} x_1, \dots, x_n - \frac{\xi_n}{\xi_1} x_1, \xi_1, \dots, \xi_n)$$

with a suitable polynomial function h of $(2n-1)$-variables. Moreover $p_\alpha(x)$ are polynomials of x with degree at most three because $p(x,\xi)$ is a polynomial of ξ with degree at most three.

Put

$$p(x,\xi) = \sum a_3^i \xi_i^3 + \sum_{i \neq j} a_{21}^{ij} \xi_i^2 \xi_j + \sum_{i<j<k} a_{111}^{ijk} \xi_i \xi_j \xi_k$$

with polynomials a_3^i, a_{21}^{ij} and a_{111}^{ijk} of x. Then the coefficients of ξ_1^4, $\xi_1^3\xi_2$, $\xi_1^2\xi_2^2$, $\xi_1^2\xi_2\xi_3$ and $\xi_1\xi_2\xi_3\xi_4$ of the equation $\{\frac{1}{2}\sum \xi_i^2, p(x,\xi)\} = 0$ show

$$\begin{aligned} \partial_1 a_3^1 &= 0, \\ \partial_2 a_3^1 + \partial_1 a_{21}^{12} &= 0, \\ \partial_2 a_{21}^{12} + \partial_1 a_{21}^{21} &= 0, \\ \partial_3 a_{21}^{12} + \partial_2 a_{21}^{13} + \partial_1 a_{111}^{123} &= 0, \\ \partial_4 a_{111}^{123} + \partial_3 a_{111}^{124} + \partial_2 a_{111}^{134} + \partial_1 a_{111}^{234} &= 0, \end{aligned} \tag{3.38}$$

respectively.

Note that the assumption of the invariance says that a_3^1 changes into $-a_3^1$ under $(x_1, x_2) \mapsto (-x_1, -x_2)$. Moreover a_3^1 is symmetric for $(x_2, \dots, x_n)$ and $\deg a_3^1 \leq 3$. Hence the condition $\partial_1 a_3^1 = 0$ proves $a_3^1 = 0$ in the cases i) and ii) and $a_3^1 = C x_2 x_3 x_4$ with $C \in \mathbb{C}$ in the case iii).

Suppose the invariance in ii) and suppose $n \geq 2$. Then $\partial_1 a_{21}^{12} = \partial_2^2 a_{21}^{12} = 0$ and we can put $a_{21}^{12} = x_2 \phi(x_3, \dots, x_n)$ because a_{21}^{12} changes into $-a_{21}^{12}$ under $x_2 \mapsto -x_2$. But $2\phi = \partial_2 a_{21}^{12} + \partial_1 a_{21}^{21} = 0$ and therefore $a_{21}^{12} = 0$. Thus $\partial_1 a_{111}^{123} = 0$ and the invariance under $x_1 \mapsto -x_1$ proves $a_{111}^{123} = 0$.

It is easy to check that (3.37) satisfies $\{\sum \xi_i^2, p(x,\xi)\} = 0$ in the case $n = 4$ and hence subtracting the right hand side of (3.37) from $p(x,\xi)$, the proof is reduced to the case $a_1^3 = 0$.

Suppose $n \geq 4$ and $a_3^1 = 0$. Then we have similarly $a_{21}^{12} = x_2 \phi(x_3, \dots, x_n)$ by the invariance under $(x_1, x_2) \mapsto (-x_1, -x_2)$, which implies $a_{21}^{12} = \partial_1 a_{111}^{123} = 0$ as in the proof of ii) and we have $a_{111}^{123} = \big(C + C'(x_4^2 + \cdots + x_n^2)\big) x_4 \cdots x_n$ because a_{111}^{123} changes into $-a_{111}^{123}$ under $(x_1, x_4) \mapsto (-x_1, -x_4)$. Here $C' = 0$ if $n > 4$. Thus we have $a_{111}^{123} = 0$ by the last equation of (3.38). $\square$

4. Determination of the potential function - type A_n

In this section we consider the case when the root system is of type A_n. We have W-invariant differential operators

$$\Delta_1 = \sum_{1\le i\le m} \partial_i,$$
$$\Delta_2 = \sum_{1\le i<j\le m} \partial_i\partial_j + R(x), \tag{4.1}$$
$$\Delta_3 = \sum_{1\le i<j<k\le m} \partial_i\partial_j\partial_k + \sum_{1\le i\le m} a_1^i\partial_i + a_0$$

satisfying $[\Delta_1,\Delta_2] = [\Delta_2,\Delta_3] = [\Delta_1,\Delta_3] = 0$ and ${}^t\Delta_i = (-1)^i\Delta_i$ for $i = 1, 2$ and 3. Theorem 2.1 says the existence of an even function $u(t)$ with

$$R(x) = \sum_{1\le i<j\le m} u(x_i - x_j). \tag{4.2}$$

Moreover we have

Lemma 4.1. *There exist a constant number C with*

$$\Delta_3 - C\Delta_1 = \sum_{1\le i<j<k\le m} \partial_i\partial_j\partial_k + \sum_{i=1}^{m} \sum_{\substack{j,k\ne i\\ 1\le j<k\le m}} u(x_j - x_k)\partial_i$$

Proof. We remark that if

$$a_1^i = \sum_{\substack{j,k\ne i\\ 1\le j<k\le m}} u(x_j - x_k) \tag{4.3}$$

the functions a_1^i satisfy (2.4) and (2.5). Hence put

$$\bar a_1^i = a_1^i - \sum_{\substack{j,k\ne i\\ 1\le j<k\le m}} u(x_j - x_k).$$

Then the commutativity implies

$$\partial_i\bar a_1^i = \partial_j\bar a_1^i + \partial_i\bar a_1^j = 0 \quad \text{for} \ \ 1\le i<j\le m$$

and by Lemma 2.2 we have $\partial_j\partial_k\bar a_1^1 = 0$ for $j,\ k = 2,\dots,m$. Since $\bar a_1^1$ is symmetric for $(x_2,\dots,x_m)$,

$$\bar a_1^1 = C + C'(x_2 + \cdots + x_m)$$

with constant numbers C and C'. Now the equation $\sum_i \partial_i\bar a_1^1 = 0$ means $C' = 0$. Since Δ_3 is skew self-adjoint, we have Lemma 4.1. $\square$

Lemma 4.1 assures that we may assume (4.3). Then the above proof shows $\operatorname{ord}[\Delta_2, \Delta_3] \leq 1$. Since $[\Delta_2, \Delta_3]$ is self-adjoint, we can prove that the condition $[\Delta_2, \Delta_3] = 0$ is equals to

$$\sum_{1 \leq i < j < k \leq m} \partial_i \partial_j \partial_k R + \sum_{1 \leq k \leq m} a_1^k \partial_k R = 0 \tag{4.4}$$

by the 0-th order term of $[\Delta_2, \Delta_3]$. Applying (4.2) and (4.3) to (4.4), we have

$$\sum_{k=1}^{m} \Big\{ \Big(\sum_{\substack{\mu,\ \nu \neq k \\ 1 \leq \mu < \nu \leq m}} u(x_\mu - x_\nu) \Big) \partial_k \Big(\sum_{1 \leq i < j \leq m} u(x_i - x_j) \Big) \Big\} = 0. \tag{4.5}$$

Since the term containing $u'(x_i - x_j)$ with $i < j$ in the left hand side of (4.5) equals

$$\begin{aligned}
&\Big(\sum_{\mu < \nu,\ \mu, \nu \neq i} u(x_\mu - x_\nu) - \sum_{\mu < \nu,\ \mu, \nu \neq j} u(x_\mu - x_\nu) \Big) u'(x_i - x_j) \\
&= \Big(\sum_{k < i < j} u(x_k - x_j) + \sum_{i < k < j} u(x_k - x_j) + \sum_{i < j < k} u(x_j - x_k) \Big) \partial_i u(x_i - x_j) \\
&\quad + \Big(\sum_{k < i < j} u(x_k - x_i) + \sum_{i < k < j} u(x_i - x_k) + \sum_{i < j < k} u(x_i - x_k) \Big) \partial_j u(x_i - x_j),
\end{aligned}$$

we have

Proposition 4.2. *Under the above notation the necessary and sufficient condition for $[\Delta_1, \Delta_2] = [\Delta_2, \Delta_3] = [\Delta_1, \Delta_3] = 0$ equals*

$$\sum_{1 \leq i < j < k \leq m} U_{ijk}(u) = 0 \tag{4.6}$$

with

$$\begin{aligned}
U_{ijk}(u) = {} & u(x_j - x_k) \partial_i \big(u(x_i - x_j) + u(x_i - x_k) \big) \\
& + u(x_i - x_k) \partial_j \big(u(x_i - x_j) + u(x_j - x_k) \big) \\
& + u(x_i - x_j) \partial_k \big(u(x_i - x_k) + u(x_j - x_k) \big).
\end{aligned} \tag{4.7}$$

Now we solve equation (4.6) for u:

Theorem 4.3. *Let $u(t)$ be an even holomorphic function for $0 < |t| \ll 1$ satisfying (4.6). Then there exist complex numbers C_0 and C_1 such that*

$$u(t) = C_1 \wp(t | 2\omega_1, 2\omega_2) + C_0 \tag{4.8}$$

Here $\wp(t | 2\omega_1, 2\omega_2)$ is Weierstrass' elliptic function with primitive periods $2\omega_1$ and $2\omega_2$.

Conversely for complex numbers C_0, C_1, ω_1 and ω_2, the function given by (4.8) *satisfies* (4.6)*. Here ω_1 and ω_2 are complex numbers which are linearly independent over $\mathbb{R}$ and allowed to be ∞.*

Proof. Note that $\wp$ and $\wp'$ are even and odd functions, respectively. Then it is clear from the addition formula (cf. [WW]) of $\wp$-function

$$(4.9)\quad \begin{vmatrix} \wp(x) & \wp'(x) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \wp(z) & \wp'(z) & 1 \end{vmatrix} = 0 \quad \text{for complex numbers } \ x, y \text{ and } z \text{ with } \ x+y+z=0$$

that the function u given by (4.8) satisfies $U_{ijk}(u) = 0$ and therefore it is a solution of (4.6).

Let $u(t)$ be an even holomorphic function for $0 < |t| \ll 1$ satisfying (4.6). Put $s = x_i - x_j$ and $t = x_j - x_k$ and suppose $0 < |s| \ll |t| \ll 1$. Then

$$(4.10)\quad \begin{aligned} U_{ijk}(u) &= u(t)\big(u'(s) + u'(s+t)\big) + u(s+t)\big(-u'(s) + u'(t)\big) \\ &\qquad + u(s)\big(-u'(s+t) - u'(t)\big) \\ &= -\Big(\big(u(s+t) - u(t)\big)u'(s) + \big(u'(s+t) + u'(t)\big)u(s)\Big) + F(s,t) \end{aligned}$$

with a function $F(s,t)$. Here we note that $F(s,t)$ is holomorphic function of s at the origin if t is fixed with the condition $0 < |t| \ll 1$.

Now put $s = x_1 - x_2$ and $t_j = x_{j-1} - x_j$ for $j = 3, \ldots, m$. Fix complex numbers $t_3, \ldots, t_m$ with $0 < |t_j| \ll 1$ and suppose $0 < |s| \ll |t_j| \ll 1$ for $j = 3, \ldots, m$. Then condition (4.6) implies

$$(4.11)\quad \begin{aligned} -\sum_{j=3}^{m} \Big(&\big(u(s + t_3 + \cdots + t_j) - u(t_3 + \cdots + t_j)\big)u'(s) \\ &+ \big(u'(s + t_3 + \cdots + t_j) + u'(t_3 + \cdots + t_j)\big)u(s)\Big) = f(s) \end{aligned}$$

with a holomorphic function $f(s)$ on a neighborhood of the origin. Now we may assume the number

$$C = \sum_{j=3}^{m} u'(t_3 + \cdots + t_j)$$

is not zero for generic $t_3, \cdots, t_m$ and from (4.11) we have

$$(4.12)\quad -(C + c_1(s)s)su'(s) - 2(C + c_2(s)s)u(s) = f(s)$$

with holomorphic functions $c_1(s)$ and $c_2(s)$ on a neighborhood of the origin. Since the origin is the regular singular point for the differential equation (4.12) for u and its characteristic exponent equals -2, the origin is at most a pole of order 2 for u.

First suppose $u(s)$ is holomorphic at the origin. We may assume $u(0) = u'(0) = 0$ because $u + C'$ is also a solution of (4.6) for $C' \in \mathbb{C}$. Then

$$\begin{aligned} U_{ijk}(u)\big|_{x_j=x_k} &= u(x_i - x_j)\partial_j u(x_i - x_j) + u(x_i - x_j)\partial_j u(x_i - x_j) \\ &= -\partial_i \Big(u(x_i - x_j)^2 \Big), \\ U_{ijk}(u)\big|_{x_i=x_j=x_k} &= 0 \end{aligned}$$

and

$$\begin{aligned} \sum_{1\le i<j<k\le m} U_{ijk}(u)\big|_{x_2=x_3=\cdots=x_m} &= \sum_{1<j<k\le m} U_{1jk}(u)\big|_{x_2=x_j=x_k} \\ &= -\binom{m-1}{2}\partial_1 \Big(u(x_1 - x_2)^2 \Big). \end{aligned}$$

Hence $\frac{d}{dt}\big(u(t)^2\big) = 0$ and therefore $u(t)$ is constant, which implies $u = 0$.

Therefore replacing u by $C'u + C''$ with suitable $C', C'' \in \mathbb{C}$, we may assume

$$u(t) = t^{-2} + C_2 t^2 + C_4 t^4 + C_6 t^6 + \cdots$$

with complex numbers C_j. Then under the same notation in equation (4.10) we have

$$\begin{aligned} U_{ijk}(u) &= \frac{\partial}{\partial t}u(t)\big(u(s+t) - u(s)\big) - \frac{\partial}{\partial s}u(s)\big(u(s+t) - u(t)\big) \\ &= \frac{\partial}{\partial t}u(t)\Big\{ \big(u(t) + \frac{u^{(1)}(t)}{1!}s + \frac{u^{(2)}(t)}{2!}s^2 + \cdots \big) \\ &\qquad\qquad - (s^{-2} + C_2 s^2 + C_4 s^4 + \cdots)\Big\} \\ &\quad - \frac{\partial}{\partial s}\Big\{ (s^{-2} + C_2 s^2 + C_4 s^4 + \cdots) \\ &\qquad\qquad \cdot \big(\frac{u^{(1)}(t)}{1!}s + \frac{u^{(2)}(t)}{2!}s^2 + \frac{u^{(3)}(t)}{3!}s^3 + \cdots \big)\Big\}. \end{aligned} \tag{4.13}$$

The coefficient of s^{-2} in the expansion (4.13) equals $-u^{(1)}(t) + u^{(1)}(t) = 0$ and therefore (4.13) is holomorphic at $s = 0$.

The coefficient of s^0 equals

$$\frac{\partial}{\partial t}u(t)^2 - \frac{1}{6}u^{(3)}(t) = \frac{d}{dt}(t^{-2} + C_2 t^2 + C_4 t^4 + \cdots)^2 - \frac{1}{6}(-4!t^{-5} + 4!C_4 t + \cdots),$$

which is holomorphic at the origin and takes the value zero at the point. Thus we have

$$\begin{aligned} \sum_{1\le i<j<k\le m} U_{ijk}(u)\Big|_{x_1=x_2} &= \sum_{2<k} \big\{ 2u(x_2 - x_k)u'(x_2 - x_k) - \frac{1}{6}u^{(3)}(x_2 - x_k) \big\} \\ &\quad + 2\sum_{2<j<k\le m} U_{2jk}(u) + \sum_{2<i<j<k\le m} U_{ijk}(u). \end{aligned}$$

and by the induction on ℓ it is easy to show

$$\Bigl(\cdots\Bigl(\sum_{1\le i<j<k\le m} U_{ijk}(u)\Big|_{x_1=x_2}\Bigr)\Big|_{x_2=x_3}\cdots\Bigr)\Big|_{x_{\ell-1}=x_\ell}$$
$$= \bigl(1+2+\cdots+(\ell-1)\bigr)\sum_{\ell<k}\bigl\{2u(x_\ell-x_k)u'(x_\ell-x_k)-\frac{1}{6}u^{(3)}(x_\ell-x_k)\bigr\}$$
$$+\ell\sum_{\ell<j<k\le m}U_{\ell jk}(u)+\sum_{\ell<i<j<k\le m}U_{ijk}(u).$$

When $\ell = m-1$, we have

$$\frac{(m-2)(m-1)}{2}\Bigl(2u(x_{m-1}-x_m)u'(x_{m-1}-x_m)-\frac{1}{6}u^{(3)}(x_{m-1}-x_m)\Bigr)=0$$

and thus

$$\begin{aligned} 2u^{(3)} &= 12(u^2)', \\ 2u^{(2)} &= 12u^2 - g_2, \\ 2u'u'' &= 12u^2u' - g_2u', \\ (u')^2 &= 4u^3 - g_2u - g_3 \end{aligned}$$

with suitable complex numbers g_2 and g_3. Since u has a pole of order 2 at the origin, this differential equation implies that u is Weierstrass' elliptic function. $\square$

Remark 4.4. The claim of Theorem 4.3 follows from the commutativity of the operators Δ_i for $i = 1$, 2 and 3. In fact we does not use the existence of the commuting operators Δ_i of order > 3 for the proof.

5. Construction of commuting families - type A_n

As in the previous section we assume the root system is of type A_n and we shall construct a commuting family of differential operators Δ_j for $j = 1, \dots, m = n+1$. In fact we shall prove the operators

$$\Delta_k = \sum_{0\le\ell\le\frac{k}{2}}\ \sum_{\substack{J\subset[1,m]\\ |J|=k-2\ell}}\Bigl\{\Bigl(\sum_{\Lambda\in\Sigma(J;\ell)}\prod_{\alpha\in\Lambda}u(\langle\alpha,x\rangle)\Bigr)\prod_{j\in J}\frac{\partial}{\partial x_j}\Bigr\} \tag{5.1}$$

form a commuting family for any function u given by Theorem 4.3. Here

$$\Sigma(J;\ell) = \bigl\{\{\beta_1,\dots,\beta_\ell\};\ \beta_1,\dots,\beta_\ell, e_j\ (j\in J)\ \text{ are orthogonal to each other and }\ \beta_i\in\Sigma^+\bigr\}$$

for $J \subset [1,m]$ and we define

$$\sum_{\Lambda\in\Sigma(J;0)}\prod_{\alpha\in\Lambda}u(\langle\alpha,x\rangle) = 1 \quad\text{and}\quad \prod_{j\in\emptyset}\frac{\partial}{\partial x_j} = 1.$$

We may write them in the following way:

$$\Delta_k = \sum_{0\le\ell\le\frac{k}{2}} \sum_{g\in\mathfrak{S}_m} \frac{1}{\#G(\ell,k-2\ell)} g(L_{\ell,k-2\ell}) \tag{5.2}$$

by putting

$$L_{i,j} = u(x_1-x_2)u(x_3-x_4)\cdots u(x_{2i-1}-x_{2i})\frac{\partial^j}{\partial x_{2i+1}\partial x_{2i+2}\cdots\partial x_{2i+j}}. \tag{5.3}$$

Here $\mathfrak{S}_m$ is the permutation group of the set $[1,m]=\{1,\ldots,m\}$ and we denote $G(i,j)=\{g\in\mathfrak{S}_m;\ g(L_{i,j})=L_{i,j}\}$.

Lemma 5.1. *These operators satisfy*

$$[\Delta_k,\Delta_1]=[\Delta_k,\Delta_2]=0,\ {}^t\Delta_k=\Delta_k^-=(-1)^k\Delta_k \quad \textit{for} \quad k=1,\ldots,m. \tag{5.4}$$

Proof. Note that $[\Delta_k,\Delta_1]=0$ and ${}^t\Delta_k=\Delta_k^-=(-1)^k\Delta_k$ are clear by definition. Furthermore it is easy to see that (5.1) implies that the commutator $[\Delta_k,\Delta_2]$ vanishes except for the terms $\partial_{j_1}\cdots\partial_{j_\ell}$ with $j_1<j_2<\cdots<j_\ell$.

Suppose $Q=[\Delta_k,\Delta_2]\neq 0$. Since ${}^tQ=(-1)^{k-1}Q$, the order of Q is odd if k is even and even otherwise. Let $k-2N-1$ be the order of Q with a nonnegative integer N and put $I=[k-2N,m]$ and $J=[1,k-2N-1]$. Then the coefficient of $\partial_1\cdots\partial_{k-2N-1}$ of $[\Delta_k,\Delta_2]$ equals

$$\begin{aligned}
&\sum_{i=k-2N}^{m}\ \sum_{\Lambda\in\Sigma(J\cup\{i\};N)} u_\Lambda(x)\partial_i\sum_{\nu<\mu}u(x_\nu-x_\mu)\\
&\quad-\sum_{i=1}^{k-2N-1}\partial(i)\sum_{\Lambda\in\Sigma(J\setminus\{i\};N+1)}u_\Lambda(x)\\
&=\sum_{i=k-2N}^{m}\ \sum_{\Lambda\in\Sigma(J\cup\{i\};N)}u_\Lambda(x)\Bigl(\sum_{\nu\neq i}u'(x_i-x_\nu)\Bigr)\\
&\quad+\sum_{i=1}^{k-2N-1}\ \sum_{j=k-2N}^{m}\ \sum_{\Lambda\in\Sigma(J\cup\{j\};N)}u_\Lambda(x)u'(x_i-x_j)
\end{aligned} \tag{5.5}$$

by denoting

$$u_\Lambda(x)=\prod_{\alpha\in\Lambda}u(\langle\alpha,x\rangle). \tag{5.6}$$

Hence for $k-2N\le\nu<\mu\le m$, the sum of the terms in (5.5) which contain $u'(x_\nu-x_\mu)$ equals

(5.7)
$$\begin{aligned}
&\Bigl(\sum_{\Lambda\in\Sigma(J\cup\{\nu\};N)}u_\Lambda(x)-\sum_{\Lambda\in\Sigma(J\cup\{\mu\};N)}u_\Lambda(x)\Bigr)u'(x_\nu-x_\mu)\\
&=\sum_{i\in I\setminus\{\nu,\mu\}}\ \sum_{\Lambda\in\Sigma(J\cup\{\mu,\nu,i\};N-1)}u_\Lambda(x)\bigl(u(x_i-x_\nu)-u(x_i-x_\mu)\bigr)u'(x_\nu-x_\mu).
\end{aligned}$$

Since we have

$$\sum_{k-2N\le\nu<\mu\le m}\ \sum_{i\in I\setminus\{\mu,\nu\}}\Big(u(x_i-x_\mu)\partial_\nu u(x_\nu-x_\mu)+u(x_i-x_\nu)\partial_\mu u(x_\nu-x_\mu)\Big)=0$$

from the addition formula (4.9), the terms (5.7) cancel out if we sum up them for all ν and μ satisfying $k-2N\le\nu<\mu\le m$. On the other hand, in the expression (5.5) it is easy to see that the terms $u'(x_i-x_\nu)$ for $i\in[k-2N,m]$ and $\nu\in[1,k-2N-1]$ vanish. This assures the vanishing of the term of order $k-2N-1$ of Q, which contradicts the assumption. Thus we have the Lemma. □

Now we can state our main theorem when the root system is of type A_n.

Theorem 5.2. *i) For Weierstrass' elliptic function $\wp(t|2\omega_1,2\omega_2)$ and any complex numbers C_0 and C_1 we put*

$$u(t)=C_1\wp(t|2\omega_1,2\omega_2)+C_0. \tag{5.8}$$

Then the differential operators Δ_k given by (5.1) *satisfy*

$$\begin{aligned}&[\Delta_i,\Delta_j]=0\quad\textit{for}\quad 1\le i<j\le m,\\ &{}^t\Delta_i=\Delta_i^-=(-1)^i\Delta_i\quad\textit{for}\quad 1\le i\le m.\end{aligned} \tag{5.9}$$

Here we note that ω_1 and ω_2 are allowed to be infinity.

ii) Let Ω be a W-invariant connected open neighborhood of the origin of $\mathbb{C}^m$. Let $\mathbb{D}(A_n)$ be a commutative algebra generated by suitable W-invariant differential operators whose coefficients are holomorphic on an open dense subset Ω' of Ω such that $\Omega\setminus\Omega'$ is an analytic subset of Ω. Suppose $\mathbb{D}(A_n)$ contains the operators

$$\Bigg(\sum_{1\le i_1<\cdots<i_k\le m}\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}}\Bigg)+R_k(x,\partial)\quad\textit{for}\quad k=1,\cdots,m. \tag{5.10}$$

Here $R_k(x,\partial)$ are differential operators of order $\le k-1$. Furthermore suppose $R_1(x,\partial)=0$, $\operatorname{ord}R_2(x,\partial)\le 0$ and $\operatorname{ord}R_3(x,\partial)\le 1$. Then $\mathbb{D}(A_n)$ coincides with $\mathbb{C}[\Delta_1,\ldots,\Delta_m]$ which is determined by a suitable function u of the form (5.8).

Proof. Owing to Theorem 3.2, Proposition 3.3, Theorem 4.3 and Lemma 5.1, we have only to prove the commutativity of Δ_j. But it follows from Lemma 3.5 and the analytic continuation for the parameters of $\wp(t)$. □

Remark 5.3. It is clear that the commuting algebra $\mathbb{C}[\Delta_1,\cdots,\Delta_m]$ in Theorem 5.2 stays invariant even if we change the constant number C_0 in (5.8).

Furthermore it is easy to show that if we consider C_0 as an element which commutes with any differential operator and consider the differential operators Δ_m and $\Delta'_m=[\Delta_m,x_1+\cdots+x_m]$ defined by (5.1) as a polynomial of C_0, then their coefficients of C_0^k for $k=0,1,\ldots$ form a complete set of generators of the commuting algebra.

6. A functional differential equation satisfied by the potential function - type B_n and D_n

Hereafter in this paper we shall study the case when the root system is of type B_n with $n > 1$ or of type D_n with $n > 3$. In this section we examine W-invariant differential operators P_1 and P_2 of the form

$$
\begin{aligned}
P_1 &= \sum_{1\le i\le n} \partial_i^2 + R(x),\\
R(x) &= \sum_{1\le i<j\le n} \big(u(x_i+x_j)+u(x_i-x_j)\big) + \sum_{1\le i\le n} v(x_i),\\
P_2 &= \sum_{1\le i<j\le n} \partial_i^2\partial_j^2 + \sum_{1\le i\le n} a_2^i\partial_i^2 + \sum_{1\le i<j\le n} a_{11}^{ij}\partial_i\partial_j + \sum_{1\le i\le n} a_1^i\partial_i + a_0
\end{aligned}
\tag{6.1}
$$

which satisfy $[P_1, P_2] = 0$ and ${}^tP_2 = P_2$.

The term ∂_i of $[P_1, P_2]$ gives

$$
\sum_{1\le\nu\le n} \partial_\nu^2 a_1^i + 2\partial_i a_0 = \sum_{\substack{\nu\ne i\\ 1\le\nu\le n}} 2\partial_i\partial_\nu^2 R + 2a_2^i\partial_i R + \sum_{\substack{\nu\ne i\\ 1\le\nu\le n}} a_{11}^{i\nu}\partial_\nu R. \tag{6.2}
$$

We may assume that a_2^i and a_{11}^{ij} are given by (2.21). Furthermore we may assume $v = 0$ if the root system is of type D_n, which follows from the argument in the last part of §2.

The condition ${}^tP_2 = P_2$ is equivalent to

$$
a_1^i = \frac{1}{2}\sum_{\substack{\nu\ne i\\ 1\le\nu\le n}} \big(u'(x_i+x_\nu)+u'(x_i-x_\nu)\big) \tag{6.3}
$$

and from (6.2) we have

$$
\begin{aligned}
2\partial_i a_0 &= \sum_{\nu\ne i} 2\partial_i\partial_\nu^2 R - \sum_{\nu}\partial_\nu^2 a_1^i + 2a_2^i\partial_i R + \sum_{\nu\ne i} a_{11}^{i\nu}\partial_\nu R\\
&= \sum_{\nu\ne i}\big(u^{(3)}(x_i+x_\nu)+u^{(3)}(x_i-x_\nu)\big)\\
&\quad + 2\Bigg(\sum_{\substack{\nu,\mu\ne i\\ \nu<\mu}}\big(u(x_\nu+x_\mu)+u(x_\nu-x_\mu)\big) + \sum_{\nu\ne i} v(x_\nu)\Bigg)\\
&\qquad \cdot\Bigg(\sum_{\nu\ne i}\big(u'(x_i+x_\nu)+u'(x_i-x_\nu)\big)+v'(x_i)\Bigg)\\
&\quad + \sum_{\nu\ne i}\Bigg\{\Big(u(x_i+x_\nu)-u(x_i-x_\nu)\Big)\\
&\qquad \cdot\Bigg(\sum_{\mu\ne\nu}\big(u'(x_\mu+x_\nu)-u'(x_\mu-x_\nu)\big)+v'(x_\nu)\big)\Bigg)\Bigg\}.
\end{aligned}
\tag{6.4}
$$

Theorem 6.1.

i) Under the above notation the condition $[P_1, P_2] = 0$ *is equivalent to the existence of a* W*-invariant function* a_0 *satisfying* (6.4).

ii) The compatibility condition of the integrability of equation (6.4) *for* a_0 *is*

(6.5)

$$\begin{aligned}
&\Big((n-2)\big(u''(x_1+x_3)+u''(x_1-x_3)\big)+v''(x_1)\Big)\Big(u(x_1+x_2)-u(x_1-x_2)\Big)\\
&+3\Big((n-2)\big(u'(x_1+x_3)+u'(x_1-x_3)\big)+v'(x_1)\Big)\Big(u'(x_1+x_2)+u'(x_1-x_2)\Big)\\
&+2\Big((n-2)\big(u(x_1+x_3)+u(x_1-x_3)\big)+v(x_1)\Big)\Big(u''(x_1+x_2)-u''(x_1-x_2)\Big)\\
&+(n-2)\Big(u''(x_1+x_3)-u''(x_1-x_3)\Big)\Big(u(x_1+x_3)-u(x_1-x_3)\Big)\\
&=\Big((n-2)\big(u''(x_2+x_3)+u''(x_2-x_3)\big)+v''(x_2)\Big)\Big(u(x_1+x_2)-u(x_1-x_2)\Big)\\
&+3\Big((n-2)\big(u'(x_2+x_3)+u'(x_2-x_3)\big)+v'(x_2)\Big)\Big(u'(x_1+x_2)-u'(x_1-x_2)\Big)\\
&+2\Big((n-2)\big(u(x_2+x_3)+u(x_2-x_3)\big)+v(x_2)\Big)\Big(u''(x_1+x_2)-u''(x_1-x_2)\Big)\\
&+(n-2)\Big(u''(x_2+x_3)-u''(x_2-x_3)\Big)\Big(u(x_2+x_3)-u(x_2-x_3)\Big).
\end{aligned}$$

Proof. Suppose there exists a W-invariant function satisfying (6.4). Since the commutator satisfies ${}^t[P_1, P_2] = -[P_1, P_2]$, the order of $[P_1, P_2]$ equals 1 or 3 or 5 if it is not zero.

It is clear that the order is smaller than 5. Furthermore equations (2.21) and (6.4) assure the vanishing of the 3-rd and and first order terms, respectively. Hence we have the first statement of the theorem.

Note that the function R and the operator $\sum_i a_1^i \partial_i$ are symmetric with respect to the coordinate. Therefore the compatibility condition for equation (6.4) equals

$$\partial_2 U(x_1, x_2, x') = \partial_1 U(x_2, x_1, x') \tag{6.6}$$

by putting

$$U(x_1, x_2, x') = \sum_{\nu\neq 1} 2\partial_1\partial_\nu^2 R - \sum_\nu \partial_\nu^2 a_1^1 + 2a_2^1\partial_1 R + \sum_{\nu\neq 1} a_{11}^{1\nu}\partial_\nu R. \tag{6.7}$$

with $x' = (x_3, \dots, x_n)$. Defining a symmetric function

$$
\begin{aligned}
S(x) = \sum_{\mu<\nu} \Big(u^{(2)}(x_\mu + x_\nu) + u^{(2)}(x_\mu - x_\nu) + 2v(x_\mu)v(x_\nu) \Big) \\
+ \Big(\sum_{\mu<\nu} \big(u(x_\mu + x_\nu) + u(x_\mu - x_\nu) \big) \Big)^2 \\
+ \frac{1}{2} \sum_{\mu<\nu} \big(u(x_\mu + x_\nu) - u(x_\mu - x_\nu) \big)^2 \\
+ 2 \sum_k v(x_k) \Big(\sum_{\mu<\nu,\ \mu,\nu\neq k} \big(u(x_\mu + x_\nu) - u(x_\mu - x_\nu) \big) \Big),
\end{aligned}
\tag{6.8}
$$

we have

(6.9)

$$
\begin{aligned}
U(x_1, x_2, x') - \partial_1 S(x) = \sum_{\nu>1} \big(u(x_1 + x_\nu) - u(x_1 - x_\nu) \big) v'(x_\nu) \\
+ 2 \sum_{\nu>1} \big(u'(x_1 + x_\nu) + u'(x_1 - x_\nu) \big) v(x_\nu) \\
- 2 \Big(\sum_{\nu>1} \big(u(x_1 + x_\nu) + u(x_1 - x_\nu) \big) \Big) \Big(\sum_{\nu>1} \big(u'(x_1 + x_\nu) + u'(x_1 - x_\nu) \big) \Big) \\
+ \Big\{ \sum_{\nu>1} \big(u(x_1 + x_\nu) - u(x_1 - x_\nu) \big) \Big(\sum_{\mu\neq 1,\nu} \big(u'(x_\mu + x_\nu) - u'(x_\mu - x_\nu) \big) \Big) \Big\}
\end{aligned}
$$

Then

(6.10)

$$
\begin{aligned}
&\partial_2 U(x_1, x_2, x') - \partial_1 \partial_2 S(x) \\
&= \big(u(x_1 + x_2) - u(x_1 - x_2) \big) v''(x_2) \\
&\quad + 3 \big(u'(x_1 + x_2) + u'(x_1 - x_2) \big) v'(x_2) \\
&\quad + 2 \big(u''(x_1 + x_2) - u''(x_1 - x_2) \big) v(x_2) \\
&\quad - 2 \big(u'(x_1 + x_2) - u'(x_1 - x_2) \big) \big(u'(x_1 + x_2) + u'(x_1 - x_2) + \partial_1 W(x_1, x') \big) \\
&\quad - 2 \big(u(x_1 + x_2) + u(x_1 - x_2) + W(x_1, x') \big) \big(u''(x_1 + x_2) - u''(x_1 - x_2) \big) \\
&\quad + \big(u'(x_1 + x_2) + u'(x_1 - x_2) \big) \partial_2 W(x_2, x') \\
&\quad + \big(u(x_1 + x_2) + u(x_1 - x_2) \big) \partial_2^2 W(x_2, x') \\
&\quad + \sum_{\nu>2} \big(u(x_1 + x_\nu) - u(x_1 - x_\nu) \big) \big(u''(x_2 + x_\nu) - u''(x_2 - x_\nu) \big)
\end{aligned}
$$

by putting

$$
W(x_1, x') = \sum_{3\leq\nu\leq n} \big(u(x_1 + x_\nu) + u(x_1 - x_\nu) \big).
\tag{6.11}
$$

Denoting

(6.12)
$$\begin{aligned}
&Q(x_1,x_2,x')\\
&=\big(\partial_1^2 W(x_1,x')+v''(x_1)\big)\big(u(x_1+x_2)-u(x_1-x_2)\big)\\
&\quad+3\big(\partial_1 W(x_1,x')+v'(x_1)\big)\big(u'(x_1+x_2)-u'(x_1-x_2)\big)\\
&\quad+2\big(W(x_1,x')+v(x_1)\big)\big(u''(x_1+x_2)-u''(x_1-x_2)\big)\\
&\quad+\sum_{\nu\geq 3}\big(u''(x_1+x_\nu)-u''(x_1-x_\nu)\big)\big(u(x_2+x_\nu)-u(x_2-x_\nu)\big),
\end{aligned}$$

the compatibility condition (6.6) can be stated as

$$Q(x_1,x_2,x')=Q(x_2,x_1,x'). \tag{6.13}$$

Then if we put $x_3=x_4=\cdots=x_n$, we obtain (6.5).

On the other hand, if (6.5) holds, the function

$$\begin{aligned}
&D(x_1,x_2,x_3)\\
&=\big(u''(x_1+x_3)+u''(x_1-x_3)\big)\big(u(x_1+x_2)-u(x_1-x_2)\big)\\
&\quad+3\big(u'(x_1+x_3)+u'(x_1-x_3)\big)\big(u'(x_1+x_2)-u'(x_1-x_2)\big)\\
&\quad+2\big(u(x_1+x_3)+u(x_1-x_3)\big)\big(u''(x_1+x_2)-u''(x_1-x_2)\big)\\
&\quad+\big(u''(x_1+x_3)-u''(x_1-x_3)\big)\big(u(x_2+x_3)-u(x_2-x_3)\big)\\
&\quad-\big(u''(x_2+x_3)+u''(x_2-x_3)\big)\big(u(x_1+x_2)-u(x_1-x_2)\big)\\
&\quad-3\big(u'(x_2+x_3)+u'(x_2-x_3)\big)\big(u'(x_1+x_2)+u'(x_1-x_2)\big)\\
&\quad-2\big(u(x_2+x_3)+u(x_2-x_3)\big)\big(u''(x_1+x_2)-u''(x_1-x_2)\big)\\
&\quad-\big(u''(x_2+x_3)-u''(x_2-x_3)\big)\big(u(x_1+x_3)-u(x_1-x_3)\big)
\end{aligned} \tag{6.14}$$

does not depend on x_3 and therefore (6.13) holds. □

Remark 6.2. If we put

$$\begin{aligned}
A(x_1,x_3)&=(n-2)\big(u(x_1+x_3)+u(x_1-x_3)\big)+v(x_1)\\
U(x_1,x_2)&=u(x_1+x_2)-u(x_1-x_2)\\
B(x_1,x_2,x_3)&=(n-2)\big(u(x_1+x_3)-u(x_1-x_3)\big)\big(u(x_2+x_3)-u(x_2-x_3)\big)\\
C(x_1,x_2,x_3)&=\frac{\partial}{\partial x_1}\bigg(2A(x_1,x_3)\frac{\partial U(x_1,x_2)}{\partial x_1}+\frac{\partial A(x_1,x_3)}{\partial x_1}U(x_1,x_2)\\
&\qquad+\frac{\partial B(x_1,x_2,x_3)}{\partial x_1}\bigg),
\end{aligned}$$

condition (6.5) is equivalent to

$$C(x_1,x_2,x_3)=C(x_2,x_1,x_3). \tag{6.15}$$

When the root system is of type B_2, we can state our result in this section as follows.

Proposition 6.3. *Suppose the root system is of type B_2.*

i) In Theorem 2.1 we can choose functions $u(t)$, $v(t)$ and $T(x,y)$ such that

$$
\begin{aligned}
&R(x,y) = u(x+y) + u(x-y) + v(x) + v(y),\\
&2\frac{\partial}{\partial y}T(x,y) = v'(x)\big(u(x+y)-u(x-y)\big) + 2v(x)\big(u'(x+y)-u'(x-y)\big)
\end{aligned}
\tag{6.16}
$$

and

$$
T(x,y) = T(y,x). \tag{6.17}
$$

ii) Assume that for given functions $u(t)$ and $v(t)$ there exists a function $T(x,y)$ satisfying (6.16) *and* (6.17)*. Then the following two differential operators are commutative.*

$$
\begin{aligned}
P_1 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + u(x+y) + u(x-y) + v(x) + v(y),\\
P_2 &= \frac{\partial^4}{\partial x^2 \partial y^2} + v(y)\frac{\partial^2}{\partial x^2} + v(x)\frac{\partial^2}{\partial y^2} + \big(u(x+y)-u(x-y)\big)\frac{\partial^2}{\partial x \partial y}\\
&\quad + \frac{1}{2}\big(u'(x+y)+u'(x-y)\big)\frac{\partial}{\partial x} + \frac{1}{2}\big(u'(x+y)-u'(x-y)\big)\frac{\partial}{\partial y}\\
&\quad + \frac{\big(u(x+y)-u(x-y)\big)^2}{4} + \frac{u''(x+y)+u''(x-y)}{2} + v(x)v(y) + T(x,y)\\
&= \left(\frac{\partial^2}{\partial x \partial y} + \frac{u(x+y)-u(x-y)}{2}\right)^2 + v(y)\frac{\partial^2}{\partial x^2} + v(x)\frac{\partial^2}{\partial y^2} + v(x)v(y)\\
&\quad + T(x,y).
\end{aligned}
$$

iii) We can choose functions $u(t)$ and $v(t)$ in Theorem 2.1 such that

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}\Big(v(x)\big(u(x+y)-u(x-y)\big)\Big) + \frac{\partial}{\partial x}\Big(v(x)\frac{\partial}{\partial x}\big(u(x+y)-u(x-y)\big)\Big)\\
&= \frac{\partial^2}{\partial y^2}\Big(v(y)\big(u(x+y)-u(x-y)\big)\Big) + \frac{\partial}{\partial y}\Big(v(y)\frac{\partial}{\partial y}\big(u(x+y)-u(x-y)\big)\Big).
\end{aligned}
\tag{6.18}
$$

iv) If a pair $\big(u(t),v(t)\big) = \big(u_0(t),v_0(t)\big)$ is a solution of (6.18)*, then the pair $\big(u(t),v(t)\big) = \big(C_1u_0(Ct)+C_2, C_1'v_0(Ct)+C_2'\big)$ for complex numbers C, C_1, C_1', C_2 and C_2' with $C \neq 0$ and the pair $\big(u(t),v(t)\big) = \big(v_0(\frac{t}{\sqrt{2}}), u_0(\sqrt{2}t)\big)$ also satisfy* (6.18).

Proof. The first and the second claims follow from the proof of Theorem 6.1. In fact putting $x = x_2$ and $y = x_1$, they follow from (2.21), (6.3), (6.4), (6.8) and (6.9) and the fact that the right hand side of (6.16) equals $\frac{\partial}{\partial y}\big(2a_0 - S - (u(x+y)-u(x-y))^2\big)$.

The third claim is obvious from Remark 6.2 and the first pair $\big(u(t),v(t)\big)$ in iv) clearly satisfies (6.18).

The last pair in iv) is obtained by the fact that the coordinate transformation

$$X = \frac{1}{\sqrt{2}}(x-y),\ Y = \frac{1}{\sqrt{2}}(x+y) \tag{6.19}$$

gives an automorphism of the Weyl group B_2 which is identified with a group of linear transformations of $\mathbb{R}^2$. In fact, we can rewrite (6.18) in the form

$$\begin{aligned}&\big(v(x)-v(y)\big)u''(x+y)+3\big(v'(x)-v'(y)\big)u'(x+y)+2\big(v''(x)-v''(y)\big)u(x+y)\\ =&\ \big(v(x)-v(y)\big)u''(x-y)+3\big(v'(x)+v'(y)\big)u'(x-y)+2\big(v''(x)-v''(y)\big)u(x-y)\end{aligned}$$

and the transformation (6.19) proves the second claim. □

When the rank of the root system is larger than 2, we have

Proposition 6.4. *i) If the root system is of type D_n with $n \geq 3$, then the function u in Theorem 2.1 satisfies* (6.18) *with $v = u$.*

ii) If the root system is of type B_n with $n \geq 3$ or of type D_n with $n \geq 3$, then we can choose the function u in Theorem 2.1 such that

$$\begin{aligned}&u^{(4)}(x)\big(u(x+y)-u(x-y)\big)+3u^{(3)}(x)\big(u^{(1)}(x+y)-u^{(1)}(x-y)\big)\\ &+2u^{(2)}(x)\big(u^{(2)}(x+y)-u^{(2)}(x-y)\big)+4u^{(3)}(x)u^{(1)}(y)\\ =&\ u^{(4)}(y)\big(u(x+y)-u(x-y)\big)+3u^{(3)}(y)\big(u^{(1)}(x+y)+u^{(1)}(x-y)\big)\\ &+2u^{(2)}(y)\big(u^{(2)}(x+y)-u^{(2)}(x-y)\big)+4u^{(3)}(y)u^{(1)}(x).\end{aligned} \tag{6.20}$$

iii) Let $u(t)$ and $E(x,y,z)$ be functions which satisfy

$$\begin{aligned}\frac{\partial}{\partial y}E(x,y,z) =&\ 2\Big(u(x+z)+u(x-z)\Big)\Big(\frac{\partial}{\partial x}\big(u(x+y)-u(x-y)\big)\Big)\\ &+\Big(\frac{\partial}{\partial x}\big(u(x+z)+u(x-z)\big)\Big)\Big(u(x+y)-u(x-y)\Big)\\ &+\Big(\frac{\partial}{\partial x}\big(u(x+z)-u(x-z)\big)\Big)\Big(u(y+z)-u(y-z)\Big),\end{aligned} \tag{6.21}$$

$$E(x,y,z) = E(y,x,z). \tag{6.22}$$

Then $D(x,y,z) = 0$ for the function D defined by (6.14).

Furthermore if $u(t)$ is holomorphic on Ω', then (6.20) *is also valid.*

Proof. The first claim is clear by putting $v = 0$ and $x_3 = 0$ in (6.15).

Applying $\frac{\partial^2}{\partial x_3^2}$ to (6.15) and moreover putting $x_1 = x$, $x_2 = y$ and $x_3 = 0$, we obtain (6.20).

Since $D(x,y,z) = \frac{\partial}{\partial x}E(x,y,z) - \frac{\partial}{\partial y}E(y,x,z)$, equations (6.21) and (6.22) imply $D(x,y,z) = 0$ and therefore we obtain (6.20) in the same way as the proof of the claim ii). □

For the uniqueness of our commuting family we have the following

Theorem 6.5. *Let $\{P_1, \dots, P_n\}$ be a family of differential operators of the form* (0.2) *which satisfies* (0.3). *Suppose the root system is of type D_n with $n \geq 4$ or of type B_n with $n \geq 2$. We may assume that the principal symbol of P_2 equals $\sum_{i<j} \xi_i^2 \xi_j^2$ and that u and v in* (2.20) *satisfy* (2.21). *Then the commuting algebra $\mathbb{C}[P_1, \dots, P_n]$ is uniquely determined by the pair (u, v). Here we put $v = 0$ in the case of type D_n.*

Proof. If ${}^tP_2 = P_2$, we have the theorem from Theorem 3.2 and the proof of Theorem 6.1. Put $P_2 = \sum_{i<j} \partial_i^2 \partial_j^2 + R_2$. Let $\sigma(R_2)$ denote the principal symbol of R_2. Then we have $\{\sum \xi_i^2, \sigma(R_2)\} = 0$.

Suppose $\operatorname{ord} R_2 = 3$. Since R_2 is W-invariant, Lemma 3.8 implies that W is of type D_4 and $\sigma(R_2)$ equals the right hand side of (3.37) with $C \neq 0$. Then we may assume

$$P_4 = \partial_1 \partial_2 \partial_3 \partial_4 + R_4$$

with $\operatorname{ord} R_4 \leq 3$. Let $\sigma_3(R_4)$ denote the symbol R_4 of order 3. Note that $[P_1, P_2] = [P_1, P_4] = [P_2, P_4] = 0$. Hence $\sigma_3(R_4) = C'\sigma(R_2)$ with some $C' \in \mathbb{C}$ and by the equality $[P_4 - C'P_2, P_2] = 0$, we have

$$\begin{aligned}&\{\xi_1\xi_2\xi_3\xi_4 - C' \sum_{1\leq i<j\leq n} \xi_i^2\xi_j^2,\\ &C \sum_{g\in\mathfrak{S}_4} g\big(\frac{1}{3!}x_2x_3x_4\xi_1^3 - \frac{1}{2!}x_1x_3x_4\xi_1^2\xi_2 - \frac{1}{3!}(x_1^2+x_2^2+x_3^2-x_4^2)x_4\xi_1\xi_2\xi_3\big)\} = 0.\end{aligned}$$

Then the coefficients of $\xi_1^5\xi_2$ in the above shows $-2CC'x_3x_4 = 0$ and hence $C' = 0$. The coefficients of $\xi_1^4\xi_2\xi_3$ prove $-Cx_2x_3 = 0$. This leads the contradiction because $C \neq 0$.

Thus we have $\operatorname{ord} R_2 \leq 2$. Put $Q = P_2 - {}^tP_2$. Suppose $Q \neq 0$, then $\operatorname{ord} Q = 1$ and $[P_1, Q] = 0$. But it is easy to see that the equation $\{\sum \xi_i^2, \sigma(Q)\} = 0$ never holds for differential operator of order 1 if $\sigma(Q)(x, \partial)$ is symmetric. Thus we can conclude $P_2 = {}^tP_2$. $\square$

7. Solutions of the functional differential equation - type B_n and D_n

In this section we want to solve the functional differential equations (6.18) and (6.20).

Lemma 7.1. *i) Suppose $u(t)$ and $v(t)$ are holomorphic functions for $0 < |t| \ll 1$ satisfying* (6.18). *Then if $u' \neq 0$ and $v' \neq 0$, the origin is at most a pole of order 2 for $u(t)$ and $v(t)$.*

ii) Let $\big(u(t), v(t)\big)$ be a meromorphic solution of (6.18) *defined on a neighborhood of the origin. Consider the Laurent developments*

$$\begin{aligned}u(t) &= U_k t^k + U_{k+2} t^{k+2} + U_{k+4} t^{k+4} + \cdots,\\ v(t) &= V_\ell t^\ell + V_{\ell+2} t^{\ell+2} + V_{\ell+4} t^{\ell+4} + \cdots.\end{aligned} \tag{7.1}$$

Here $U_i \in \mathbb{C}$, $V_j \in \mathbb{C}$, and k and ℓ are nonzero even integers. If $U_k \neq 0$ and $V_\ell \neq 0$, then (k, ℓ) equals $(-2,-2)$, $(-2,2)$, $(-2,4)$, $(-2,6)$, $(2,2)$, $(2,-2)$, $(4,-2)$ *or* $(6,-2)$.

Proof. Using the Laurent development

$$u(x+y) - u(x-y) = 2\Big(\frac{u^{(1)}(x)}{1!}y + \frac{u^{(3)}(x)}{3!}y^3 + \cdots\Big). \tag{7.2}$$

with respect to y, it follows from (6.18) that

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\Big(v(x)(\frac{u^{(1)}(x)}{1!}y + \frac{u^{(3)}(x)}{3!}y^3 + \cdots)\Big) \\
&\quad + \frac{\partial}{\partial x}\Big(v(x)\frac{\partial}{\partial x}(\frac{u^{(1)}(x)}{1!}y + \frac{u^{(3)}(x)}{3!}y^3 + \cdots)\Big) \\
&= \frac{\partial^2}{\partial y^2}\Big(v(y)(\frac{u^{(1)}(x)}{1!}y + \frac{u^{(3)}(x)}{3!}y^3 + \cdots)\Big) \\
&\quad + \frac{\partial}{\partial y}\Big(v(y)\frac{\partial}{\partial y}(\frac{u^{(1)}(x)}{1!}y + \frac{u^{(3)}(x)}{3!}y^3 + \cdots)\Big).
\end{aligned} \tag{7.3}$$

To prove i) we fix x with $0 < |x| \ll 1$ and $u'(x) \neq 0$. Suppose $0 < |y| \ll |x|$. Then (7.3) implies

$$f(x,y) = y(u'(x) + yc_2(x,y))v''(y) + 3(u'(x) + yc_1(x,y))v'(x) + c_0(x,y)v(x)$$

with suitable holomorphic functions $f(x,y)$, $c_0(x,y)$, $c_1(x,y)$ and $c_2(x,y)$ of y defined on a neighborhood of the origin. Since this equation for v has a regular singularity at the origin with the characteristic exponents 0 and -2, the origin is at most a pole of order 2 for the solution v.

On the other hand Proposition 6.3 iv) assures that the origin is also at most a pole of order 2 for u and moreover that we may suppose $\ell \geq k$ to prove the second part of the lemma.

Suppose $\ell \geq 4$ and $\ell \geq k$. Then the coefficients of y in equation (7.3) shows

$$\frac{\partial^2}{\partial x^2}\big(v(x)u^{(1)}(x)\big) + \frac{\partial}{\partial x}\big(v(x)u^{(2)}(x)\big) = 0.$$

Expanding this into the Laurent series of x, the coefficients of $x^{k+\ell-3}$ proves

$$k(k+\ell-1)(k+\ell-2)V_\ell U_k + k(k-1)(k+\ell-2)V_\ell U_k = 0$$

and hence we can conclude that k equals $2-\ell$ or $1-\frac{\ell}{2}$, from which we have $(\ell,k) = (4,-2)$ or $(6,-2)$ because of the assumption. $\square$

Now we want to get solutions of (6.18). Suppose $\big(u(t), v(t)\big)$ is a holomorphic solution of (6.18) defined for $0 < |t| \ll 1$. Furthermore suppose $u' \neq 0$, $v' \neq 0$, $u(-t) = u(t)$ and $v(-t) = v(t)$. Then Lemma 7.1 assures that we may assume

$k = \ell = -2$ in (7.1). Here U_k may be 0 and V_ℓ may be 0. Subtracting constant numbers from u and v, respectively, we may moreover assume $U_0 = V_0 = 0$.

Then (7.3) equals

(7.4)
$$\begin{aligned}
&\frac{\partial^2}{\partial x \partial y}\Big\{ v'(x)\big(\frac{u^{(1)}}{2!}y^2 + \frac{u^{(3)}}{4!}y^4 + \cdots\big) + 2v(x)\big(\frac{u^{(2)}}{2!}y^2 + \frac{u^{(4)}}{4!}y^4 + \cdots\big)\Big\} \\
&= \frac{\partial^2}{\partial x \partial y}\Big\{ \big(-2V_{-2}y^{-3} + 2V_2 y^1 + 4V_4 y^3 + \cdots\big)\big(\frac{u^{(0)}}{1!}y + \frac{u^{(2)}}{3!}y^3 + \cdots\big) \\
&\quad + \big(2V_{-2}y^{-2} + 2V_2y^2 + 2V_4y^4 + \cdots\big)\big(\frac{u^{(0)}}{0!} + \frac{u^{(2)}}{2!}y^2 + \cdots\big)\Big\}.
\end{aligned}$$

Comparing the coefficients of y^1 and y^3 in the above, we have

Lemma 7.2. *Under the above notation*

$$u^{(1)}v' + 2u^{(2)}v = \frac{2}{3 \cdot 5}V_{-2}u^{(4)} + 2 \cdot 2^2 V_2 u + C_1 \tag{7.5}$$

and

$$u^{(3)}v' + 2u^{(4)}v = \frac{2}{5 \cdot 7}V_{-2}u^{(6)} + 2 \cdot 4^2 V_2 u^{(2)} + \frac{2 \cdot 4! \cdot 3}{1!}V_4 u + C_2 \tag{7.6}$$

with suitable constant numbers C_1 and C_2.

Now we give solutions of equation (6.18). The claim i) in Proposition 7.3 is not necessary for our later purpose if we have Proposition 7.8. The proof of Proposition 7.8 is similar as that of Proposition 7.3 i). The both proofs are elementary but the latter one is more complicated. Hence we shall also give the former one for the reader's convenience. In fact, it is useful for the calculation in the proofs to have the aid of a computer with an algebraic programming system such as Reduce, Maple, Mathematica.

Proposition 7.3. *Let $\big(u(t), v(t)\big)$ be a holomorphic solution of* (6.18) *defined for $0 < |t| \ll 1$. Assume that $u' \neq 0$, $v' \neq 0$, $u(-t) = u(t)$ and $v(-t) = v(t)$.*

i) If $u = v$, then there exist complex numbers A_2, A_1, A_0, ω_1 and ω_2 such that

$$u(t) = A_1 \wp(t|2\omega_1, 2\omega_2) + A_0 \tag{7.7}$$

or

$$u(t) = A_1 t^2 + A_2 t^{-2} + A_0. \tag{7.8}$$

ii) Suppose $u(t) = \wp(t|2\omega_1, 2\omega_2)$. Then there exist complex numbers C_0, C_1, C_2, C_3 and C_4 such that

$$v(t) = \frac{C_4\wp(t)^4 + C_3\wp(t)^3 + C_2\wp(t)^2 + C_1\wp(t) + C_0}{\wp'(t)^2}. \tag{7.9}$$

On the other hand for any complex numbers C_i, there exists a function $T(x,y)$ satisfying (6.16) *and* (6.17) *if the function v is defined by* (7.9).

iii) Suppose $u(t) = t^2 + Ct^{-2}$ with a complex number C. Then there exist complex numbers C_0, C_1 and C_2 such that

$$v(t) = C_0 + C_1 t^{-2} + C_2 t^2. \tag{7.10}$$

Conversely for any complex numbers C_0, C_1 and C_2 and the function v given by (7.10)*, there exists a function $T(x,y)$ which satisfies* (6.16) *and* (6.17).

Proof. First we shall prove the claim iii). Put

$$\begin{cases} u(t) &= \alpha t^{-2} + \beta t^2, \\ v(t) &= \gamma t^{-2} + \delta t^2 \end{cases} \tag{7.11}$$

with complex numbers α, β, γ and δ. Then

$$u(x+y) - u(x-y) = 2\frac{\partial}{\partial y}\Bigl(-\alpha\frac{x}{x^2-y^2} + \beta xy^2\Bigr)$$

and

$$v'(x)\bigl(u(x+y) - u(x-y)\bigr) + 2v(x)\bigl(u'(x+y) - u'(x-y)\bigr) = 2\frac{\partial T}{\partial y} \tag{7.12}$$

with

$$T(x,y) = \frac{4\alpha\gamma + 4\alpha\delta x^2y^2}{(x^2-y^2)^2} + 4\beta\delta x^2y^2. \tag{7.13}$$

Hence (6.18) is clear from (6.16) and (6.17).

Next suppose $u(t) = t^2 + Ct^{-2}$. We want to prove that v is of the form (7.10). Subtracting a suitable function of the form of the right hand side of (7.10) from v, we may assume $\ell = 4$ in (7.1). We shall show $v = 0$, which proves Proposition 7.3 iii).

If $C = 0$, then (7.5) means

$$2(t\frac{\partial}{\partial t} + 2)v(t) = C'$$

with a constant number C' and therefore we have $v = 0$.

Hence we may assume $C \neq 0$. Multiplying the both sides of (7.6) by $\frac{1}{24}t^6$, we get

$$C(-t\frac{\partial}{\partial t} + 10)v(t) = 6V_4(Ct^4 + t^8) + \frac{C'}{24}t^6$$

with a constant C'. This proves that

$$v(t) = V_4t^4 + V_6t^6 + \frac{3V_4}{C}t^8 + V_{10}t^{10}.$$

Since

$$u^{(1)}v' + 2u^{(2)}v = 4CV_4 + (16V_6 - 8CV_{10})t^6 + \frac{60V_4}{C}t^8 + 24V_{10}t^{10},$$

equation (7.5) assures $V_4 = V_6 = V_{10} = 0$.

Next we shall prove the claim ii). Suppose $u(t) = \wp(t)$ and $v(t)$ is given by (7.10). We shall show equation (6.18). Put

$$Q(t) = C_4t^4 + C_3t^3 + C_2t^2 + C_1t + C_0.$$

Then we have the following lemma by direct calculation.

Lemma 7.4.

$$2Q(s) - Q'(s)(s-t) = 2B(s,t) - (2C_4 s^2 + C_3 s)(s-t)^2.$$

by denoting

$$B(s,t) = C_4 s^2 t^2 + C_3 st\frac{s+t}{2} + C_2 st + C_1\frac{s+t}{2} + C_0 \tag{7.14}$$

Since $\wp$ satisfies

$$\wp(x+y) - \wp(x-y) = \frac{\partial}{\partial y}\left(\frac{\wp'(x)}{\wp(y)-\wp(x)}\right) \tag{7.15}$$

(cf. [WW]), we have

(7.16)

$$\begin{aligned}
&v'(x)\big(u(x+y)-u(x-y)\big) + 2v(x)\big(u'(x+y)-u'(x-y)\big)\\
&= \frac{\partial}{\partial y}\left\{\Big(\frac{\partial}{\partial x}\frac{Q(\wp(x))}{\wp'(x)^2}\Big)\frac{\wp'(x)}{\wp(y)-\wp(x)} + \frac{2Q(\wp(x))}{\wp'(x)^2}\frac{\partial}{\partial x}\Big(\frac{\wp'(x)}{\wp(y)-\wp(x)}\Big)\right\}\\
&= \frac{\partial}{\partial y}\left\{\frac{2Q(\wp(x)) - Q'(\wp(x))(\wp(x)-\wp(y))}{(\wp(x)-\wp(y))^2}\right\}\\
&= 2\frac{\partial}{\partial y}\frac{B(\wp(x),\wp(y))}{\big(\wp(x)-\wp(y)\big)^2}.
\end{aligned}$$

Since $B(s,t)$ is symmetric for (s,t), we obtain (6.16) and (6.18).

Next suppose $(u(t), v(t))$ satisfies (6.18) with $u(t) = \wp(t)$. Subtracting a suitable function of the form of the right hand side of (7.8) from $v(t)$, we may assume $\ell \geq 8$ in (7.1) to prove the claim ii). But Lemma 7.1 assures that $v = 0$ and we have the claim.

Now we shall prove i) and hence we suppose $u = v$. Note that if u is given by (7.7) or (7.8) and $v = u$, then u and v satisfy (6.18) (cf. (1.8)).

Use the developments (7.1) and equations (7.5) and (7.6). Then we may assume $k = -2$ or $k = 2$ and moreover $U_k = 1$ and $U_0 = 0$ by virtue of Lemma 7.1.

Comparing the coefficients of t^{2j} in equation (7.5), we have

$$\begin{aligned}
&\sum_{\nu=-1}^{j+2} 4\nu(j+\nu)U_{2\nu}U_{2j+2-2\nu}\\
&- \frac{2}{15}(2j+1)(2j+2)(2j+3)(2j+4)U_{-2}U_{2j+4} - 8U_2U_{2j} = 0
\end{aligned} \tag{7.17}$$

for any positive integer j.

First suppose $k = 2$. Then for $j \geq 2$ we have

$$4(j+1)U_{2j} + 4j(2j)U_{2j} + \sum_{\nu=2}^{j-1} 4\nu(j+\nu)U_{2\nu}U_{2j+2-2\nu} - 8U_{2j-2} = 0.$$

and therefore

$$4(2j-1)(j+1)U_{2j}=-4\sum_{\nu=2}^{j-1}\nu(j+\nu)U_{2\nu}U_{2j+2-2\nu}.$$

Hence by the induction on j, we have $U_{2j}=0$ for $j\geq 2$, which means $u(t)=t^2$.

Next suppose $k=-2$. Then in equation (7.17) there only appear U_ν for $\nu\leq 2j+4$. We can prove that if $j\geq 2$, then U_{2j+4} are inductively determined by U_ν with $\nu<2j+4$. In fact, since the term containing U_{2j+4} in (7.17) equals

$$\begin{aligned}
&2(-2)(2j+4)U_{-2}U_{2j+4}+2(-2)(-3)U_{-2}U_{2j+4}+2(2j+4)(2j+3)U_{2j+4}U_{-2}\\
&\quad-\frac{2}{15}(2j+1)(2j+2)(2j+3)(2j+4)U_{-2}U_{2j+4}\\
&=-\frac{2}{15}(2j-2)(2j+7)(4j^2+10j+9)U_{2j+4},
\end{aligned}$$

we have

$$\frac{1}{15}(\ell-3)(2\ell+3)(4\ell^2-6\ell+5)U_{2\ell}=\sum_{\nu=1}^{\ell-2}\nu(\ell+\nu-2)U_{2\nu}U_{2(\ell-\nu-1)}-2U_2U_{2(\ell-2)} \tag{7.18}$$

for $\ell=j+2\geq 3$. By putting $\ell=4$, 5, 6 and 7 in (7.18), we obtain

$$U_8=\frac{3}{11}U_2U_4, \tag{7.19}$$

$$U_{10}=\frac{2}{13}U_2U_6+\frac{1}{13}U_4^2, \tag{7.20}$$

$$U_{12}=\frac{35}{3729}U_2^2U_4+\frac{11}{113}U_4U_6, \tag{7.21}$$

$$U_{14}=\frac{270}{3604}U_2U_{10}+\frac{75}{1802}U_4U_8+\frac{30}{901}U_6^2, \tag{7.22}$$

respectively. In general, if $\ell\geq 4$, U_ℓ are determined by U_ν with $\nu<\ell$ and therefore the solution of (6.19) with $u=v$ is uniquely determined by the numbers U_2, U_4 and U_6.

Similarly the coefficients of t^6 and t^8 in (7.6) mean

$$U_{12}=\frac{140}{3883}U_2^2U_4+\frac{26}{353}U_4U_6, \tag{7.23}$$

$$U_{14}=\frac{90}{833}U_2U_{10}+\frac{50}{833}U_4U_8+\frac{15}{833}U_6^2. \tag{7.24}$$

Now it follows from (7.21) and (7.23) that

$$U_4(U_2^2-3U_6)=0. \tag{7.25}$$

Note that $U_6=\frac{1}{3}U_2^2$ if u is a $\wp$-function. Since $u(t)=\wp(t)$ is a solution of (6.18) and since $U_{2\ell}$ with $\ell\geq 4$ are uniquely determined by U_2, U_4 and U_6, we can conclude that $u(t)$ is Weierstrass' elliptic function if $U_2^2=3U_6$.

Hence to prove the proposition we may assume $U_2^2 \neq 3U_6$. Then we have $U_4 = 0$ and $U_{10} = \frac{2}{13}U_2U_6$ by (7.20). Combining this with (7.22) and (7.24), we get

$$U_6(U_2^2 - 3U_6) = 0. \tag{7.26}$$

Then by the assumption $U_2^2 \neq 3U_6$ we have $U_4 = U_6 = 0$ and therefore we can conclude $u(t) = t^{-2} + U_2t^2$ by the same reason as in the case $U_2^2 = 3U_6$. $\square$

Remark 7.5. i) Suppose $u(t) = \wp(t|2\omega_1, 2\omega_2)$ and put $\omega_3 = -\omega_1 - \omega_2$. If ω_1 and ω_2 are finite complex numbers, then the condition that v is of the form (7.9) is equivalent to say that

$$v(t) = C_0'\wp(t) + C_1'\wp(t+\omega_1) + C_2'\wp(t+\omega_2) + C_3'\wp(t+\omega_3) + C_4' \tag{7.27}$$

with suitable complex numbers C_0', C_1', C_2', C_3' and C_4' (cf. [WW]).

ii) For complex numbers C_1, C_2 and C_3, the pair

$$(u(t), v(t)) = (C_1\wp(t), C_2\wp(t) + C_3\wp(2t)) \tag{7.28}$$

satisfies equation (6.18), which follows from the duplication formula for $\wp(t)$ (cf. [WW]).

Corollary 7.6. *Suppose the root system is of type B_2 in Theorem 2.1.*

i) Suppose $u = 0$. Then (6.18) always holds and our commuting differential operators are

$$P_1 = Q_1 + Q_2,\ P_2 = Q_1Q_2$$

with

$$Q_j = \partial_j^2 + v(x_j)$$

for $j = 1$ and 2.

ii) The case when $v = 0$ is also trivial. It corresponds to the case when $v = 0$ by the symmetry given by Proposition 6.3 iv).

iii) Suppose

$$(u(t), v(t)) = (\alpha t^{-2} + \beta t^2, \gamma t^{-2} + \delta t^2) \tag{7.29}$$

or

$$(u(t), v(t)) = (A\wp(t), \frac{C_4\wp(t)^4 + C_3\wp(t)^3 + C_2\wp(t)^2 + C_1\wp(t) + C_0}{\wp'(t)^2}). \tag{7.30}$$

Then there exists a commuting algebra $\mathbb{C}[P_1, P_2]$, where P_1 and P_2 are defined by Proposition 6.3 i) through (7.13) or (7.16).

Now we shall solve equation (6.20). Suppose $u(t)$ is meromorphic at the origin. We may assume that $u(t)$ has the form given in (7.1).

Suppose $k \leq -2$ and $U_k = 1$. Then by using (7.2), the coefficient of y^{k-3} of (6.20) means

$$\big((k)(k-1)(k-2)(k-3) + (k)(k-1)(k-2)(3+2)\big)u'(x) = 0$$

and therefore we have $k = -2$.

Hence we may assume $k = -2$ in the expansion (7.1) of $u(t)$ if we allow $U_k = 0$. Furthermore we may assume $U_0 = 0$ by subtracting a constant from u. Then expanding (6.20) into the Laurent series of y, we have

$$\begin{aligned}
&u^{(4)}\left\{\frac{u^{(1)}}{1!}y + \frac{u^{(3)}}{3!}y^3 + \cdots\right\}\\
&+ 3u^{(3)}\left\{\frac{u^{(2)}}{1!}y + \frac{u^{(4)}}{3!}y^3 + \cdots\right\}\\
&+ 4u^{(4)}\left\{\frac{u^{(3)}}{1!}y + \frac{u^{(5)}}{3!}y^3 + \cdots\right\}\\
&= \Big((-2)(-3)(-4)(-5)U_{-2}y^{-6} + 4\cdot 3\cdot 2\cdot 1U_4 + \cdots\Big)\left\{\frac{u^{(1)}}{1!}y + \frac{u^{(3)}}{3!}y^3 + \cdots\right\}\\
&+ \Big((-2)(-3)(-4)U_{-2}y^{-5} + 4\cdot 3\cdot 2U_4 y + \cdots\Big)\left\{3\Big(\frac{u^{(1)}}{1!} + \frac{u^{(3)}}{3!}y^2 + \cdots\Big) + 2u^{(1)}\right\}\\
&+ \Big((-2)(-3)U_{-2}y^{-4} + 2U_2 + 4\cdot 3U_4 y^2 + \cdots\Big)\left\{2\Big(\frac{u^{(3)}}{1!}y + \frac{u^{(5)}}{3!}y^3 + \cdots\Big)\right\}\\
&+ \Big((-2)U_{-2}y^{-3} + 2U_2 y + 4U_4 y^3 + \cdots\Big)\left\{-2u^{(3)}\right\}.
\end{aligned}$$

The coefficients of y and y^3 in the above equation imply

$$\begin{aligned}
5u^{(3)}u^{(2)} + u^{(4)}u^{(1)} &= \frac{U_{-2}}{42}u^{(7)} + 144U_4u^{(1)},\\
\frac{1}{3}u^{(5)}u^{(2)} + \frac{2}{3}u^{(4)}u^{(3)} &= \frac{U_{-2}}{1080}u^{(9)} + \frac{2U_2}{3}u^{(5)} + 56U_4u^{(3)} + 960U_6u^{(1)},
\end{aligned}$$

respectively. Integrating the above equations, we have

Lemma 7.7. *Suppose a meromorphic function*

$$u(t) = U_{-2}t^{-2} + U_2t^2 + U_4t^4 + U_6t^6 + \cdots \tag{7.31}$$

defined on a neighborhood of the origin satisfies (6.20). Then it also satisfies the differential equations

$$\frac{U_{-2}}{42}u^{(6)} - 2(u'')^2 - u^{(3)}u' + 144U_4u = C \tag{7.32}$$

and

$$\frac{U_{-2}}{1080}u^{(8)} - \frac{1}{3}u^{(4)}u^{(2)} - \frac{1}{6}(u^{(3)})^2 + \frac{2U_2}{3}u^{(4)} + 56U_4u^{(2)} + 960U_6u = C' \tag{7.33}$$

with suitable constants C and C'.

Now the following proposition solves the equation (6.20).

Proposition 7.8. *Let $u(t)$ be a meromorphic function defined on a neighborhood of the origin. Suppose $u(t)$ satisfies* (6.20). *Then $u(t)$ is of the form* (7.7) *or* (7.8).

Conversely any function $u(t)$ of the form (7.7) *or* (7.8) *satisfies* (6.20).

Now we prepare

Lemma 7.9. *Let $u(t)$ be a function of the form* (7.7) *or* (7.8). *Then $D(x_1, x_2, x_3) = 0$ with the function $D(x_1, x_2, x_3)$ defined by* (6.14).

Proof. First suppose $u(t)$ is given by (7.7). Then Theorem 5.2 assures the existence of the commuting algebra $\mathbb{D}(A_n)$ corresponding to the potential function $R(x)$ defined by u. Since $A_3 \simeq D_3$, we have $D(x_1, x_2, x_3) = 0$ from Theorem 2.1 and Theorem 6.1

Next suppose $u(t)$ is given by (7.8). Put

(7.34)
$$\begin{aligned} E(x,y,z) &= 8A_1^2 \frac{2x^2y^2 + x^2z^2 + y^2z^2}{(x^2-y^2)^2(x^2-z^2)(y^2-z^2)} \\ &+ 8A_1A_2\left(\frac{2x^2y^2 + x^2z^2 + y^2z^2}{(x^2-y^2)^2} + \frac{z^2}{x^2-z^2} + \frac{z^2}{y^2-z^2}\right) \\ &+ 16A_2^2(x^2y^2 + x^2z^2 + y^2z^2) \\ &+ 8A_0A_1\frac{x^2+y^2}{(x^2-y^2)^2} + 8A_0A_2(x^2+y^2). \end{aligned}$$

Then we can prove equality (6.21) by direct calculation. Hence Lemma 7.9 follows from Proposition 6.4 iii). $\square$

Proof of Proposition 7.8. First expand equations (7.32) and (7.33) into the Laurent series of t. Then the coefficients of t^4, t^6, t^8, t^{10}, t^{12} and t^{14} in (7.32) show

$$13U_{-2}U_{10} - 2U_2U_6 - U_4^2 = 0, \tag{7.35}$$
$$195U_{-2}U_{12} - 14U_2U_8 - 24U_4U_6 = 0, \tag{7.36}$$
$$2159U_{-2}U_{14} - 90U_2U_{10} - 170U_4U_8 - 105U_6^2 = 0, \tag{7.37}$$
$$2888U_{-2}U_{16} - 77U_2U_{12} - 1532U_4U_{10} - 202U_6U_8 = 0, \tag{7.38}$$
$$20070U_{-2}U_{18} - 364U_2U_{14} - 735U_4U_{12} - 1020U_6U_{10} - 560U_8^2 = 0, \tag{7.39}$$

(7.40)
$$97635U_{-2}U_{20} - 1260U_2U_{16} - 2576U_4U_{14} - 3675U_6U_{12} - 4270U_8U_{10} = 0$$

by dividing 1680, 560, 168, 336, 112 and 48, respectively.

In general, comparing the coefficients of t^{2k-6} in (7.32), we obtain

(7.41)
$$\begin{aligned} &\frac{1}{42}2k(2k-2)(2k-8)(2k+3)(4k^2-16k+43)U_{-2}U_{2k} \\ &= \sum_{\nu=1}^{k-2}(2\nu)(2\nu-1)(4k-2\nu-2)(2k-2\nu-8)U_{2\nu}U_{2(k-\nu-1)} - 144U_4U_{2(k-3)} \end{aligned}$$

for $k \geq 4$. This equation implies that U_{2k} are uniquely determined by $U_{2\nu}$ with $\nu < k$ if $k \geq 5$ and $U_{-2} \neq 0$. Hence we see that $u(t)$ is uniquely determined by U_2, U_4, U_6 and U_8 if $U_{-2} \neq 0$. In fact, from (7.35), (7.36) and (7.36) we have

$$U_{10} = \frac{1}{13}(2U_2U_6 + U_4^2), \tag{7.42}$$

$$U_{12} = \frac{2}{195}(7U_2U_8 + 12U_4U_6), \tag{7.43}$$

$$U_{14} = \frac{5}{28067}(36U_2^2U_6 + 18U_2U_4^2 + 442U_4U_8 + 273U_6^2), \tag{7.44}$$

if $U_{-2} = 1$.

Similarly the coefficients of t^6, t^8, t^{10} and t^{12} in (7.33) mean

$$221U_{-2}U_{14} - 20U_4U_8 - 15U_6^2 = 0, \tag{7.45}$$

$$57U_{-2}U_{16} - 3U_4U_{10} - 5U_6U_8 = 0, \tag{7.46}$$

$$13515U_{-2}U_{18} - 462U_4U_{12} - 795U_6U_{10} - 466U_8^2 = 0, \tag{7.47}$$

$$93955U_{-2}U_{20} - 2184U_4U_{14} - 3885U_6U_{12} - 4690U_8U_{10} = 0, \tag{7.48}$$

by dividing 1008, 21600, 336 and 144, respectively.

First suppose $U_{-2} = 1$. Then substituting U_{14} in (7.45) by the left hand side of (7.44), we have

$$\frac{30}{127}(6U_2^2U_6 + 3U_2U_4^2 - 11U_4U_8 - 18U_6^2) = 0$$

and hence if $U_4 \neq 0$

$$U_8 = \frac{3}{11U_4}(2U_2^2U_6 + U_2U_4^2 - 6U_6^2). \tag{7.49}$$

Now suppose $U_{-2} = 1$ and $U_4 \neq 0$. Then from (7.43), (7.44), (7.38) and (7.49) we have

$$U_{12} = \frac{3}{715U_4}(14U_2^3U_6 + 7U_2^2U_4^2 - 42U_2U_6^2 + 44U_4^2U_6), \tag{7.50}$$

$$U_{14} = \frac{15}{2431}(8U_2^2U_6 + 4U_2U_4^2 - 13U_6^2), \tag{7.51}$$

$$\begin{aligned} U_{16} = \frac{1}{1032460U_4}&(1078U_2^4U6 + 539U_2^3U_4^2 + 36156U_2^2U_6^2 \\ &+ 31443U_2U_4^2U_6 + 4180U_4^4 - 118170U_6^3). \end{aligned} \tag{7.52}$$

Then applying (7.42), (7.49) and (7.52) to (7.46), we have

$$(3U_6 - U_2^2)(130U_6^2 - 14U_2^2U_6 - 7U_2U_4^2) = 0. \tag{7.53}$$

Now we suppose $U_{-2} = 1$, $U_4 \neq 0$ and $3U_6 \neq U_2^2$. If $U_2 = 0$, then (7.53) implies $U_6 = 0$, which contradicts the assumption just we have made. Hence we conclude $U_2 \neq 0$. Then from (7.53) we have

$$U_4^2 = \frac{1}{7U_2}(130U_6^2 - 14U_2^2U_6). \tag{7.54}$$

In this case we get

$$U_{10} = \frac{10U_6^2}{7U_2}, \quad U_{12} = \frac{16U_6^3}{7U_2U_4}, \quad U_{14} = \frac{45U_6^2}{119}, \quad U_{16} = \frac{20U_6^3}{931U_2^2U_4}(7U_2^2 + 65U_6),$$
$$U_{20} = \frac{48U_6^3}{14716849U_2U_4}(154390U_6 - 5523U_2^2).$$

Applying these equations to (7.48), we obtain

$$\frac{19918080}{489307U_2U_4}U_6^3(3U_6 - U_2^2) = 0$$

and therefore $U_6 = 0$ because $3U_6 \neq U_2^2$. Then from (7.53) we have $U_2U_4^2 = 0$, which contradicts our assumption.

Thus we have proved that if $U_{-2} = 1$ and $U_4 \neq 0$, then $U_6 = \frac{1}{3}U_2^2$ and u is uniquely determined by U_2 and U_4. Since Weierstrass' elliptic function is a solution of (6.20), we can conclude in this case that $u(t)$ is Weierstrass' elliptic function.

Next we assume that $U_{-2} = 1$ and $U_4 = 0$. In this case we have

$$\begin{gathered} U_{10} = \frac{2U_2U_6}{13}, \quad U_{12} = \frac{14U_2U_8}{195}, \quad U_{14} = \frac{15U_6}{28067}(12U_2^2 + 91U_6), \\ U_{16} = \frac{U_8}{281580}(539U_2^2 + 19695U_6), \\ U_{18} = \frac{2}{56330469}(3276U_2^3U_6 + 245061U_2U_6^2 + 785876U_8^2). \end{gathered} \tag{7.55}$$

from (7.35), (7.42), (7.43), (7.44) and (7.39). Applying these equations with $U_4 = 0$ to (7.45), we have

$$\frac{180U_6}{127}(U_2^2 - 3U_6) = 0.$$

If $U_6 = 0$, then (7.47) is reduced to

$$\frac{-47432U_8^2}{669} = 0$$

by using (7.55) and therefore $U_8 = 0$ and we can conclude $u(t) = t^{-2} + U_2t^2$ in the same way as in the case when $U_{-2} = 1$ and $U_4 \neq 0$.

Consider the case when $U_4 = 0$, $U_6 = \frac{1}{3}U_2^2 \neq 0$. In this case (7.46) is similarly reduced to

$$\frac{-847U_2^2U_8}{3705} = 0.$$

and hence $U_8 = 0$. Then we can similarly conclude that $u(t)$ is the $\wp$-function.

Thus we have proved the proposition when $U_{-2} = 1$. Since we can reduce the proof of the proposition to this special case if $U_{-2} \neq 0$, we may assume $U_{-2} = 0$.

Choose a positive integer ℓ such that $U_{2\ell} \neq 0$ and $U_{2\nu} = 0$ if $\nu < \ell$. Suppose $\ell \geq 3$. Then the equation (7.41) with $k = 2\ell + 1$ says

$$2\ell(2\ell - 1)(6\ell + 2)(2\ell - 6)U_{2\ell}^2 = 0,$$

which implies $\ell = 3$.

Hence we can conclude that the condition $U_{-2} = U_2 = U_4 = U_6 = 0$ assures $u(t) = 0$.

If $U_{-2} = U_2 = 0$, we have $U_4 = 0$ from (7.35) and therefore $U_6 = 0$ from (7.45) and we can conclude $u(t) = 0$.

Suppose $U_{-2} = 0$ and $U_2 \neq 0$. Then from (7.35) and (7.36) we have

$$U_6 = \frac{-U_4^2}{2U_2}, \quad U_8 = \frac{-12U_4U_6}{7U_2} = \frac{6U_4^4}{7U_2^2}$$

and (7.45) is reduced to

$$\frac{-7020U_4^2}{U_2^2} = 0.$$

Hence $U_4 = U_6 = U_8 = 0$ and we can conclude $u(t) = U_2t^2$ by the similar argument as before.

Thus we have completed the proof of the proposition. $\square$

Now we state our main result in this section. For any even function $w(t)$ we can define the following *trivial commuting family*

$$\begin{aligned} Q_k &= \sum_{1\le i_1<i_2<\cdots<i_k\le n} \prod_{\nu=1}^{k} \left(\frac{\partial^2}{\partial x_{i_\nu}^2} + w(x_{i_\nu})\right) \quad \text{for} \quad k = 1, \ldots, n \\ \mathbb{D}(w) &= \mathbb{C}[Q_1, \ldots, Q_n]. \end{aligned} \tag{7.56}$$

Theorem 7.10. *Suppose there exist a W-invariant connected open neighborhood Ω of the origin of $\mathbb{C}^n$ such that the potential function $R(x)$ in (0.4) is a holomorphic function defined on an open dense subset Ω' of Ω. Here $\Omega \setminus \Omega'$ is an analytic subset of Ω.*

i) If the root system is of type D_n with $n \ge 3$, then the function $u(t)$ in Theorem 2.1 equals $A_1\wp(t|2\omega_1, 2\omega_2) + A_0$ or $A_1t^2 + A_2t^{-2} + A_0$ with suitable complex numbers A_i and ω_k.

ii) Suppose the root system is of type B_n with $n \ge 3$ and suppose $\mathbb{C}[P_1, \ldots, P_n]$ is not equal to any trivial commuting algebra $\mathbb{D}(w)$. Then there exist complex numbers A_i, C_j and ω_k such that

$$\begin{aligned} u(t) &= A_1\wp(t|2\omega_1, 2\omega_2) + A_0, \\ v(t) &= \frac{C_4\wp(t)^4 + C_3\wp(t)^3 + C_2\wp(t)^2 + C_1\wp(t) + C_0}{\wp'(t)^2}. \end{aligned} \tag{7.57}$$

or

$$\begin{aligned} u(t) &= A_1t^2 + A_2t^{-2} + A_0, \\ v(t) &= C_1t^2 + C_2t^{-2} + C_0. \end{aligned} \tag{7.58}$$

Here we remark that ω_k $(k = 1, 2)$ may be infinite.

Proof. The theorem is clear from Proposition 6.4, Proposition 7.3 and Proposition 7.8. $\square$

Remark 7.11. The proof of Theorem 7.10 shows that when $n > 2$, (7.57) and (7.58) give all the solutions of (6.5) such that $u(t)$ and $v(t)$ are holomorphic for $0 < |t| \ll 1$.

When the root system is of type B_2, Theorem 7.10 is not valid (cf. [OOS]). On the other hand, we have the following result under the assumption that the coefficients of the differential operators have expansions of Harish-Chandra type.

Theorem 7.12. *i) Assume the root system is of type B_2 in Theorem 2.1. Suppose $u' \neq 0$, $v' \neq 0$ and the functions $u(\log s)$ and $v(\log s)$ are meromorphic for $|s| \ll 1$ under the notation* (6.1)*. Then there exist a positive integer r and complex numbers $C_1, \ldots, C_8$ with*

$$C_2C_6 = C_4C_6 = 0 \tag{7.59}$$

such that $\big(u(t), v(t)\big)$ or $\big(v(t), u(2t)\big)$ equals

$$\begin{aligned}\big(C_1 \sinh^{-2}\frac{r}{2}t + C_2 \sinh^{-2} rt + C_3 \cosh rt + C_4 \cosh 2rt + C_5,& \\ C_6 \sinh^{-2}\frac{r}{2}t + C_7 \sinh^{-2} rt + C_8\big).&\end{aligned} \tag{7.60}$$

ii) If $\big(u(t), v(t)\big)$ equals (7.60) *with complex numbers $C_1, \ldots, C_8$ satisfying* (7.59)*. Then $u(t)$ and $v(t)$ satisfy the assumption in Proposition 6.3 ii) and therefore we have commuting differential operators.*

Proof. Suppose the meromorphic functions

$$\begin{cases} u(\log t) &= \sum_{i \geq r} U_i t^i \\ v(\log s) &= \sum_{j \geq r'} V_j s^j \end{cases}$$

satisfies (6.18). Here r and r' are integers and U_i and V_j are complex numbers with $U_r \neq 0$. By subtracting constant numbers from u and v, we may assume $U_0 = V_0 = 0$ and $r \neq 0$.

Since

$$u(\log s + \log t) - u(\log s - \log t) = \sum_{i \geq r} U_i (t^i - t^{-i}) s^i,$$

for $0 < |s| \ll |t| \ll 1$ it follows from (6.18) that

$$\begin{aligned}&(s\frac{\partial}{\partial s})^2 \Big\{ \big(\sum_{j \geq r'} V_j s^j\big)\big(\sum_{i \geq r} U_i (t^i - t^{-i}) s^i\big)\Big\} \\ &\quad + (s\frac{\partial}{\partial s})\Big\{ \big(\sum_{j \geq r'} V_j s^j\big) s\frac{\partial}{\partial s}\big(\sum_{i \geq r} U_i (t^i - t^{-i}) s^i\big)\Big\} \\ &= (t\frac{\partial}{\partial t})^2 \Big\{ \big(\sum_{j \geq r'} V_j t^j\big)\big(\sum_{i \geq r} U_i (t^i - t^{-i}) s^i\big)\Big\} \\ &\quad + (t\frac{\partial}{\partial t})\Big\{ \big(\sum_{j \geq r'} V_j t^j\big) t\frac{\partial}{\partial t}\big(\sum_{i \geq r} U_i (t^i - t^{-i}) s^i\big)\Big\}\end{aligned}$$

and therefore

$$
\begin{aligned}
&\sum_{\substack{i\geq r\\ j\geq r'}}(i+j)(2i+j)U_iV_j(t^i-t^{-i})s^{i+j}\\
(7.61)\qquad &=\sum_{\substack{i\geq r\\ j\geq r'}}\Big((2i+j)(i+j)U_iV_jt^{i+j}-(2i-j)(i-j)U_iV_jt^{-i+j}\Big)s^i.
\end{aligned}
$$

If $V_{r'}\neq 0$ and $r'<0$, the coefficients of $t^rs^{r+r'}$ in (7.61) means

$$(r+r')(2r+r')U_rV_{r'}=0$$

and therefore $r=-r'$ or $r=-\frac{r'}{2}$. Hence Proposition 6.3 iv) assures that we may assume $r>0$ by replacing $(u(t),v(t))$ by $(v(t),u(2t))$ if necessary.

Admitting $V_{r'}$ to be 0, we may assume

$$r>0 \quad \text{and} \quad r'=-2r.$$

When $j<0$, the coefficients of t^rs^{r+j} means $(r+j)(2r+j)U_rV_j=0$ and therefore

$$V_j=0 \quad \text{for} \quad -2r<j<-r \ \text{ and } \ -r<j\leq 0. \tag{7.62}$$

The terms in (7.61) corresponding to s^r imply

$$
\begin{aligned}
&\sum_{r\leq i\leq 3r} r(i+r)U_iV_{r-i}(t^i-t^{-i})\\
=&\sum_{j\geq -2r}(2r+j)(r+j)U_rV_jt^{r+j}-\sum_{j\geq -2r}(2r-j)(r-j)U_rV_jt^{-r+j}
\end{aligned}
$$

and hence from (7.62) we have

$$
\begin{aligned}
&3r^2U_{2r}V_{-r}(t^{2r}-t^{-2r})+4r^2U_{3r}V_{-2r}(t^{3r}-t^{-3r})\\
(7.63)\qquad &=\sum_{k\geq -3r}\Big((r+k)kU_rV_{k-r}-(r-k)kU_rV_{k+r}\Big)t^k
\end{aligned}
$$

by denoting $V_j=0$ for $j<-2r$.

If $k\neq\pm 2r$ and $k\neq\pm 3r$, then by the coefficients of t^k of (7.63) we have

$$(r+k)kU_rV_{k-r}=(r-k)kU_rV_{k+r}$$

and hence

$$V_j=0 \quad \text{if} \quad j\not\equiv 0 \mod r \tag{7.64}$$

and

$$V_{jr}=\frac{j}{j-1}V_{(j-1)r} \quad \text{for} \quad j>4. \tag{7.65}$$

Furthermore the coefficients of t^{-2r}, t^{2r} and t^{3r} in (7.63) mean

$$\begin{aligned} -3r^2U_{2r}V_{-r} &= -(2r)(-3r)U_rV_{-r} \\ 3r^2U_{2r}V_{-r} &= 6r^2U_rV_r - 2r^2U_rV_{3r} \end{aligned}$$

and

$$4r^2U_{3r}V_{-2r} = 12r^2U_rV_{2r} - 6r^2U_rV_{4r},$$

respectively, and thus we have

$$V_{-r}(U_{2r} - 2U_r) = 0, \tag{7.66}$$

$$V_{3r} = 3(V_r - V_{-r}), \tag{7.67}$$

$$V_{4r} = 2(V_{2r} - V_{-2r}). \tag{7.68}$$

On the other hand, the coefficients of $t^{-r}s^{2r}$ in (7.61) says

$$-6r^2U_rV_r = -3r^2U_{2r}V_r$$

and therefore

$$V_r(U_{2r} - 2U_r) = 0. \tag{7.69}$$

Now we remark that relations (7.64), (7.65), (7.67) and (7.68) show that the numbers V_{-2r}, V_{-r}, V_r and V_{2r} uniquely determine the function $v(\log s)$ because we have assumed $V_0 = 0$.

On the other hand, if V_j are the coefficients of t^j of the function $\frac{t^r}{(1-t^r)^2}$, they satisfy (7.64), (7.65), (7.67) and (7.68). In fact it is clear from the equation

$$\frac{t}{(1-t)^2} = \sum_{k=1}^{\infty} kt^k.$$

Similarly it is easy to see that the functions $\frac{t^{2r}}{(1-t^{2r})^2}$, $t^r + t^{-r}$ and $t^{2r} + t^{-2r}$ have the same property.

Thus we can conclude that

$$v(t) = C_1 \sinh^{-2} \frac{r}{2}t + C_2 \sinh^{-2} rt + C_3 \cosh rt + C_4 \cosh 2rt \tag{7.70}$$

with some constant numbers $C_1, \ldots, C_4$.

Next we shall show

$$u(t) = C_6 \sinh^{-2} \frac{r}{2}t + C_7 \sinh^{-2} rt \tag{7.71}$$

with some constant numbers C_6 and C_7, which proves the first part of the theorem by virtue of relations (7.66) and (7.69). Here we note that we have assumed that $U_r \neq 0$ with $r > 0$.

If $v(\log s)$ is holomorphic at the origin $s = 0$, it follows from (7.70) that $u(t)$ is of the form (7.71) because $(v(t), u(2t))$ is also a solution of (6.18).

To examine the case when $v(\log s)$ is not holomorphic at the origin, we shall study the solution $(v(t), u(2t))$ of (6.18) and the proof of Theorem 7.12 i) is reduced to the determination of $u(t)$ satisfying (7.61) under the assumption $r < 0$, $U_r \neq 0$ and $r' > 0$ by replacing r if necessary. Under this assumption, the terms in (7.61) corresponding to s^r prove

$$\sum_{j \geq r'} \Big((2r+j)(r+j)U_r V_j t^{r+j} - (2r-j)(r-j)U_r V_j t^{-r+j} \Big) = 0$$

and furthermore by the coefficients of t^{j-r} in the above we have

$$(j-r)(j-2r)V_j = (j-r)jV_{j-2r} \quad \text{for} \quad j > 0,$$

which means

$$v(t) = C_6' \sinh^{-2} \frac{r}{2} t + C_7' \sinh^{-2} rt$$

with some complex numbers C_6' and C_7'. Thus we have completed the proof of Theorem 7.12 i).

First suppose $C_6 = 0$ to prove the second part of the theorem. If $\big(v(2t), u(t)\big)$ equals (7.60), $\big(u(t), v(t)\big)$ is a special case given in Theorem 7.3 ii) and therefore it satisfies the assumption in Proposition 6.3 ii). Hence the second part follows from Proposition 6.3 iv) when $C_6 = 0$.

Next suppose $C_2 = C_4 = 0$. We have proved that if $\big(u(t), v(t)\big)$ equals

$$\Big(C_1 \sinh^{-2} \frac{r}{2} t + C_3 \cosh rt + C_5, C_7 \sinh^{-2} rt\Big),$$

or

$$\big(C_1 \sinh^{-2} r't + C_3 \cosh 2r't + C_5, C_6 \sinh^{-2} r't + C_8\big),$$

with suitable positive numbers r and r', it satisfies the assumption in Proposition 6.3 ii).

Putting $r' = \frac{r}{2}$, it is clear that

$$\begin{aligned} u(t) &= C_1 \sinh^{-2} \frac{r}{2} t + C_3 \cosh rt + C_5, \\ v(t) &= C_6 \sinh^{-2} \frac{r}{2} t + C_7 \sinh^{-2} rt + C_8 \end{aligned}$$

satisfy the same assumption. Thus we have completed the proof of Theorem 7.12 owing to Proposition 6.3 iv).

Combining Proposition 3.6, Theorem 5.2, Theorem 7.10 and Theorem 7.12, we have

Theorem 7.13. *Let u and v be functions in Theorem 2.1. Suppose u and v are holomorphic except some isolated singular points and suppose $u(\log s)$ and $v(\log s)$ are holomorphically extended to the point $s = 0$.*

i) If the root system is of type A_n with $n > 1$ or of type D_n with $n > 2$, then

$$u(t) = C_1 \sinh^{-2} kt + C_0. \tag{7.72}$$

ii) Suppose the root system is of type B_n and suppose $u' \neq 0$ and $v' \neq 0$. If $n > 2$, then

$$\begin{aligned} u(t) &= C_1 \sinh^{-2} kt + C_0, \\ v(t) &= A_1 \sinh^{-2} kt + A_2 \sinh^{-2} 2kt + A_0 \end{aligned} \tag{7.73}$$

and if $n = 2$, then (7.73) *holds or*

$$\begin{aligned} u(t) &= A_1 \sinh^{-2} kt + A_2 \sinh^{-2} 2kt + A_0, \\ v(t) &= C_1 \sinh^{-2} 2kt + C_0. \end{aligned} \tag{7.74}$$

In i) and ii), A_0, A_1, A_2, C_0 and C_1 are complex numbers and $2k$ is a positive integer.

iii) Suppose P_j are invariant under the parallel translation $x_1 \mapsto x_1 + 2\pi\sqrt{-1}$. Then u and v in Theorem 2.1 determine the commuting algebra $\mathbb{C}[P_1, \dots, P_n]$.

Remark 7.14. The assumption in Theorem 7.13 gives a characterization for the commuting algebra $\mathbb{C}[P_1, \dots, P_n]$ to be equal to the one constructed by [D1], [H1], [H2], [Op1], [Op2] and [Sj].

8. Examples

In this paper we have studied the potential function $R(x)$ of a Laplacian which allows a commuting family of differential operators invariant under the action of a classical Weyl group. In this section we first consider the one-dimensional analogue of the potential function we have obtained. That is the ordinal differential equation

$$\frac{d^2 y}{dt^2} + u_\alpha(t) y = 0 \tag{8.1}$$

for the function u_α in (0.8).

Then the most general potential function in Theorem 7.10 gives

$$\frac{d^2 y}{dt^2} + \frac{C_4 \wp(t)^4 + C_3 \wp(t)^3 + C_2 \wp(t)^2 + C_1 \wp(t) + C_0}{\wp'(t)^2} y = 0. \tag{8.2}$$

Note that

$$\begin{aligned} [\wp']^2 &= 4\wp^3 - g_2 \wp - g_3 \\ &= 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \end{aligned}$$

with some complex numbers e_1, e_2 and e_3 and then

$$\begin{aligned}\wp'' &= 6\wp^2 - \frac{g_2}{2}\\ &= 2\big\{(\wp - e_2)(\wp - e_3) + (\wp - e_3)(\wp - e_1) + (\wp - e_1)(\wp - e_2)\big\},\\ \frac{\wp''}{[\wp']^2} &= \frac{1}{2}\Big(\frac{1}{\wp - e_1} + \frac{1}{\wp - e_2} + \frac{1}{\wp - e_3}\Big).\end{aligned}$$

Putting $x = \wp(t)$, we have $\frac{d}{dt} = \wp'(t)\frac{d}{dx}$ and

$$\frac{d^2}{dt^2} = [\wp']^2\Big\{\frac{d^2}{dx^2} + \frac{1}{2}\Big(\frac{1}{\wp - e_1} + \frac{1}{\wp - e_2} + \frac{1}{\wp - e_3}\Big)\frac{d}{dx}\Big\}. \tag{8.3}$$

Hence equation (8.2) equals

$$\frac{d^2y}{dx^2} + \frac{1}{2}\Big(\frac{1}{x - e_1} + \frac{1}{x - e_2} + \frac{1}{x - e_3}\Big)\frac{dy}{dx} + \frac{C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0}{16(x - e_1)^2(x - e_2)^2(x - e_3)^2}y = 0. \tag{8.4}$$

Suppose $e_1 \neq e_2 \neq e_3 \neq e_1$. Then (8.4) can be written as

$$\begin{aligned}&\frac{d^2y}{dx^2} + \frac{1}{2}\Big(\frac{1}{x - e_1} + \frac{1}{x - e_2} + \frac{1}{x - e_3}\Big)\frac{dy}{dx}\\ &\quad + \Big(\frac{A_1}{(x - e_1)^2} + \frac{A_2}{(x - e_2)^2} + \frac{A_3}{(x - e_3)^2} + \frac{B_1}{x - e_1} + \frac{B_2}{x - e_2} + \frac{B_3}{x - e_3}\Big)y = 0\end{aligned} \tag{8.5}$$

with some complex numbers A_1, A_2, A_3, B_1, B_2 and B_3 satisfying

$$B_1 + B_2 + B_3 = 0. \tag{8.6}$$

Equation (8.5) is a Fuchsian equation on $\mathbb{P}^1(\mathbb{C})$ which has the four regular singular points e_1, e_2, e_3 and ∞. The indicial equations for the singular points are

$$\begin{aligned}\rho_j^2 - \frac{1}{2}\rho_j + A_j = 0 \quad &\text{at } x = e_j \quad \text{for } j = 1,\ 2 \text{ and } 3,\\ \rho_\infty^2 - \frac{1}{2}\rho_\infty + \sum_{j=1}^{3}(A_j + e_jB_j) = 0 \quad &\text{at } x = \infty.\end{aligned} \tag{8.7}$$

By the transformation $y \mapsto (x - e_1)^{\lambda_1}(x - e_2)^{\lambda_2}(x - e_3)^{\lambda_3}y$ with complex numbers λ_1, λ_2 and λ_3, the equation is transformed into Huen's equation (cf. [WW]) and moreover we obtain any Fuchsian equation on $\mathbb{P}^1(\mathbb{C})$ of order 2 which has the four regular singular points.

On the other hand, if

$$u_\alpha(t) = C_1\sinh^{-2}t + C_2\sinh^{-2}2t + C_5 \tag{8.8}$$

or

$$u_\alpha(t) = C_3\cosh 2t + C_5 \tag{8.9}$$

or

$$u_\alpha(t) = A_1 t^2 + A_2 t^{-2} + A_0 \tag{8.10}$$

(cf. Theorem 7.10 and Theorem 7.12), (8.1) is isomorphic to the Gauss hypergeometric equation or the modified Mathieu equation or the equation of the paraboloid of revolution which is equivalent to the equation of Whittaker functions, respectively.

When the root system is of type A_n, Theorem 4.3 says $u_\alpha = C_1\wp + C_0$ and the corresponding equation (8.1) is the Weierstrassian form of Lamé's equation, which corresponds to $A_1 = A_2 = A_3 = 0$ in (8.5). In particular if $u_\alpha(t) = C_1 \sinh^{-2} t + C_0$ or $u_\alpha(t) = C_1 t^{-2} + C_0$, the equation is reduced to the Legendre equation or the Bessel equation, respectively.

Next consider the case when the root system is of type A_2. First remark that

$$\begin{aligned}
\wp(2s) &= \frac{1}{4}\frac{\wp''(s)^2}{\wp'(s)^2} - 2\wp(s),\\
\wp(s+t) + \wp(s-t) &= \frac{\wp'(s)^2 + \wp'(t)^2}{2\big(\wp(s) - \wp(t)\big)^2} - 2\wp(s) - 2\wp(t),\\
\wp(s+t) - \wp(s-t) &= \frac{\wp'(s)\wp'(t)}{\big(\wp(s) - \wp(t)\big)^2}.
\end{aligned} \tag{8.11}$$

For $(x_1, x_2, x_3) \in \mathbb{C}^3$, we consider the coordinate system (X, Y, Z) with

$$2X = x_1 - x_2,\ X + Y = x_1 - x_3,\ Z = x_3. \tag{8.12}$$

Then $2Y = x_1 + x_2 - 2x_3$ and

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial X} + \frac{1}{2}\frac{\partial}{\partial Y},\ \frac{\partial}{\partial x_2} = -\frac{\partial}{\partial X} + \frac{1}{2}\frac{\partial}{\partial Y},\ \frac{\partial}{\partial x_3} = \frac{\partial}{\partial Y} + \frac{\partial}{\partial Z}.$$

The commuting family in this case is generated by

(8.13)

$$\begin{aligned}
\Delta_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3},\\
\Delta_2 &= \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_3} + \frac{\partial^2}{\partial x_1 \partial x_3} + C\wp(x_1 - x_2) + C\wp(x_2 - x_3) + C\wp(x_1 - x_3),\\
\Delta_3 &= \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} + C\wp(x_2 - x_3)\frac{\partial}{\partial x_1} + C\wp(x_1 - x_3)\frac{\partial}{\partial x_2} + C\wp(x_1 - x_2)\frac{\partial}{\partial x_3}.
\end{aligned}$$

Let J be the left ideal of the ring of differential operators generated by $\Delta_1 = \frac{\partial}{\partial Z}$ and put $x = \wp(X)$, $y = \wp(Y)$ and $z = \wp(Z)$. Then

(8.14)

$$\Delta_2 = -\frac{\partial^2}{\partial X^2} - \frac{3}{4}\frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Y \partial Z} + C\wp(2X) + C\wp(X - Y) + C\wp(X + Y)$$

$$
\begin{aligned}
&\equiv -(4x^3-4g_2x-g_3)\frac{\partial^2}{\partial x^2}-\frac{3}{4}(4y^3-4g_2y-g_3)\frac{\partial^2}{\partial y^2}\\
&\quad-(6x^2-\frac{g_2}{2})\frac{\partial}{\partial x}-\frac{3}{4}(6y^2-\frac{g_2}{2})\frac{\partial}{\partial y}+C\frac{\bigl(6x^2-\frac{g_2}{2}\bigr)^2}{4\bigl(4x^3-g_2x-g_3\bigr)^2}\\
&\quad+C\frac{4x^3+4y^3-g_2x-g_2y-2g_3}{2\bigl(x-y\bigr)^2}-4Cx-2Cy \qquad \bmod J,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_3 &= \left(\frac{\partial^2}{\partial X^2}-\frac{1}{4}\frac{\partial^2}{\partial Y^2}\right)\left(\frac{\partial}{\partial Y}-\frac{\partial}{\partial Z}\right)+C\wp(X-Y)\left(\frac{\partial}{\partial X}+\frac{1}{2}\frac{\partial}{\partial Y}\right)\\
&\quad+C\wp(X+Y)\left(-\frac{\partial}{\partial X}+\frac{1}{2}\frac{\partial}{\partial Y}\right)+C\wp(2X)\frac{\partial}{\partial Y}\\
&\equiv \left(\frac{\partial^2}{\partial X^2}-\frac{1}{4}\frac{\partial^2}{\partial Y^2}\right)\frac{\partial}{\partial Y}-C\bigl(\wp(X+Y)-\wp(X-Y)\bigr)\frac{\partial}{\partial X}\\
&\quad+C\left(\frac{\wp(X+Y)+\wp(X-Y)}{2}+\wp(2X)\right)\frac{\partial}{\partial Y} \qquad \bmod J\\
&= \sqrt{4y^3-g_2y-g_3}\biggl[(4x^3-g_2x-g_3)\frac{\partial^3}{\partial x^2\partial y}+(6x^2-\frac{g_2}{2})\frac{\partial^2}{\partial x\partial y}\\
&\quad-\frac{1}{4}(4y^3-g_2y-g_3)\frac{\partial^3}{\partial y^3}-\frac{3}{4}(6y^2-\frac{g_2}{2})\frac{\partial^2}{\partial y^2}\\
&\quad-C\frac{4x^3-g_2-g_3}{(x-y)^2}\frac{\partial}{\partial x}\\
&\quad+\frac{C}{4}\left(4x^3+4y^3-(g_2+8)x-g_2y-2g_3+\frac{12x^2-g_2}{8x^3-2g_2x+2g_3}\right)\frac{\partial}{\partial y}\biggr].
\end{aligned}
$$

Now consider the case when the root system is of type B_2. Use the coordinate system $(s,t)\in\mathbb{C}^2$ and put $x=\wp(s)$ and $y=\wp(t)$. Let

$$
\begin{aligned}
u(t) &= A\wp(t),\\
v(t) &= \frac{C_4\wp(t)^4+C_3\wp(t)^3+C_2\wp(t)^2+C_1\wp(t)+C_0}{\wp'(t)^2}
\end{aligned}
\tag{8.15}
$$

in Proposition 6.3 and Proposition 7.3. Then by (7.16) we have

(8.16)

$$
\begin{aligned}
P_1 &= \frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}+A(\wp(s+t)+\wp(s-t))\\
&\quad+\frac{C_4x^4+C_3x^3+C_2x^2+C_1x+C_0}{4x^3-g_2x-g_3}+\frac{C_4y^4+C_3y^3+C_2y^2+C_1y+C_0}{4y^3-g_2y-g_3}\\
&= (4x^3-g_2x-g_3)\frac{\partial^2}{\partial x^2}+(6x^2-\frac{g_2}{2})\frac{\partial}{\partial x}+(4y^3-g_2y-g_3)\frac{\partial^2}{\partial y^2}\\
&\quad+(6y^2-\frac{g_2}{2})\frac{\partial}{\partial y}+\frac{A(6x^2+6y^2-g_2)}{(x-y)^2}-2Ax-2Ay\\
&\quad+\frac{C_4x^4+C_3x^3+C_2x^2+C_1x+C_0}{4x^3-g_2x-g_3}+\frac{C_4y^4+C_3y^3+C_2y^2+C_1y+C_0}{4y^3-g_2y-g_3},
\end{aligned}
$$

$$
\begin{aligned}
P_2 &= \left[\frac{\partial^2}{\partial s\partial t} + \frac{u(s+t)-u(s-t)}{2}\right]^2 + v(t)\frac{\partial^2}{\partial s^2} + v(s)\frac{\partial^2}{\partial t^2} + v(s)v(t)\\
&\quad + \frac{2AC_4x^2y^2 + AC_3xy(x+y) + 2AC_2xy + AC_1(x+y) + 2AC_0}{2(x-y)^2}\\
&= \Bigg[\sqrt{(4x^3-g_2x-g_3)(4y^3-g_2y-g_3)}\frac{\partial^2}{\partial x\partial y}\\
&\qquad + \frac{A\sqrt{(4x^3-g_2x-g_3)(4y^3-g_2y-g_3)}}{2(x-y)^2}\Bigg]^2\\
&\quad + \frac{C_4y^4+C_3y^3+C_2y^2+C_1y+C_0}{4y^3-g_2y-g_3}\left((4x^3-g_2x-g_3)\frac{\partial^2}{\partial x^2} + (6x^2-\frac{g_2}{2})\frac{\partial}{\partial x}\right)\\
&\quad + \frac{C_4x^4+C_3x^3+C_2x^2+C_1x+C_0}{4x^3-g_2x-g_3}\left((4y^3-g_2y-g_3)\frac{\partial^2}{\partial y^2} + (6y^2-\frac{g_2}{2})\frac{\partial}{\partial y}\right)\\
&\quad + \frac{(C_4x^4+C_3x^3+C_2x^2+C_1x+C_0)(C_4y^4+C_3y^3+C_2y^2+C_1y+C_0)}{(4x^3-g_2x-g_3)(4y^3-g_2y-g_3)}\\
&\quad + \frac{2AC_4x^2y^2 + AC_3xy(x+y) + 2AC_2xy + AC_1(x+y) + 2AC_0}{2(x-y)^2}.
\end{aligned}
$$

Here we note that the coefficients of the differential operator P_2 are rational functions under the coordinate (x,y).

On the other hand, if

$$
\big(u(t), v(t)\big) = \big(\alpha t^{-2} + \beta t^2, \gamma t^{-2} + \delta t^2\big), \tag{8.17}
$$

the commuting operators are

(8.18)

$$
\begin{aligned}
P_1 &= \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} + 2\alpha\frac{s^2+t^2}{(s^2-t^2)^2} + (2\beta+\delta)(s^2+t^2) + \gamma(s^{-2}+t^{-2}),\\
P_2 &= \left[\frac{\partial^2}{\partial s\partial t} - 2\alpha\frac{st}{(s^2-t^2)^2} + 2\beta st\right]^2 + (\gamma t^{-2}+\delta t^2)\frac{\partial^2}{\partial s^2} + (\gamma s^{-2}+\delta s^2)\frac{\partial^2}{\partial t^2}\\
&\quad + (\gamma s^{-2}+\delta s^2)(\gamma t^{-2}+\delta t^2) + \frac{4\alpha\delta s^2t^2 + 4\alpha\gamma}{(s^2-t^2)^2} + 4\beta\delta s^2t^2
\end{aligned}
$$

from Proposition 6.3 and (7.16). In particular, if $\alpha = \gamma = 0$, we have

(8.19)

$$
\begin{aligned}
P_1 &= \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} + \lambda(s^2+t^2),\\
P_2 &= \left[\frac{\partial^2}{\partial s\partial t} + (\lambda-\delta)st\right]^2 + \delta\left(t^2\frac{\partial^2}{\partial s^2} + s^2\frac{\partial^2}{\partial t^2}\right) + (2\lambda-\delta)\delta s^2t^2
\end{aligned}
$$

by putting $\lambda = 2\beta + \delta$.

Lastly we consider the operators when

$$
\big(u(t),\ v(t)\big) = \big(\alpha\sinh^{-2}t + \beta\cosh 2t,\ \gamma\sinh^{-2}t + \delta\sinh^{-2}2t\big), \tag{8.20}
$$

which is given by Theorem 7.12. Putting $x = \sinh^2 s$ and $y = \sinh^2 t$, we have

$$\begin{aligned}
u(s+t) + u(s-t) &= 2\alpha \frac{x+y+2xy}{(x-y)^2} + 2\beta(1+2x)(1+2y), \\
u(s+t) - u(s-t) &= \left(\frac{-4\alpha}{(x-y)^2} + 8\beta \right) \sqrt{s(1+x)y(1+y)}, \\
v(s) &= \frac{\gamma}{x} + \frac{\delta}{4x(1+x)}, \\
\frac{\partial}{\partial s} &= 4\sqrt{x(1+x)} \frac{\partial}{\partial x}
\end{aligned}$$

and

$$\begin{aligned}
&2\frac{\partial}{\partial t}\left(2\alpha\gamma \frac{2+x+y}{(x-y)^2} + \frac{\alpha\delta}{(x-y)^2} + 4\beta\gamma(x+y) \right) \\
&\quad = v'(s)\big(u(s+t) - u(s-t)\big) + 2v(s)\big(u'(s+t) - u'(s-t)\big).
\end{aligned}$$

Thus by Proposition 6.3 we have

(8.21)

$$\begin{aligned}
P_1 &= \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \\
&\quad + \alpha\big(\sinh^{-2}(s+t) + \sinh^{-2}(s-t)\big) + \beta\big(\cosh 2(s+t) + \cosh 2(s-t)\big) \\
&\quad + \gamma\big(\sinh^{-2} s + \sinh^{-2} t\big) + \delta\big(\sinh^{-2} 2s + \sinh^{-2} 2t\big) \\
&= 16x(1+x)\frac{\partial^2}{\partial x^2} + 8(1+2x)\frac{\partial}{\partial x} + 16y(1+y)\frac{\partial^2}{\partial y^2} + 8(1+2y)\frac{\partial}{\partial y} \\
&\quad + 2\alpha\frac{x+y+2xy}{(x-y)^2} + 2\beta(1+2x)(1+2y) \\
&\quad + \gamma\left(\frac{1}{x} + \frac{1}{y}\right) + \delta\left(\frac{1}{4x(1+x)} + \frac{1}{4y(1+y)}\right), \\
P_2 &= \Bigg[\frac{\partial^2}{\partial s \partial t} \\
&\qquad + \frac{\alpha\big(\sinh^{-2}(s+t) - \sinh^{-2}(s-t)\big) + \beta\big(\cosh 2(s+t) - \cosh 2(s-t)\big)}{2}\Bigg]^2 \\
&\quad + \big(\gamma \sinh^{-2} t + \delta \sinh^{-2} 2t\big)\frac{\partial^2}{\partial s^2} + \big(\gamma \sinh^{-2} s + \delta \sinh^{-2} 2s\big)\frac{\partial^2}{\partial t^2} \\
&\quad + \big(\gamma \sinh^{-2} s + \delta \sinh^{-2} 2s\big)\big(\gamma \sinh^{-2} t + \delta \sinh^{-2} 2t\big) \\
&\quad + \frac{2\alpha\gamma(2 + \sinh s + \sinh t) + \alpha\delta}{\sinh^2(s+t)\sinh^2(s-t)} + 4\beta\gamma(\sinh^2 s + \sinh^2 t) \\
&= \left[16\sqrt{x(1+x)y(1+y)}\frac{\partial^2}{\partial x \partial y} + \left(\frac{-2\alpha}{(x-y)^2} + 4\beta\right)\sqrt{x(1+x)y(1+y)}\right]^2 \\
&\quad + \left(\frac{\gamma}{y} + \frac{\delta}{4y(1+y)}\right)\frac{\partial^2}{\partial x^2} + \left(\frac{\gamma}{x} + \frac{\delta}{4x(1+x)}\right)\frac{\partial^2}{\partial y^2}
\end{aligned}$$

$$+\Big(\frac{\gamma}{x}+\frac{\delta}{4x(1+x)}\Big)\Big(\frac{\gamma}{y}+\frac{\delta}{4y(1+y)}\Big)+\frac{2\alpha\gamma(2+x+y)+\alpha\delta}{(x-y)^2}+4\beta\gamma(x+y).$$

Here if we use the symmetric coordinate system

$$\begin{cases} X = x+y, \\ Y = xy \end{cases} \tag{8.22}$$

then by

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y},\ \frac{\partial}{\partial y}=\frac{\partial}{\partial X}+x\frac{\partial}{\partial Y},$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial x\partial y} &= \Big(\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}\Big)\Big(\frac{\partial}{\partial X}+x\frac{\partial}{\partial Y}\Big)\\
&= \frac{\partial^2}{\partial X^2}+X\frac{\partial^2}{\partial X\partial Y}+Y\frac{\partial^2}{\partial Y^2}+\frac{\partial}{\partial Y}\\
f(x,y)\frac{\partial}{\partial x}+f(y,x)\frac{\partial}{\partial y} &= f(x,y)\Big(\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}\Big)+f(y,x)\Big(\frac{\partial}{\partial X}+x\frac{\partial}{\partial Y}\Big)\\
&= \Big(f(x,y)+f(y,x)\Big)\frac{\partial}{\partial X}+\Big(yf(x,y)+xf(y,x)\Big)\frac{\partial}{\partial Y}\\
f(x,y)\frac{\partial^2}{\partial x^2}+f(y,x)\frac{\partial^2}{\partial y^2} &= \Big(f(x,y)+f(y,x)\Big)\frac{\partial^2}{\partial X^2}\\
&\quad +2\Big(yf(x,y)+xf(y,x)\Big)\frac{\partial^2}{\partial X\partial Y}+\Big(y^2f(x,y)+x^2f(y,x)\Big)\frac{\partial^2}{\partial Y^2},
\end{aligned}$$

these operators are

(8.23)

$$\begin{aligned}
P_1 &= 16(X+X^2-2Y)\frac{\partial^2}{\partial X^2}+16(2+X)Y\frac{\partial^2}{\partial X\partial Y}+16(X+2Y)Y\frac{\partial^2}{\partial Y^2}\\
&\quad +16(1+X)\frac{\partial}{\partial X}+8(X+4Y)\frac{\partial}{\partial Y}\\
&\quad +2\alpha\frac{X+2Y}{X^2-4Y}+2\beta(1+2X+4Y)+\gamma\frac{X}{Y}+\delta\frac{X+X^2-2Y}{4Y(1+X+Y)},\\
P_2 &= \Big[16\sqrt{Y(1+X+Y)}\Big(\frac{\partial^2}{\partial X^2}+X\frac{\partial^2}{\partial X\partial Y}+Y\frac{\partial^2}{\partial Y^2}+\frac{\partial}{\partial Y}\Big)\\
&\qquad +\Big(\frac{-2\alpha}{X^2-4Y}+4\beta\Big)\sqrt{Y(1+X+Y)}\Big]^2\\
&\quad +\Big(\gamma\frac{X}{Y}+\delta\frac{X+X^2-2Y}{4Y(1+X+Y)}\Big)\frac{\partial^2}{\partial X^2}+\Big(2\gamma+\delta\frac{2+X}{4(1+X+Y)}\Big)\frac{\partial^2}{\partial X\partial Y}\\
&\quad +\Big(\gamma X+\delta\frac{X+2Y}{4(1+X+Y)}\Big)\frac{\partial^2}{\partial Y^2}\\
&\quad +\frac{\gamma^2}{Y}+\frac{\gamma\delta(2+X)}{4Y(1+X+Y)}+\frac{\delta^2}{16Y(1+X+Y)}+\frac{2\alpha\gamma(2+X)+\alpha\delta}{X^2-4Y}+4\beta\gamma X.
\end{aligned}$$

References

[D] A. Debiard, *Systèm différentiel hypergeómétrique et parties radiales des espaces symétriques de type BC_p*, Springer Lecture Notes in Math. **1296** (1988), 42–124.

[I] V. I. Inozemtsev, *Finite Toda lattice*, Commun. Math. Phys. **121** (1989), 629–638.

[IM] V. I. Inozemtsev and D. V. Meshcheryakov, *Extensions of the class of integrable dynamical systems connected with semisimple Lie algebras*, Lett. Math. Phys. **9** (1985), 13–18.

[HC] Harish-Chandra, *Representations of semisimple Lie groups IV*, Amer. J. Math. **77** (1955), 743–777.

[H1] G. J. Heckman, *Root system and hypergeometric functions II*, Comp. Math. **64** (1987), 353–373.

[H2] ———, *An elementary approach to the hypergeometric shift operators of Opdam*, Invent. Math. **103** (1991), 341–350.

[HO] G. J. Heckman and E. M. Opdam, *Root system and hypergeometric functions I*, Comp. Math. **64** (1987), 329–352.

[OO] H. Ochiai and T. Oshima, *Commuting differential operators of type B_2*, preprint, 1994, UTMS 94–65, Dept. of Mathematical Science, University of Tokyo.

[OOS] H. Ochiai, T. Oshima and H. Sekiguchi, *Commuting families of symmetric differential operators*, Proc. Japan Acad. **70 A** (1994), 62–66.

[OP] M. A. Olshanetsky and A. M. Perelomov, *Completely integrable Hamiltonian systems connected with semisimple Lie algebras*, Invent. Math. **37** (1976), 93–108.

[OP2] ———, *Quantum system connected with root systems and the radial parts of Laplace operators*, Funct. Anal. Appl. **12, No. 2** (1978), 60–68.

[OP3] ———, *Quantum integrable systems related to Lie algebras*, Phys. Rep. **94** (1983), 313–404.

[Op1] E. M. Opdam, *Root system and hypergeometric functions III*, Comp. Math. **67** (1988), 21–49.

[Op2] ———, *Root system and hypergeometric functions IV*, Comp. Math. **67** (1988), 191–209.

[O] T. Oshima, *Completely integrable systems with a symmetry in coordinates*, preprint, 1994, UTMS 94–6, Dept. of Mathematical Science, University of Tokyo.

[P] A. M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhaüser, 1990.

[Sj] J. Sekiguchi, *Zonal spherical functions on some symmetric spaces*, Rubl. RIMS Kyoto Univ. **12 Suppl.** (1977), 455–459.

[Sh] H. Sekiguchi, *Radial parts of Casimir operators on semisimple symmetric spaces*, RIMS KôkyûRoku **816** (1992), 155–168. (Japanese)

[WW] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis, Fourth Edition*, Cambridge University Press, 1927.

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