COMMUTING FAMILIES OF SYMMETRIC DIFFERENTIAL OPERATORS

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Introduction

Many commuting families of differential operators or completely integrable quantum systems have been constructed in connection with root systems (cf. [OP2] and references therein). Such families often have a certain symmetry in coordinates.

The radial parts of invariant differential operators on symmetric spaces give a good example of a commuting family of differential operators (cf. [HC]). In this case some parameters take only some discrete values determined by the dimensions of the root spaces for the symmetric spaces. On the other hand, [Sj] generalized them to complex parameters for the root system of type A_n. The same generalization was given by [H1], [H2], [HO], [Op1], [Op2] for general root systems. If the root system is of classical type, their operators give examples of the commuting families studied in this note (cf. Remark 3 iii)). Namely we shall determine all the families under the assumption of a symmetry in coordinates.

Let W be the Weyl group of type A_{n-1} with $n \geq 3$ or of type B_n with $n \geq 2$ or of type D_n with $n \geq 4$. We identify W with the group of the coordinate transformations

$$(x_1, \dots, x_n) \mapsto (\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)})$$

of $\mathbb{R}^n$, where σ are the elements of the n-th symmetric group $\mathfrak{S}_n$ and

$$\begin{cases} \varepsilon_1 = \cdots = \varepsilon_n = 1 & \text{if } W \text{ is of type } A_{n-1}, \\ \varepsilon_1 = \pm 1, \cdots, \varepsilon_n = \pm 1 & \text{if } W \text{ is of type } B_n, \\ \varepsilon_1 = \pm 1, \cdots, \varepsilon_n = \pm 1 \text{ and } \#\{i\,;\,\varepsilon_i = -1\} \text{ is even} & \text{if } W \text{ is of type } D_n. \end{cases}$$

We examine the Laplacian

$$P = -\frac{1}{2} \sum_{1 \leq j \leq n} \frac{\partial^2}{\partial x_j^2} + V(x)$$

on $\mathbb{R}^n$ with a W-invariant potential $V(x)$ which has enough W-invariant commuting differential operators. To be precise we assume that there exist W-invariant differential operators $P_1, \dots, P_n$ with

$$P \in \mathbb{C}[P_1, \dots, P_n]$$

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and

$$[P_i, P_j] = 0 \quad \text{for} \quad 1 \le i < j \le n$$

such that

$$P_j = \sum_{1 \le i_1 < \cdots < i_j \le n} \partial_{i_1} \cdots \partial_{i_j} + R_j \quad \text{with} \quad \operatorname{ord} R_j < j \quad \text{for } 1 \le j \le n$$

or

$$P_j = \sum_{1 \le i_1 < \cdots < i_j \le n} \partial_{i_1}^2 \cdots \partial_{i_j}^2 + R_j \quad \text{with} \quad \operatorname{ord} R_j < 2j \quad \text{for } 1 \le j \le n$$

or

$$\begin{cases} P_n = \partial_1 \cdots \partial_n + R_n \quad \text{with} \quad \operatorname{ord} R_n < n, \\ P_j = \displaystyle\sum_{1 \le i_1 < \cdots < i_j \le n} \partial_{i_1}^2 \cdots \partial_{i_j}^2 + R_j \quad \text{with} \quad \operatorname{ord} R_j < 2j \quad \text{for } 1 \le j \le n-1, \end{cases}$$

if the type of W is A_{n-1} or B_n or D_n, respectively. Here $\mathbb{C}[P_1, \ldots, P_n]$ is the commutative algebra over $\mathbb{C}$ generated by $P_1, \ldots, P_n$ and for simplicity we put $\partial_i = \frac{\partial}{\partial x_i}$ and $\operatorname{ord} R_j$ are the orders of differential operators R_j.

In this note, we assume that the coefficients of the differential operators are extended to holomorphic functions on a Zariski open subset Ω' of an open connected neighborhood Ω of the origin of the complexification $\mathbb{C}^n$ of $\mathbb{R}^n$. Namely there exists a non-zero holomorphic function ϕ on Ω with $\Omega' = \{x \in \Omega\,;\, \phi(x) \neq 0\}$.

Determination of the commuting families

The first theorem says that the potential $V(x)$ is only allowed to be a special function.

Theorem 1. *Under the assumption in the introduction, we can conclude*

$$\begin{aligned} V(x) &= \sum_{1 \le i < j \le n} u(x_i - x_j) && \text{if } W \text{ is of type } A_{n-1}, \\ V(x) &= \sum_{1 \le i < j \le n} \Big(u(x_i - x_j) + u(x_i + x_j) \Big) + \sum_{1 \le j \le n} v(x_j) && \text{if } W \text{ is of type } B_n, \\ V(x) &= \sum_{1 \le i < j \le n} \Big(u(x_i - x_j) + u(x_i + x_j) \Big) && \text{if } W \text{ is of type } D_n. \end{aligned}$$

Here $u(t)$ and $v(t)$ are following functions with complex numbers $C_1, C_2, \ldots$:

If W is of type A_{n-1} with $n \ge 3$,

$$u(t) = C_1 \wp(t) + C_2. \tag{1}$$

If W is of type B_n with $n \ge 3$,

$$\begin{cases} u(t) = C_1 \wp(t) + C_2, \\ v(t) = \dfrac{C_3 \wp(t)^4 + C_4 \wp(t)^3 + C_5 \wp(t)^2 + C_6 \wp(t) + C_7}{\wp'(t)^2} \end{cases} \tag{2}$$

or

$$u(t)=C_1t^{-2}+C_2t^2+C_3 \quad \text{and} \quad v(t)=C_4t^{-2}+C_5t^2+C_6 \tag{3}$$

or

$$u(t)=C_1 \quad \text{and} \quad v(t) \ \text{ is any even function.} \tag{4}$$

If W is of type D_n with $n\geq 4$, then u is (2) or (3).

If W is of type B_2, then $(u(t),v(t))$ is (2) or (3) or (4) or

$$\begin{cases} u(t) &= \dfrac{C_3\wp(\frac{t}{2})^4+C_4\wp(\frac{t}{2})^3+C_5\wp(\frac{t}{2})^2+C_6\wp(\frac{t}{2})+C_7}{\wp'(\frac{t}{2})^2}, \\ v(t) &= C_1\wp(t)+C_2 \end{cases} \tag{5}$$

or

$$\begin{cases} u(t) &= C_1\wp(t)+C_2\dfrac{\big(\wp(\frac{t}{2})-e_3\big)^2}{\wp'(\frac{t}{2})^2}+C_3, \\ v(t) &= C_4\wp(t)+\dfrac{C_5}{\wp(t)-e_3}+C_6 \end{cases} \tag{6}$$

or

$$v(t)=C_1 \quad \text{and} \quad u(t) \ \text{ is any even function.} \tag{7}$$

In the above theorem, $\wp(t)$ is the Weierstrass elliptic function $\wp(t|2\omega_1,2\omega_2)$ with primitive half-periods ω_1 and ω_2 which are allowed to be infinity and e_3 is a complex number satisfying $\wp'^2=4(\wp-e_1)(\wp-e_2)(\wp-e_3)$ (cf. [WW]). In particular

$$\wp(t|\sqrt{-1}\pi,\infty)=\sinh^{-2}t+\tfrac{1}{3} \quad \text{and} \quad \wp(t|\infty,\infty)=t^{-2}.$$

Then we note that $(u(t),v(t))$ in (2) has 9 complex parameters including the periods.

Theorem 2. i) *If W is of type B_n, the expression of $V(x)$ by u and v is not unique and then we may assume that the coefficient of $\partial_1\partial_2$ of P_2 equals $2u(x_1-x_2)-2u(x_1+x_2)$ without changing the commuting algebra $\mathbb{C}[P_1,\ldots,P_n]$.*

ii) *If W is not of type A_{n-1} or if W is of type A_{n-1} and $\operatorname{ord}R_3<2$, then $\mathbb{C}[P_1,\ldots,P_n]$ is uniquely determined by u or (u,v).*

iii) *The commuting differential operators $P_1,\ldots,P_n$ exist for P with the potential $V(x)$ defined by u and v of the form (1), (2), (4), (5), (6) and (7) according to the type of W, where $C_1,\ldots$ are any complex numbers.*

If W is of type A_{n-1}, the commuting differential operators are given by

$$P_k=\sum_{0\leq j\leq[\frac{k}{2}]}\frac{1}{2^j j!(k-2j)!}\sum_{\sigma\in\mathfrak{S}_n}\sigma\Big(u(x_1-x_2)u(x_3-x_4)\cdots \cdot u(x_{2j-1}-x_{2j})\partial_{2j+1}\partial_{2j+2}\cdots\partial_k\Big)$$

for $k = 1, \ldots, n$ (cf. [OP2] and [OS]).

If W is of type B_n and

$$u(t) = C_5 \wp(t), \quad v(t) = \sum_{j=1}^{4} C_j \wp(t + \omega_j) - \frac{C_0}{2}$$

with complex numbers $C_0, \ldots, C_5$ and $\omega_3 = -(\omega_1 + \omega_2)$ and $\omega_4 = 0$, then the commuting operators are given by

$$P_n(C_0) = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{\sigma \in \mathfrak{S}_n} \sigma\Big(q_{\{1,\ldots,k\}} \Delta^2_{\{k+1,\ldots,n\}} \Big)$$

(cf. [O]), where

$$\begin{aligned}
\Delta_{\{1,\ldots,k\}} &= \sum_{0 \le j \le [\frac{k}{2}]} \frac{1}{2^{k-j} j!(k-2j)!} \sum_{w \in W(B_k)} \varepsilon(w) w\Big(u(x_1 - x_2) u(x_3 - x_4) \cdots \\
&\qquad \cdot u(x_{2j-1} - x_{2j}) \partial_{2j+1} \partial_{2j+2} \cdots \partial_k \Big), \\
q_{\{1,\ldots,k\}} &= \sum_{I_1 \amalg \cdots \amalg I_\nu = \{1,\ldots,k\}} T_{I_1} \cdots T_{I_\nu}, \quad q_\varnothing = 1, \\
T_{\{1,\ldots,k\}} &= (-C_5)^{k-1} \Big(\frac{C_0}{2} T^0_{\{1,\ldots,k\}}(1) - \sum_{j=1}^{4} C_j T^0_{\{1,\ldots,k\}}(\wp(t + \omega_j)) \Big), \\
T^0_{\{1,\ldots,k\}}(\psi) &= \sum_{I_1 \amalg \cdots \amalg I_\nu = \{1,\ldots,k\}} (-1)^{\nu-1} (\nu - 1)! S_{I_1}(\psi) \cdots S_{I_\nu}(\psi), \\
S_{\{1,\ldots,k\}}(\psi) &= \sum_{w \in W(B_k)} w\Big(\psi(x_1) \wp(x_1 - x_2) \wp(x_2 - x_3) \cdots \wp(x_{k-1} - x_k) \Big).
\end{aligned}$$

Here $W(B_k)$ and $W(D_k)$ are the Weyl groups of type B_k and D_k, respectively, $W(B_k)$ and $W(D_k)$ and $\mathfrak{S}_k$ are realized as groups of coordinate transformations of $\mathbb{R}^k$. For $w \in W(B_k)$, $\varepsilon(w) = 1$ if $w \in W(D_k)$ and -1 otherwise, the sums for $I_1, \ldots, I_\nu$ run over all the partitions of $\{1, \ldots, k\}$, and for a subset I of $\{1, \ldots, n\}$, we define $\Delta_I = \sigma(\Delta_{\{1,\ldots,k\}})$ etc. by $\sigma \in \mathfrak{S}_n$ and $k = \#I$ with $\sigma(\{1, \ldots, k\}) = I$.

Expanding $P_n(C_0)$ into a polynomial function of the parameter C_0, the operators P_j are given by the coefficients of C_0^{n-j} in the expansion. In fact we have $[P_n(C_0), P_n(C_0')] = 0$.

If W is of type D_n, we have only to put $C_1 = C_2 = C_3 = C_4 = 0$ and $P_n = \Delta_{\{1,\ldots,n\}}$ in the above definition. See [OO] for other cases of type B_2.

Remark 3. i) If (u, v) is of the form (3), P_j do not exist in general and we need operators of higher order (cf. [OP2]).

ii) If (u, v) is given by (4), then $\mathbb{C}[P_1, \ldots, P_n]$ equals the totality of $\mathfrak{S}_n$-invariants in $\mathbb{C}[-\frac{1}{2}\partial_1^2 + v(x_1), \ldots, -\frac{1}{2}\partial_n^2 + v(x_n)]$.

iii) If $2\omega_1 = \sqrt{-1}\lambda^{-1}\pi$ and $\omega_2 = \infty$ with $\lambda \neq 0$, (2) is reduced to

$$\begin{cases} u(t) = C_1' \sinh^{-2} \lambda t + C_2', \\ v(t) = C_3' \sinh^{-2} \lambda t + C_4' \sinh^{-2} 2\lambda t + C_5' \sinh^2 \lambda t + C_6' \sinh^2 2\lambda t + C_7'. \end{cases}$$

The commuting differential operators studied by Heckman-Opdam correspond to this case with $C'_5 = C'_6 = 0$. Moreover if $\omega_1 = \omega_2 = \infty$, then (2) is reduced to

$$\begin{cases} u(t) = C'_1 t^{-2} + C'_2, \\ v(t) = C'_3 t^{-2} + C'_4 t^2 + C'_5 t^4 + C'_6 t^6 + C'_7. \end{cases}$$

iv) Some results stated in this note were announced in [Sh]. The precise statements and arguments will be given in [OS], [O] and [OO].

v) Replacing ∂_i, x_j, $[\ ,\ ]$ and ord by $\sqrt{-1}p_i$, q_j, the Poisson bracket $\{\ ,\ \}$ and the degree for p, respectively, we have the same statements as in Theorem 1 and Theorem 2, and moreover the operators $P_1, \ldots, P_n$ give the integrals of the Hamiltonian corresponding to the Laplacian P (cf. [OP1] for completely integrable classical systems).

References

[HC] Harish-Chandra, *Representations of semisimple Lie groups IV*, Amer. J. Math. **77** (1955), 743-777.

[H1] G. J. Heckman, *Root system and hypergeometric functions II*, Comp. Math. **64** (1987), 353-373.

[H2] ———, *An elementary approach to the hypergeometric shift operators of Opdam*, Invent. Math. **103** (1991), 341-350.

[HO] G. J. Heckman and E. M. Opdam, *Root system and hypergeometric functions I*, Comp. Math. **64** (1987), 329-352.

[O] T. Oshima, *Completely integrable systems with a symmetry in coordinates*, preprint, 1994, UTMS 94-6, Dept. of Mathematical Sciences, Univ. of Tokyo.

[OO] H. Ochiai and T. Oshima, *Commuting families of differential operators invariant under the action of a Weyl group II*, in preparation.

[Op1] E. M. Opdam, *Root system and hypergeometric functions III*, Comp. Math. **67** (1988), 21-49.

[Op2] ———, *Root system and hypergeometric functions IV*, Comp. Math. **67** (1988), 191-209.

[OP1] M. A. Olshanetsky and A. M. Perelomov, *Classical integrable finite dimensional systems related to Lie algebras*, Phys. Rep. **71** (1981), 313-400.

[OP2] ———, *Quantum integrable systems related to Lie algebras*, Phys. Rep. **94** (1983), 313-404.

[OS] T. Oshima and H. Sekiguchi, *Commuting families of differential operators invariant under the action of a Weyl group*, preprint, 1993, UTMS 93-43, Dept. of Mathematical Sciences, Univ. of Tokyo.

[Sj] J. Sekiguchi, *Zonal spherical functions on some symmetric spaces*, Rubl. RIMS Kyoto Univ. **12 Suppl.** (1977), 455-459.

[Sh] H. Sekiguchi, *Radial parts of Casimir operators on semisimple symmetric spaces*, RIMS KôkyûRoku **816** (1992), 155-168. (Japanese)

[WW] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis, Fourth Edition*, Cambridge University Press, 1927.