

不確定特異点をもつ常微分方程式の解析

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§ Generalized Riemann scheme

$$\mathcal{M} : \boxed{Pu = 0} \quad (P = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_0(x) \in W[x])$$

$$W[x] = \mathbb{C}[x, \partial], \quad \partial = \frac{d}{dx}, \quad \vartheta = x\partial$$

$$c_0 = \infty, c_1, \dots, c_p : \text{the singular points of } \mathcal{M} \quad (a_n(c_j) = 0)$$

If $c_1 = 0$, we assume $\mathcal{M}$ has (formal) local solutions

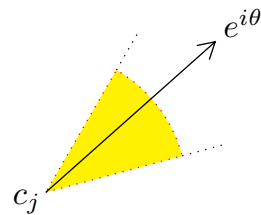
$$(E) \quad \boxed{u_\nu(x) = \phi(x)x^{\lambda_{\nu,0}^o} \exp\left(-\frac{\lambda_{\nu,r}^o}{rx^r} - \frac{\lambda_{\nu,r-1}^o}{(r-1)x^{r-1}} - \cdots - \frac{\lambda_{\nu,1}^o}{x}\right)} \quad (\nu = 1, \dots, n)$$

$$\phi(x) = 1 + a_1x + a_2x^2 + \cdots \in \mathbb{C}[[x]]$$

$$\lambda_\nu(x) := \lambda_{\nu,r}^o x^r + \cdots + \lambda_{\nu,1}^o x + \lambda_{\nu,0}^o \in \mathbb{C}[x], \quad \lambda_\nu \neq \lambda_{\nu'} \quad (\Leftarrow \nu \neq \nu')$$

Theorem [Hukuhara,...] $\theta \in \mathbb{R} \Rightarrow \exists R > 0, \theta_1, \theta_2$ and solutions u_ν to $\mathcal{M}$ such that $\theta \in (\theta_1, \theta_2)$ and u_ν have asymptotic (E) for $x \rightarrow c_j = 0$ on

$$V_{c_j, R, (\theta_1, \theta_2)} := \{x \in \mathbb{C} \mid |x - c_j| < R, \theta_1 < \arg(x - c_j) < \theta_2\}$$



Characteristic exponents $\lambda_{j,\nu} \in \mathbb{C}[x]$ of $\mathcal{M}$ at $x = c_j$ are defined through $c_j \mapsto 0$

- No logarithmic term ($\Leftarrow$ for simplicity)
- No ramified irregular singularity ($\Leftarrow$ essential)

$\{[\lambda_{j,1}]_{(m_1)}, \dots, [\lambda_{j,n_j}]_{(m_{n_j})}\} : \text{characteristic exponents of } \mathcal{M} \text{ at } x = c_j$

$$\lambda_{j,\nu} \in \mathbb{C}[x], \quad n = m_{j,1} + \dots + m_{j,n_j} \quad (j = 0, \dots, p),$$

$$[\lambda]_{(m)} := \{\lambda, \lambda + 1, \dots, \lambda + m - 1\} \quad (\lambda \in \mathbb{C}[x])$$

Generalized Riemann scheme (GRS)

$$(\star) : \left\{ \begin{array}{c} x = c_j \quad (j = 0, \dots, p) \\ [\lambda_{j,1}]_{(m_{j,1})} \\ \vdots \\ [\lambda_{j,n_j}]_{(m_{j,n_j})} \end{array} \right\}, \quad \left\{ \begin{array}{ccc} x = 0 & x = 1 & x = \infty \\ 0 & [0]_{(n-1)} & \alpha_1 \\ 1 - \beta_1 & & \alpha_2 \\ \vdots & & \vdots \\ 1 - \beta_{n-1} & -\beta_n & \alpha_n \end{array} \right\} \quad \begin{array}{l} 1^n, (n-1)1, 1^n \\ {}_nF_{n-1}(\alpha; \beta; x) \\ \sum a_\nu = \sum b_\nu \end{array}$$

- $\deg \lambda_{j,\nu} = 0 \quad (\nu = 1, \dots, n_j) \Rightarrow c_j \text{ is a regular singular point}$
- Versal unfolding of (GRS) $\hookrightarrow$ (GRS) of a Fuchsian equation (cf. [Ov])
 $\Rightarrow$ spectral type (a tuple of partition of $\text{ord } P$)

Example : spectral type 211|22|22, 31

$Pu = 0$: $\text{ord } P = 4$ with $x = \infty$ (Poincaré rank 2) and 0 (regular singularity)

$$u(x) \approx x^{-a_0} (1 + o(x^{-1})) e^{-a_1 x - \frac{1}{2} a_2 x^2}, \quad x^{-a_0-1} e^{-a_1 x - \frac{1}{2} a_2 x^2} \quad (x \rightarrow +\infty)$$

$$\approx x^{-b_0} e^{-b_1 x}, \quad x^{-c_0} e^{-b_1 x} \quad (x \rightarrow +\infty)$$

$$\approx (1 + o(x^2)) x^{c_1}, \quad (x + o(x))^{c_1+1}, \quad x^{c_1+2}, \quad x^{c_2} \quad (x \rightarrow +0)$$

Fuchs-Hukuhara relation : $2a_0 + b_0 + c_0 + 2c_1 + c_2 + c_3 = 4$

$$\left\{ \begin{array}{cc} x = \infty & x = 0 \\ [a_0 + a_1 x + a_2 x^2]_{(2)} & [c_1]_{(3)} \\ b_0 + b_1 x & c_2 \\ c_0 + b_1 x & \end{array} \right\} = \left\{ \begin{array}{cccc} x = \infty & (1) & (2) & x = 0 \\ [a_0]_{(2)} & [a_1]_2 & [a_2]_2 & [c_1]_{(3)} \\ b_0 & [b_1]_2 & [0]_2 & c_2 \\ c_0 & & & \end{array} \right\}$$

$\text{idx}(211|22|22, 31) = 0 \Rightarrow 1 (= 1 - \frac{\text{idx}}{2})$ accessory parameter $\rightsquigarrow$ versal Heun

$$\left\{ \begin{array}{cccc} x = \infty & x = \frac{1}{t_1} & x = \frac{1}{t_2} & x = 0 \\ [a_0 - \frac{a_1}{t_1} + \frac{a_2}{t_1 t_2}]_{(2)} & [\frac{a_1}{t_1} + \frac{a_2}{t_1(t_1-t_2)}]_{(2)} & [\frac{a_2}{t_2(t_2-t_1)}]_{(2)} & [c_1]_{(3)} \\ b_0 - \frac{b_1}{t_1} & [\frac{b_1}{t_1}]_{(2)} & [0]_{(2)} & c_2 \\ c_0 - \frac{b_1}{t_1} & & & \end{array} \right\}$$

§ Index of rigidity and operations

$$\begin{aligned}\text{idx } \mathbf{m} &= 2n^2 - \sum (n^2 - m_{j,\nu}^2) - \sum \deg(\lambda_{j,\nu} - \lambda_{j,\nu'}) \cdot m_{j,\nu} m_{j,\nu'} \\ &= 2n^2 - \sum (n^2 - (m_{j,\nu}^{(r)})^2)\end{aligned}$$

$$\begin{aligned}d_1(\mathbf{m}) &= 2n - \sum (n - m_{j,1}) - \sum \deg(\lambda_{j,\nu} - \lambda_{j,1}) \cdot m_{j,\nu} \\ &= 2n - \sum (n - m_{j,1}^{(r)})\end{aligned}$$

$$(\text{FC}) \quad \sum m_{j,\nu} \lambda_{j,\nu}(0) - \text{ord } \mathbf{m} + \frac{1}{2} \text{idx } \mathbf{m} := |\{[\lambda_{j,\nu}^{(r)}]_{(m_{j,1}^{(r)})}\}| = 0$$

Addition

$$\begin{aligned}u(x) \mapsto v(x) = \varphi(x)u(x) &\Rightarrow \partial := \frac{d}{dx} \mapsto \text{Ad}(\varphi)\partial = \varphi \circ \partial \circ \varphi^{-1} = \partial - \frac{\varphi'}{\varphi} \\ \text{Ad}((x-c)^\lambda)\partial &= \partial - \frac{\lambda}{x-c}, \quad \vartheta := x\partial \mapsto \text{Ad}(x^\lambda)\vartheta = \vartheta - \lambda\end{aligned}$$

Versal addition

$$\begin{aligned}\text{Ad} \left(\prod_{r=0}^{r_j} (x - c_j - t_r)^{\sum_{\substack{k=r \\ s \neq r}}^{r_j} \frac{\lambda_{j,\nu,k}}{\prod_{0 \leq s \leq k} (t_r - t_s)}} \right) &= \text{Ad} \left(e^{\int \sum_{r=0}^{r_j} \frac{\lambda_{j,\nu,r}}{\prod_{k=0}^r (x - c_j - t_k)} dx} \right) \\ \partial &\mapsto \partial - \sum_{r=0}^{r_j} \frac{\lambda_{j,\nu,r}}{\prod_{k=0}^r (x - c_j - t_k)}, \quad x \mapsto x\end{aligned}$$

Middle convolution : a microlocal operator

$$u(x) \mapsto v(x) = \partial^{-\mu} u = \frac{1}{\Gamma(\mu)} \int_c^x u(t)(x-t)^{\mu-1} dt \quad (c : \text{singular point}, \mu \in \mathbb{C})$$

$$Pu(x) = 0 \Rightarrow \text{mc}_{\mu}(P)v(x) = 0 \quad (\text{mc}_{\mu}(\partial) = \partial, \text{mc}_{\mu}(\vartheta) = \vartheta - \mu)$$

$$P = \sum_{j=0}^n a_j(x) \partial^j = \sum_{j=0}^n \sum_{i=0}^N c_{ij} x^i \partial^j \in W[x] \quad (a_j(x) \in \mathbb{C}[x], c_{ij} \in \mathbb{C})$$

$$\partial^N P = \sum_{j=0}^n \sum_{i=0}^N \partial^N c_{ij} x^i \partial^j = \sum_{j=0}^n \sum_{i=0}^N c_{ij} \partial^{N-i} (\vartheta + 1 + i)_i \partial^j$$

$$\text{mc}_{\mu}(P) := \partial^{-L} \left(\sum_{j=0}^n \sum_{i=0}^N c_{ij} \partial^{N-i} (\vartheta + 1 + i - \mu)_i \partial^j \right) = \partial^{N-L} \partial^{-\mu} P \partial^{\mu} \in W[x]$$

$$(L \in \mathbb{Z}_{\geq 0} : \text{largest}) \quad (a)_i := (a)(a+1) \cdots (a+i-1)$$

$$\text{Ad}(\varphi^{-1}) \circ \text{Ad}(\varphi) = \text{id}, \quad \text{mc}_{-\mu} \circ \text{mc}_{\mu} = \text{id} \Rightarrow \text{invertible operators}$$

They keep the index of rigidity ($\text{idx } \mathbf{m}$)

Spectral type $\mathbf{m}$ of $Pu = 0$ is rigid ($\text{idx } \mathbf{m} = 2$)

$\Rightarrow Pu = 0$ is transformed into trivial Eq. $u' = 0$ by these operations

$\Rightarrow$ Existence and construction of $Pu = 0$ for a given (GRS)

§ Properties of middle convolution

$$(\star) : \left\{ \begin{array}{cccc} x = c_0 = \infty & x = c_1 & \cdots & x = c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \cdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{array} \right\}$$

$$\text{mc}_\mu : \{[\lambda_{j,\nu}]_{(m_{j,\nu})}\} \mapsto \{[\lambda'_{j,\nu}]_{(m_{j,\nu})}\}, \quad Pu = 0 \mapsto P'v = 0 \quad (P' = \text{mc}_\mu(P))$$

Theorem. Suppose $\lambda_{0,1} = \mu + 1$ ($\deg \lambda_{0,1} = 0$), $\lambda_{1,1} = \cdots = \lambda_{p,1} = 0$ and other exponents are generic ($m_{j,1}$ may be 0). Then ([Of, Hi])

$$m'_{j,\nu} = m_{j,\nu} - \delta_{\nu,1} \cdot d_1(\mathbf{m}) \quad (1 \leq \nu \leq n_j, \ 0 \leq j \leq p),$$

$$\lambda'_{j,1} = \delta_{j,0} \cdot (1 - \mu) \quad (j = 0, \dots, p),$$

$$\lambda'_{j,\nu} = \lambda_{j,\nu} + (\deg \lambda_{j,\nu} + (-1)^{\delta_{j,0}}) \cdot \mu \quad (2 \leq \nu \leq n_j, \ 0 \leq j \leq p)$$

and the irreducibility is kept.

By suitable (versal) additions and transpositions of the indices, we may assume

$$(\star) \text{ with } (\sharp) \quad \sum_r m_{j,1}^{(r)} \geq \sum_r m_{j,\nu}^{(r)} \quad (j = 0, \dots, p, \ \nu = 2, \dots, n_j).$$

Lemma. $(\sharp) \text{ and } \text{idx } \mathbf{m} > 0 \Rightarrow d_1(\mathbf{m}) > 0 \Rightarrow \text{ord } P' = \text{ord } P - d_1(\mathbf{m}) < \text{ord } P$

$$(\partial^{-\mu} u)(x) = (I_a^\mu u)(x) := \frac{1}{\Gamma(\mu)} \int_a^x u(t)(x-t)^{\mu-1} dt$$

(a : a singular point of $u(x)$)

$$I_0^\mu(x^\lambda) = \frac{1}{\Gamma(\mu)} \int_0^x t^\lambda (x-t)^{\mu-1} dt = \frac{x^{\lambda+\mu}}{\Gamma(\mu)} \int_0^1 s^\lambda (1-s)^{\mu-1} ds \quad (t = xs)$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu}$$

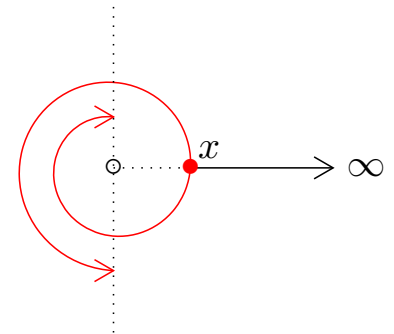
$$\tilde{I}_\infty^\mu(x^\lambda e^{-x}) = \frac{1}{\Gamma(\mu)} \int_x^\infty t^\lambda e^{-t} (t-x)^{\mu-1} dt = \frac{1}{\Gamma(\mu)} \int_0^\infty (x+s)^\lambda e^{-x-s} s^{\mu-1} ds$$

$$= \frac{x^\lambda e^{-x}}{\Gamma(\mu)} \int_0^\infty \sum_{n=0}^\infty (-\lambda)_n \left(-\frac{s}{x}\right)^n s^{\mu-1} e^{-s} ds$$

$$\sim \frac{x^\lambda e^{-x}}{\Gamma(\mu)} \sum_{n=0}^\infty (-\lambda)_n \left(-\frac{1}{x}\right)^n \int_0^\infty s^{\mu+n} e^{-s} ds$$

$$\sim x^\lambda e^{-x} \sum_{n=0}^\infty \frac{(-\lambda)_n (\mu)_n}{n!} \left(-\frac{1}{x}\right)^n = x^\lambda e^{-x} {}_2F_0\left(-\lambda, \mu; -\frac{1}{x}\right)$$

($|x| \rightarrow \infty$, $|\arg x| < \frac{3\pi}{2}$) : (Whittaker function)



$${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; x) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n n!} x^n, \text{ which satisfies}$$

$$\prod_{\nu=1}^q (\vartheta + \beta_{\nu}) \cdot \partial u = \prod_{\nu=1}^p (\vartheta + \alpha_{\nu}) u \qquad \left\{ \begin{array}{l} (n+\beta_1) \cdots (n+\beta_q) (n+1) a_{n+1} x^n \\ \qquad \qquad \qquad = (n+\alpha_1) \cdots (n+\alpha_p) a_n x^n \end{array} \right. \Leftrightarrow u = \sum a_n x^n$$

$$\text{Radius of the convergence} = \begin{cases} \infty & (p \leq q) \\ 1 & (p = q + 1) \\ 0 & (p > q + 1) \end{cases}$$

$$e^x = {}_0F_0(x), \quad (1-x)^{-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} x^n = {}_1F_0(\lambda; x),$$

$$\begin{aligned} &I_0^\mu(x^\lambda \cdot {}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \dots, \beta_q; \pm x)) \\ &= x^{\lambda+\mu} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} \cdot {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, \lambda+1; \beta_1, \dots, \beta_q, \lambda+\mu+1; \pm x) \end{aligned}$$

$$I_0^\mu(x^\lambda e^{-x}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu} \cdot {}_1F_1(\lambda+1; \lambda+\mu+1; -x) \quad (\text{Kummer})$$

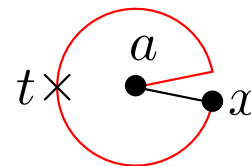
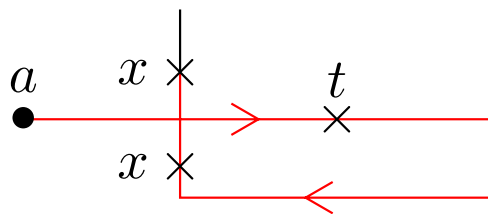
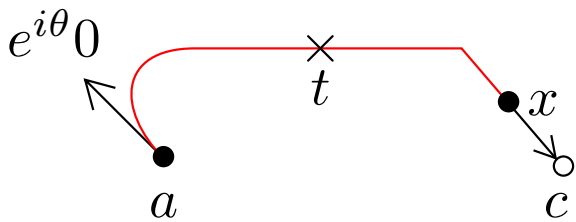
$$I_0^\mu(x^\lambda (1-x)^{\lambda'}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu} \cdot {}_2F_1(\lambda+1, -\lambda'; \lambda+\mu+1; x) \quad (\text{Gauss})$$

§ Riemann-Liouville transform

$$(I_{a+e^{i\theta}0}^\mu u)(x) := \frac{1}{\Gamma(\mu)} \int_{a+e^{i\theta}0}^x u(t)(x-t)^{\mu-1} dt = \frac{1}{\Gamma(\mu)} \int_L u(t)(x-t)^{\mu-1} dt$$

$$\textcolor{red}{L} : [\alpha, \beta] \ni t \mapsto L(t) \in \mathbb{C}, \quad L(\alpha) = a, \quad L(\beta) = x, \quad \theta = \arg L'(\alpha),$$

$$L(s) \neq L(t) \quad \text{for } \alpha \leq s < t \leq \beta$$



$$\tilde{L} : [\alpha, \gamma] \ni t \rightarrow \tilde{L}(t) \in \mathbb{C}, \ a = \tilde{L}(\alpha) \text{ and } c = \tilde{L}(\gamma),$$

$$\tilde{L}(s) \neq \tilde{L}(t) \text{ for } \alpha \leq s < t < \gamma.$$

- $u(x)$ is holomorphic for $x = \tilde{L}(t_0)$ ($t_0 \in (\alpha, \gamma)$), $L := \tilde{L}|_{[\alpha, t_0]}$

Want to know the asymptotics of $(I_a^\mu u)(x)$ for $t_0 \rightarrow \alpha$ and $t_0 \rightarrow \gamma$

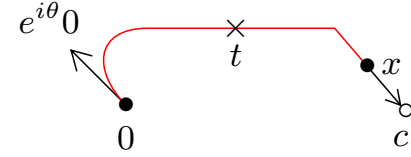
in terms of those of $u(x)$.

$$(\tilde{I}_{a-e^{i\theta_0}0}^\mu u)(x) = \frac{1}{\Gamma(\mu)} \int_x^{a-e^{i\theta_0}0} u(t)(t-x)^{\mu-1} dt$$



§ Riemann-Liouville transform (Main Theorem)

$$(I_{e^{i\theta}0}^\mu u)(x) = \frac{1}{\Gamma(\mu)} \int_{e^{i\theta}0}^x u(t)(x-t)^{\mu-1} dt$$



$$(1.1) \quad u(x) \sim \left(\sum_{n=0}^{\infty} a_n x^n \right) x^\lambda \quad (x \rightarrow e^{i\theta}0) \quad (\phi(x) := a_0 + a_1 x + \dots \in \mathbb{C}[[x]])$$

$$\Rightarrow (I_{e^{i\theta}0}^\mu u)(x) \sim \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} \left(\sum_{n=0}^{\infty} \frac{(\lambda+1)_n a_n}{(\lambda+\mu+1)_n} x^n \right) x^{\lambda+\mu}$$

$$(1.2) \quad u(x) \sim \phi(x) x^\lambda \exp\left(-\frac{C_0}{x^{m_0}} - \frac{C_1}{x^{m_1}} - \dots\right) \quad (x \rightarrow e^{i\theta}0) \quad \text{and} \quad \operatorname{Re} C_0 e^{-im_0\theta} > 0$$

$$\Rightarrow (I_{e^{i\theta}0}^\mu u)(x) \sim (m_0 C_0)^{-\mu} \psi(x) x^{\lambda+(m_0+1)\mu} \exp\left(-\frac{C_0}{x^{m_0}} - \frac{C_1}{x^{m_1}} - \dots\right)$$

$$\phi, \psi \in \mathbb{C}[[x]], \phi(0) = \psi(0), m_0 > m_1 > \dots > m_k > 0 \text{ and } m_j \in \mathbb{Z}_{>0}$$

$$(2.1) \quad u(x) \approx (c-x)^{\lambda'} \quad (x \rightarrow c - e^{i\theta'}0) \quad \text{and} \quad \operatorname{Re}(\lambda' + \mu) < 0$$

$$\Rightarrow (I_{e^{i\theta}0}^\mu u)(x) \approx \frac{\Gamma(-\lambda' - \mu)}{\Gamma(-\lambda')} (c-x)^{\lambda'+\mu} \Leftrightarrow (\operatorname{Re}(\lambda - \lambda') > 0 \Leftrightarrow c = 0, (1.1))$$

$$(2.2) \quad u(x) \approx (c-x)^{\lambda'} \exp\left(\frac{C'_0}{(c-x)^{m'_0}} + \frac{C'_1}{(c-x)^{m'_1}} + \dots\right) \quad (x \rightarrow c - e^{i\theta'}0) \quad \text{and}$$

$$\operatorname{Re} C'_0 e^{-im'_0\theta} > 0$$

$$\Rightarrow (I_{e^{i\theta}0}^\mu u)(x) \approx (m'_0 C'_0)^{-\mu} (c-x)^{\lambda'+(m'_0+1)\mu} \exp\left(\frac{C'_0}{(c-x)^{m'_0}} + \frac{C'_1}{(c-x)^{m'_1}} + \dots\right)$$

$$(2.3) \quad \lim_{x \rightarrow c - e^{i\theta'}0} u(x)(c-x)^{-\operatorname{Re} \mu} = 0 \Rightarrow \lim_{x \rightarrow c - e^{i\theta'}0} u(x) = (I_{e^{i\theta}0}^\mu u)(c - e^{i\theta'}0) \in \mathbb{C}$$

§ Outline of the Proof of (2.2) ($\Leftarrow$ [Or])

$$\int_0^x (1-t)^{\lambda} e^{\frac{C}{(1-t)^m} + \frac{C_1}{(1-t)^{m_1}} + \dots} (x-t)^{\mu-1} \phi(t) dt \quad (\phi \in C[0,1], \quad m > m_1 > \dots)$$

$$(t = x(1 - (1-x)^{m+1}s), \quad \operatorname{Re} C > 0, \quad \operatorname{Re} \mu > 0 \Leftarrow I_0^\mu u = I_0^{\mu+k} \partial^k u)$$

$$= x^\mu (1-x)^{\lambda + (m+1)\mu} e^{\frac{C}{(1-x)^m} + \frac{C_1}{(1-x)^{m_1}} + \dots}$$

$$\times \int_0^{\frac{1}{(1-x)^{m+1}}} (1 + x(1-x)^m s)^\lambda s^{\mu-1} e^{-f(x, (1-x)^m s)} \phi(x - x(1-x)^{m+1}s) ds,$$

$$f(x, s_2) = \frac{C}{(1-x)^m} \left(1 - \frac{1}{(1+x s_2)^m}\right) + \frac{C_1}{(1-x)^{m_1}} \left(1 - \frac{1}{(1+x s_2)^{m_1}}\right) + \dots,$$

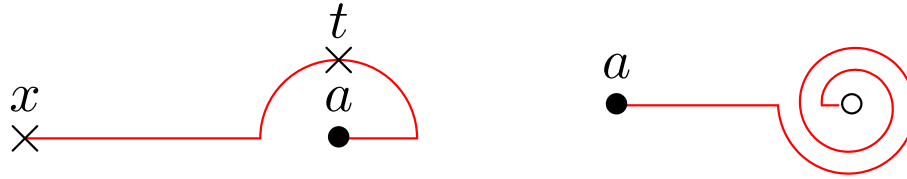
$$\bullet \int_{\frac{1}{(1-x)^m}}^{\frac{1}{(1-x)^{m+1}}} \dots ds \xrightarrow{x \rightarrow 1-0} 0,$$

$$\bullet (f(x, (1-x)^m s), \phi(x - x(1-x)^{m+1}s)) \xrightarrow{x \rightarrow 1-0} (mCs, \phi(1)) \Rightarrow$$

$$\int_0^{\frac{1}{(1-x)^m}} \dots ds \xrightarrow[\text{Lebergue's Dom. Cov. Th.}]{x \rightarrow 1-0} \int_0^\infty s^{\mu-1} e^{-mCs} \phi(1) ds = \Gamma(\mu) \phi(1)$$

$$(1.1), (2.1), (2.3) \Leftarrow [\text{Of}], (1.2) \Leftarrow [\text{Ov}]$$

Several Path

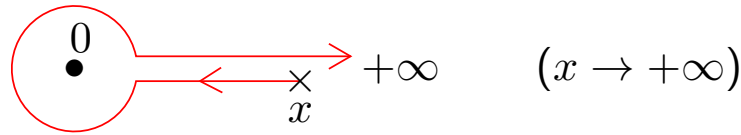
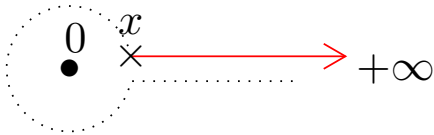


$$\tilde{I}_a^\mu u = \frac{1}{\Gamma(\mu)} \int_x^a u(t)(t-x)^{\mu-1} dt = e^{\mu\pi i} I_a^\mu u = \frac{e^{\mu\pi i}}{\Gamma(\mu)} \int_a^x u(t)(x-t)^{\mu-1} dt$$

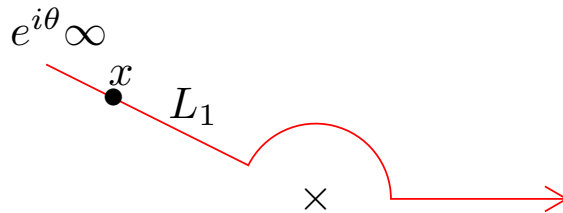
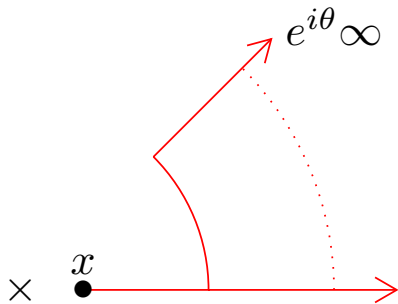
Singularity at ∞

$$(\tilde{I}_a^\mu u)(x) = x^{\mu-1} \cdot \left(I_{\frac{1}{a}}^\mu x^{-\mu-1} u\left(\frac{1}{x}\right) \right) \left(\frac{1}{x}\right)$$

$$a = e^{i\theta} \infty \Rightarrow \frac{1}{a} = e^{-i\theta} 0$$



$(x \rightarrow +\infty)$



$$\frac{1}{\Gamma(\mu)} \int_x^{+\infty} t^\lambda e^{-t} (t-x)^{\mu-1} dt$$

Asymptotic at ∞ (Theorem)

$$(1.1) \quad u(x) \sim \left(\sum_{n=0}^{\infty} a_n x^{-n} \right) x^{\lambda} \quad (x \rightarrow e^{i\theta} \infty) \quad (\phi(x) := a_0 + a_1 x + \dots \in \mathbb{C}[[x]])$$

$$\Rightarrow (\tilde{I}_{e^{i\theta}\infty}^{\mu} u)(x) \sim \frac{\Gamma(-\lambda - \mu)}{\Gamma(-\lambda)} \left(\sum_{n=0}^{\infty} \frac{\Gamma(-\lambda - \mu)_n a_n}{\Gamma(-\lambda)_n} x^{-n} \right) x^{\lambda + \mu}$$

$$(1.2) \quad u(x) \sim \phi\left(\frac{1}{x}\right) x^{\lambda} \exp(-C_0 x^{m_0} - C_1 x^{m_1} - \dots) \quad (x \rightarrow e^{i\theta} \infty) \quad \text{and} \\ \operatorname{Re} C_0 e^{im_0\theta} > 0, \phi \in \mathbb{C}[[x]]$$

$$\Rightarrow (I_0^{\mu} u)(x) \sim (m_0 C_0)^{-\mu} \psi\left(\frac{1}{x}\right) x^{\lambda + (1-m_0)\mu} \exp(-C_0 x^{m_0} - C_1 x^{m_1} - \dots), \\ \psi \in \mathbb{C}[[x]], \phi(0) = \psi(0), m_0 > m_1 > \dots > m_k > 0 \text{ and } m_j \in \mathbb{Z}_{>0}$$

$$(2.1) \quad u(x) \approx x^{\lambda'} \quad (x \rightarrow e^{i\theta'} \infty) \quad \text{and} \quad \operatorname{Re} \lambda' > -1$$

$$\Rightarrow (I_0^{\mu} u)(x) \approx \frac{\Gamma(\lambda' + 1)}{\Gamma(\lambda' + \mu + 1)} x^{\lambda' + \mu}$$

$$(2.2) \quad u(x) \approx x^{\lambda'} \exp(C'_0 x^{m'_0} + C'_1 x^{m'_1} + \dots) \quad (x \rightarrow e^{i\theta'} \infty) \quad \text{and} \quad \operatorname{Re} C'_0 e^{im'_0\theta} > 0$$

$$\Rightarrow (I_0^{\mu} u)(x) \approx (m'_0 C'_0)^{-\mu} x^{\lambda' + (1-m'_0)\mu} \exp(C'_0 x^{m'_0} + C'_1 x^{m'_1} + \dots)$$

§ Example (${}_{n-1}F_{n-1}$: a confluence of ${}_nF_{n-1}$)

$1^n | (n-1)1, 1^n$ (n=2 $\Rightarrow$ Kummer's Eq.)

$$G^\pm := \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} (1 \leq \nu < n) & \lambda_{0,\nu} (1 \leq \nu \leq n) \\ \pm x - \lambda_{1,n} & \end{array} \right\} \quad \left(\sum_{\nu=1}^n \lambda_{0,\nu} = \sum_{\nu=1}^n \lambda_{1,\nu} \right)$$

$$\xrightarrow{\text{Ad}(x^{-\lambda_{0,1}})} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} + \lambda_{0,1} (1 \leq \nu < n) & 0 \\ \pm x - \lambda_{1,n} + \lambda_{0,1} & \lambda_{0,\nu} - \lambda_{0,1} (2 \leq \nu \leq n) \end{array} \right\}$$

$$\xrightarrow{\text{mc} \lambda_{0,1} - \lambda_{1,1}} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} + \lambda_{1,1} (2 \leq \nu < n) & \lambda_{0,\nu} - \lambda_{1,1} (2 \leq \nu \leq n) \\ \pm x - \lambda_{1,n} + \lambda_{0,1} & \end{array} \right\}$$

$$\xrightarrow{\text{Ad}(x^{\lambda_{1,1} - \lambda_{0,2}})} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} + \lambda_{0,2} (3 \leq \nu < n) & 0 \\ \pm x - \lambda_{1,n} + \lambda_{0,1} + \lambda_{0,2} - \lambda_{1,1} & \lambda_{0,\nu} - \lambda_{0,2} (3 \leq \nu \leq n) \end{array} \right\}$$

$$\dots \xrightarrow{\text{Ad}(x^{\lambda_{0,n-1} - \lambda_{1,n-2}})}$$

$$\left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \pm x - \lambda_{1,n} + \lambda_{0,1} + \sum_{\nu=2}^{n-1} (\lambda_{0,\nu} - \lambda_{1,\nu-1}) & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\}$$

$$= G_2^\pm := \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \pm x + \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\} \quad (\text{Kummer})$$

$$G_2^\pm = \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \pm x + \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\} \quad (\text{Kummer})$$

$$\xrightarrow{\text{mc} \lambda_{0,n-1} - \lambda_{1,n-1}} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ \pm x + \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{1,n-1} \end{array} \right\} \ni x^{\lambda_{0,n} - \lambda_{1,n-1}} e^{\mp x}$$

$$G_2^\pm \xrightarrow{x \mapsto -x} G_2^\mp = \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \mp x + \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\}$$

$$\xrightarrow{\text{Ad}(e^{\mp x})} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ \pm x + 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\}$$

$$\xrightarrow{\text{mc} \lambda_{1,n-1} - \lambda_{0,n-1}} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ \pm x + 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & \lambda_{1,n-1} - \lambda_{0,n-1} - 1 \end{array} \right\}$$

$$G^\pm := \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} (1 \leq \nu < n) & \lambda_{0,\nu} (1 \leq \nu \leq n) \\ \pm x - \lambda_{1,n} & \end{array} \right\}$$

$$G^+ : u_{\infty, x - \lambda_{1,n}}(x) = \text{Ad}(x^{\lambda_{0,1}}) \tilde{I}_\infty^{\lambda_{1,1} - \lambda_{0,1}} \text{Ad}(x^{\lambda_{0,2} - \lambda_{1,1}}) \tilde{I}_\infty^{\lambda_{1,2} - \lambda_{0,2}} \dots$$

$$\dots \text{Ad}(x^{\lambda_{0,n-1} - \lambda_{1,n-2}}) \tilde{I}_\infty^{\lambda_{1,n-1} - \lambda_{0,n-1}} (x^{\lambda_{0,n} - \lambda_{1,n-1}} e^{-x})$$

$$\approx x^{\lambda_{1,n}} e^{-x} \quad (|x| \rightarrow \infty, \quad |\arg x| < \frac{3\pi}{2}),$$

$$u_{\infty, 1 - \lambda_{1,\nu}}^\pm(x) \approx (e^{\pm \pi i} x)^{\lambda_{1,\nu} - 1} \quad (|x| \rightarrow \infty, \quad |\arg x \pm \pi| < \frac{3\pi}{2}),$$

$$u_{0, \lambda_{0,\nu}}(x) \approx x^{\lambda_{0,\nu}} \quad (x \rightarrow 0) \quad (\Leftarrow \tilde{I}_\infty^\mu \mapsto I_0^\mu)$$

$$G^+ : \textcolor{red}{u}_{\infty, x-\lambda_{1,n}}$$

$$\begin{aligned} u_1(x) &:= \tilde{I}_{\infty}^{\lambda_{1,n-1}-\lambda_{0,n-1}} (x^{\lambda_{0,n}-\lambda_{1,n-1}} e^{-x}) \\ &\sim \sum_{m_1=0}^{\infty} \frac{(\lambda_{1,n-1}-\lambda_{0,n-1})_{m_1} (\lambda_{1,n-1}-\lambda_{0,n})_{m_1} (-1)^{m_1}}{m_1!} x^{\lambda_{0,n}-\lambda_{1,n-1}-m_1} e^{-x} \\ &= x^{\lambda_{0,n}-\lambda_{1,n-1}} e^{-x} {}_2F_0(\lambda_{1,n-1}-\lambda_{0,n-1}, \lambda_{1,n-1}-\lambda_{0,n}; -\frac{1}{x}) \quad (x \rightarrow +\infty) \\ &\approx \frac{\Gamma(\lambda_{0,n-1}-\lambda_{0,n})}{\Gamma(\lambda_{1,n-1}-\lambda_{0,n})} x^{\lambda_{0,n}-\lambda_{0,n-1}} \quad (x \rightarrow +0, \operatorname{Re} \lambda_{0,n} < \operatorname{Re} \lambda_{0,n-1}), \\ u_2(x) &:= \tilde{I}_{\infty}^{\lambda_{1,n-2}-\lambda_{0,n-2}} \operatorname{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}}) u_1(x) \end{aligned}$$

.....

$$\begin{aligned} u_{n-1}(x) &\sim \sum_{m_{n-1}=0}^{\infty} \cdots \sum_{m_1=0}^{\infty} \frac{\prod_{k=1}^{n-1} (m_1 + \cdots + m_{k-1} + \sum_{\nu=1}^k (\lambda_{1,n-k} - \lambda_{0,n+1-k}))_{m_k}}{m_1! \cdots m_{n-1}!} \\ &\quad \times \prod_{k=1}^{n-1} (\lambda_{1,n-k} - \lambda_{0,n-k})_{m_k} \left(-\frac{1}{x}\right)^{m_1+\cdots+m_{n-1}} \cdot x^{\sum_{k=1}^{n-1} (\lambda_{0,n+1-k} - \lambda_{1,n-k})} e^{-x} \\ &\approx x^{\lambda_{1,n}-\lambda_{0,1}} e^{-x} \quad (x \rightarrow +\infty) \\ &\approx \prod_{k=1}^{n-1} \frac{\Gamma(\lambda_{0,k}-\lambda_{0,n})}{\Gamma(\lambda_{1,k}-\lambda_{0,n})} x^{\lambda_{0,n}-\lambda_{0,1}} \quad (x \rightarrow +0, \operatorname{Re} \lambda_{0,n} < \operatorname{Re} \lambda_{0,k} \text{ for } k=1, \dots, n-1), \end{aligned}$$

$$\textcolor{red}{u}_{\infty, x-\lambda_{1,n}} = x^{\lambda_{0,1}} u_{n-1} = \sum_{m=1}^n \frac{\prod_{k \in I_{n,m}} \Gamma(\lambda_{0,k} - \lambda_{0,m})}{\prod_{k \in J_{n-1}} \Gamma(\lambda_{1,k} - \lambda_{0,m})} u_{0, \lambda_{0,m}} \quad \begin{cases} I_{n,m} := \{1, \dots, n\} \setminus \{m\} \\ J_{n-1} := \{1, \dots, n-1\} \end{cases}$$

$$G^{\pm} : u_{0,\lambda_0,k}^{\pm}$$

$$\begin{aligned} v_1^{\pm}(x) &:= I_0^{\lambda_{1,n-1}-\lambda_{0,n-1}}(x^{\lambda_{0,n}-\lambda_{1,n-1}} e^{\mp x}) \\ &= \frac{\Gamma(\lambda_{0,n}-\lambda_{1,n-1}+1)}{\Gamma(\lambda_{0,n}-\lambda_{0,n-1}+1)} x^{\lambda_{0,n}-\lambda_{0,n-1}} {}_1F_1(\lambda_{0,n}-\lambda_{1,n-1}+1; \lambda_{0,n}-\lambda_{0,n-1}+1; \mp x) \end{aligned}$$

$$v_1^-(x) \approx x^{\lambda_{0,n}-\lambda_{1,n-1}} e^x \quad (x \rightarrow +\infty),$$

$$v_1^+(x) \in \mathbb{C} e^{-x} I_0^{\lambda_{0,n}-\lambda_{1,n-1}}(x^{\lambda_{1,n-1}-\lambda_{0,n-1}-1} e^x) \Rightarrow$$

$$v_1^+(x) \approx x^{\lambda_{1,n-1}-\lambda_{0,n-1}-1} \quad (x \rightarrow +\infty)$$

$${}_1F_1(\alpha; \gamma; x) = e^x {}_1F_1(\gamma - \alpha; \gamma; -x) \approx \frac{\Gamma(\gamma)}{\Gamma(\alpha)} x^{\alpha-\gamma} e^x \quad (x \rightarrow +\infty)$$

$$v_2^{\pm}(x) := I_0^{\lambda_{1,n-2}-\lambda_{0,n-2}} \text{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}}) v_1^{\pm}(x)$$

.....

$$\begin{aligned} v_{n-1}^{\pm}(x) &= \prod_{\nu=1}^{n-1} \frac{\Gamma(\lambda_{0,n}-\lambda_{1,\nu}+1)}{\Gamma(\lambda_{0,n}-\lambda_{0,\nu}+1)} x^{\lambda_{0,n}-\lambda_{0,1}} \\ &\quad \times {}_{n-1}F_{n-1}(\lambda_{0,n}-\lambda_{1,\nu}+1 \quad (1 \leq \nu < n); \lambda_{0,n}-\lambda_{0,\nu}+1 \quad (1 \leq \nu < n); \mp x), \end{aligned}$$

$$v_{n-1}^-(x) \approx x^{\lambda_{1,n}-\lambda_{0,1}} e^x \quad (x \rightarrow +\infty)$$

$$v_{n-1}^+(x) \approx \prod_{\nu=1}^{n-2} \frac{\Gamma(\lambda_{1,n-1}-\lambda_{1,\nu})}{\Gamma(\lambda_{1,n-1}-\lambda_{0,\nu})} x^{\lambda_{1,n-1}-\lambda_{0,1}-1} \quad (x \rightarrow +\infty, \text{Re } \lambda_{1,n-1} > \text{Re } \lambda_{1,m}, \quad m \in J_{n-2})$$

$$u_{0,\lambda_0,k}(x) = x^{\lambda_{0,k}} {}_{n-1}F_{n-1}(\{\lambda_{0,k}-\lambda_{1,\nu}+1\}_{\nu \in I_{n-1}}; \{\lambda_{0,k}-\lambda_{0,\nu}+1\}_{\nu \in I_{n,k}}; -x)$$

$$G^- : w_{\infty, 1-\lambda_{1,m}}, u_{\infty, 1-\lambda_{1,m}}^{\pm}(x) = w_{\infty, 1-\lambda_{1,m}}(e^{\pm\pi i}x)$$

$$\begin{aligned} w_1(x) &= e^x \tilde{I}_{\infty}^{\lambda_{0,n}-\lambda_{1,n-1}+1} (x^{\lambda_{1,n-1}-\lambda_{0,n-1}-1} e^{-x}) \\ &= x^{\lambda_{1,n-1}-\lambda_{0,n-1}-1} {}_2F_0(\lambda_{0,n}-\lambda_{1,n-1}+1, \lambda_{0,n-1}-\lambda_{1,n-1}+1; -\frac{1}{x}) \\ &\approx \frac{\Gamma(\lambda_{0,n-1}-\lambda_{0,n}+1)}{\Gamma(\lambda_{0,n-1}-\lambda_{1,n-1}+1)} x^{\lambda_{0,n}-\lambda_{0,n-1}} \quad (x \rightarrow 0, \operatorname{Re} \lambda_{0,n} < \operatorname{Re} \lambda_{0,n-1}), \end{aligned}$$

$$w_2(x) = \tilde{I}_{\infty}^{\lambda_{1,n-2}-\lambda_{0,n-2}} \operatorname{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}}) w_1(x)$$

.....

$$\begin{aligned} w_{n-1}(x) &= \prod_{\nu=1}^{n-2} \frac{\Gamma(\lambda_{0,\nu}-\lambda_{1,n-1}+1)}{\Gamma(\lambda_{1,\nu}-\lambda_{1,n-1}+1)} \cdot x^{\lambda_{1,n-1}-\lambda_{0,1}-1} \\ &\quad \times {}_nF_{n-2}(\{\lambda_{0,\nu}-\lambda_{1,n-1}+1\}_{\nu \in I_n}; \{\lambda_{1,\nu}-\lambda_{1,n-1}+1\}_{\nu \in J_{n-2}}; -\frac{1}{x}) \\ &\approx \frac{\prod_{\nu=1}^{n-1} \Gamma(\lambda_{0,\nu}-\lambda_{0,n})}{\Gamma(\lambda_{0,n-1}-\lambda_{1,n-1}+1) \prod_{\nu=1}^{n-2} \Gamma(\lambda_{1,\nu}-\lambda_{0,n})} x^{\lambda_{0,n}-\lambda_{0,1}} \\ &\quad (x \rightarrow +0, \operatorname{Re} \lambda_{0,n} < \operatorname{Re} \lambda_{0,\nu} \text{ for } 1 \leq \nu < n) \end{aligned}$$

$$\begin{aligned} w_{\infty, 1-\lambda_{1,n-1}}(x) &= x^{\lambda_{0,1}} \prod_{\nu=1}^{n-2} \frac{\Gamma(\lambda_{1,\nu}-\lambda_{1,n-1}+1)}{\Gamma(\lambda_{0,\nu}-\lambda_{1,n-1}+1)} w_{n-1}(x) \\ &= x^{\lambda_{1,n-1}-1} \cdot {}_nF_{n-2}(\{\lambda_{0,\nu}-\lambda_{1,n-1}+1\}_{\nu \in I_n}; \{\lambda_{1,\nu}-\lambda_{1,n-1}+1\}_{\nu \in J_{n-2}}; -\frac{1}{x}) \\ &\approx \frac{\prod_{\nu=1}^{n-2} \Gamma(\lambda_{1,\nu}-\lambda_{1,n-1}+1) \prod_{\nu=1}^{n-1} \Gamma(\lambda_{0,\nu}-\lambda_{0,n})}{\prod_{\nu=1}^{n-1} \Gamma(\lambda_{0,\nu}-\lambda_{1,n-1}+1) \prod_{\nu=1}^{n-2} \Gamma(\lambda_{1,\nu}-\lambda_{0,n})} x^{\lambda_{0,n}} \\ &\quad (x \rightarrow +0, \operatorname{Re} \lambda_{0,n} < \operatorname{Re} \lambda_{0,\nu} \text{ for } 1 \leq \nu < n). \end{aligned}$$

Connection formula:

$$\begin{aligned}
 (u_{\infty, 1-\lambda_{1,m}}^{\pm}(x) &= w_{\infty, 1-\lambda_{1,m}}(e^{\pm\pi i}x)) \\
 u_{\infty, 1-\lambda_{1,m}}^{\pm}(x) &= \sum_{k=1}^n c_{\Gamma}(\infty:1-\lambda_{1,m} \rightsquigarrow 0:\lambda_{0,k}) e^{\pm\lambda_{0,k}\pi i} u_{0,\lambda_{0,k}}(x) \\
 &\approx (e^{\pm\pi i}x)^{\lambda_{1,m}-1} \quad (|x| \rightarrow \infty, \quad |\arg x \pm \pi| < \frac{3\pi}{2}) \\
 u_{\infty, x-\lambda_{1,n}}(x) &= \sum_{k=1}^n c_{\Gamma}(\infty:x-\lambda_{1,n} \rightsquigarrow 0:\lambda_{0,k}) u_{0,\lambda_{0,k}}(x) \\
 &\approx x^{\lambda_{1,n}} e^{-x} \quad (|x| \rightarrow \infty, \quad |\arg x| < \frac{3\pi}{2}) \\
 u_{0,\lambda_{0,k}}(x) &= \sum_{m=1}^n c_{\Gamma}(0:\lambda_{0,k} \rightsquigarrow \infty:1-\lambda_{1,m}) e^{\pm(1-\lambda_{1,m})\pi i} u_{\infty, 1-\lambda_{1,m}}^{\pm}(x) \\
 &\quad + c_{\Gamma}(0:\lambda_{0,k} \rightsquigarrow \infty:x-\lambda_{1,n}) e^{\pm(\lambda_{1,n}-\lambda_{0,k})\pi i} u_{\infty, x-\lambda_{1,n}}(x) \\
 &\approx x^{\lambda_{0,k}} \quad (x \rightarrow +0)
 \end{aligned}$$

Stokes relations:

$$\begin{aligned}
 &e^{\mp(\lambda_{1,m}-1)\pi i} \cdot u_{\infty, 1-\lambda_{1,m}}^{\pm}(e^{\mp 2\pi i}x) - e^{\pm(\lambda_{1,m}-1)\pi i} \cdot u_{\infty, 1-\lambda_{1,m}}^{\pm}(x) \\
 &= \pm 2\pi i \cdot c_{\Gamma}(\infty:1-\lambda_{1,m} \rightsquigarrow \infty:x-\lambda_{1,n}) \cdot u_{\infty, x-\lambda_{1,n}}(x), \\
 &e^{-\lambda_{1,n}\pi i} \cdot u_{\infty, x-\lambda_{1,n}}(e^{\pi i}x) - e^{\lambda_{1,n}\pi i} \cdot u_{\infty, x-\lambda_{1,n}}(e^{-\pi i}x) \\
 &= -2\pi i \sum_{m=1}^{n-1} c_{\Gamma}(\infty:x-\lambda_{1,n} \rightsquigarrow \infty:1-\lambda_{1,m}) \cdot u_{\infty, 1-\lambda_{1,m}}^{\pm}(e^{\mp\pi i}x)
 \end{aligned}$$

$$n = 2 \Rightarrow {}_2F_0(\alpha, \beta; -\frac{1}{e^{\pm\pi i}x}) - {}_2F_0(\alpha, \beta; -\frac{1}{e^{\mp\pi i}x}) = \frac{2\pi i x^{\alpha+\beta-1} e^{-x}}{\Gamma(1-\alpha)\Gamma(1-\beta)} {}_2F_0(1-\alpha, 1-\beta; -\frac{1}{x})$$

Γ -factors:

$$c_{\Gamma}(\infty:1-\lambda_{1,m} \rightsquigarrow 0:\lambda_{0,k}) = \frac{\prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,\nu} - \lambda_{0,k}) \prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{1,\nu} - \lambda_{1,m} + 1)}{\prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,\nu} - \lambda_{1,m} + 1) \prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{1,\nu} - \lambda_{0,k})}$$

$$c_{\Gamma}(\infty:x-\lambda_{1,n} \rightsquigarrow 0:\lambda_{0,k}) = \frac{\prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,\nu} - \lambda_{0,k})}{\prod_{\nu \in J_{n-1}} \Gamma(\lambda_{1,\nu} - \lambda_{0,k})}$$

$$c_{\Gamma}(0:\lambda_{0,k} \rightsquigarrow \infty:1-\lambda_{1,m}) = \frac{\prod_{\nu \in I_{n-1,m}} \Gamma(\lambda_{1,m} - \lambda_{1,\nu}) \prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,k} - \lambda_{0,\nu} + 1)}{\prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{0,k} - \lambda_{1,\nu} + 1) \prod_{\nu \in I_{n,k}} \Gamma(\lambda_{1,m} - \lambda_{0,\nu})}$$

$$c_{\Gamma}(0:\lambda_{0,k} \rightsquigarrow \infty:x-\lambda_{1,n}) = \frac{\prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,n} - \lambda_{0,\nu} + 1)}{\prod_{\nu \in J_{n-1}} \Gamma(\lambda_{0,k} - \lambda_{1,\nu} + 1)}$$

$$c_{\Gamma}(\infty:x-\lambda_{1,n} \rightsquigarrow \infty:1-\lambda_{1,m}) = \frac{\prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{1,\nu} - \lambda_{1,m} + 1)}{\prod_{\nu \in I_n} \Gamma(\lambda_{0,\nu} - \lambda_{1,m} + 1)}$$

$$c_{\Gamma}(\infty:1-\lambda_{1,m} \rightsquigarrow \infty:x-\lambda_{1,n}) = \frac{\prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{1,m} - \lambda_{1,\nu})}{\prod_{\nu \in I_n} \Gamma(\lambda_{1,m} - \lambda_{0,\nu})}$$

Irreducible $\Leftrightarrow \lambda_{0,k} - \lambda_{1,m} \notin \mathbb{Z} \quad (1 \leq k \leq n, \ 1 \leq m < n)$

$$\left\{ \begin{array}{cc} x = \infty & 0 \\ 1 - \lambda_{1,\nu} & \lambda_{0,\nu} \\ x - \lambda_{1,n} & \end{array} \right\} \xrightarrow{\times x^{-\lambda_{0,1}} e^x} \overset{\text{Kummer}}{\left\{ \begin{array}{cc} x = \infty & 0 \\ -x + \gamma - \alpha & 0 \\ \alpha & 1 - \gamma \end{array} \right\}} \xrightarrow{\times x^{\frac{\gamma}{2}} e^{-\frac{x}{2}}} \overset{\text{Whittaker}}{\left\{ \begin{array}{cc} x = \infty & 0 \\ -\frac{x}{2} - k & \frac{1}{2} - m \\ \frac{x}{2} + k & \frac{1}{2} + m \end{array} \right\}}$$

§ Evaluation of connection coefficients $c(\lambda)$

$$\text{Gauss : } \left\{ \begin{array}{ccc} x = 0 & x = 1 & x = \infty \\ \textcolor{red}{0} & \textcolor{red}{0} & \alpha \\ 1 - \gamma & \gamma - \alpha - \beta & \beta \end{array} ; x \right\}$$

$$F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta - 1; 1 - x) + O((1 - x)^{\gamma - \alpha - \beta})$$

1. Evaluation of an integral expression of the corresponding solution

$$F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tx)^{-\alpha} dt \quad (t \rightarrow 1-0)$$

2. Difference equation of $c(\lambda)$ ($\Leftarrow$ contiguous relation) and asymptotic of $c(\lambda)$

$$c(\alpha, \beta, \gamma + 1) = \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma(\gamma - \alpha - \beta)} c(\alpha, \beta, \gamma), \quad \lim_{\gamma \rightarrow +\infty} c(\alpha, \beta, \gamma) = 1$$

3. Possible poles of $c(\lambda)$ ($\Leftarrow$ difference of characteristic exponents) and known zeros of $c(\lambda)$ ($\Leftarrow$ reducibility) and asymptotic of $c(\lambda)$

Poles: $1 - \gamma = 1, 2, \dots$ and $\gamma - \alpha - \beta = 0, 1, \dots$ and zeros: $\gamma - \alpha, \gamma - \beta = 0, -1, \dots$

4. Change of asymptotic of solution under the Riemann-Liouville transform

$$I_0^\mu(x^\lambda(1-x)^{\lambda'}), \quad x \rightarrow +0, \quad 1-0$$

5. Irregular singularities $\Rightarrow$ regular singularities (by versal unfolding)

$$\lim_{\beta \rightarrow +\infty} F(\alpha, \beta, \gamma; \frac{x}{\beta}) = {}_1F_1(\alpha; \gamma; x), \quad \lim_{\gamma \rightarrow +\infty} F(\alpha, \beta, \gamma; \gamma x) = {}_2F_0(\alpha, \beta; x)$$

6. Extension to several variables

$$\text{Jordan-Pochhammer} \xrightarrow{\text{extension}} \text{Lauricella's } F_D \xrightarrow{\text{restriction}} \text{Gauss}$$

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Thank you for your attention!