STABLE HYPERPLANE ARRANGEMENTS

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Abstract. We classify complex hyperplane arrangements $\mathcal{A}$ whose intersection posets $L(\mathcal{A})$ satisfy $L(\mathcal{A}) = \pi_i^{-1} \circ \pi_i\bigl(L(\mathcal{A})\bigr)$ for $i = 1, \dots, n$. Here π_i denotes the projection from $\mathbb{C}^n$ onto $\mathbb{C}^{n-1}$ that forgets the coordinate x_i of $(x_1, \dots, x_n) \in \mathbb{C}^n$, and $\pi_i\bigl(L(\mathcal{A})\bigr) = \{\pi_i(S) \mid S \in L(\mathcal{A})\}$. We show that such arrangements $\mathcal{A}$ arise as pullbacks of the mirror hyperplanes of complex reflection groups of type A or B.

1. Introduction

Let $\mathcal{A}$ be a hyperplane arrangement in $V = \mathbb{C}^n$. That is, $\mathcal{A}$ is a finite union of hyperplanes :

$A \ni H := \{f_H(x) = 0 \mid x \in V\}$ where each $f_H(x)$ is a polynomial of degree 1.

If $0 \in H$ for every $H \in \mathcal{A}$, we say that $\mathcal{A}$ is *homogeneous.*

Definition 1.1. We denote by $L(\mathcal{A})$, or simply by $\mathcal{L}$, the set or poset of affine subspaces of V obtained as intersections of hyperplanes in $\mathcal{A}$

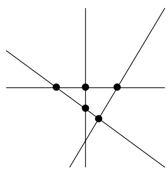
$$\mathcal{L} = L(\mathcal{A}) := \Bigl\{ \bigcap_{H \in \mathcal{B}} H \neq \emptyset \mid \mathcal{B} \subset \mathcal{A} \Bigr\}$$

and put

$$\mathcal{L}^{(k)} = L(\mathcal{A})^{(k)} := \{S \in L(\mathcal{A}) \mid \operatorname{codim} S = k\}, \quad \mathcal{L}^{(0)} = \{V\}, \quad \mathcal{L}^{(1)} = \mathcal{A}.$$

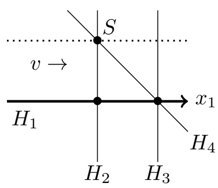
For affine subspaces S, $S' \in \mathcal{L}$, we define

$$\mathcal{L}^{(k)}_{\subset S} = L^{(k)}_{\subset S}(\mathcal{A}) := \{T \in \mathcal{L}^{(k)} \mid T \subset S\},$$
$$\mathcal{L}^{(k)}_{\supset S} = L^{(k)}_{\supset S}(\mathcal{A}) := \{T \in \mathcal{L}^{(k)} \mid T \supset S\},$$
$$\mathcal{A}_S := \mathcal{L}^{(1)}_{\supset S} = \{H \in \mathcal{A} \mid H \supset S\}.$$



For a non-zero vector $v \in V$ and S, $S' \in \mathcal{L}$, we define

$$\langle v, S\rangle := \{tv + y \mid t \in \mathbb{C},\ y \in S\},$$
$$\mathcal{A}_v := \{H \in \mathcal{A} \mid \langle v, H\rangle = V\},$$
$$\mathcal{A}_v^c := \mathcal{A} \setminus \mathcal{A}_v,$$
$$\operatorname{mc}_v \mathcal{A} := \mathcal{A} \cup \{\langle v, S\rangle \mid \operatorname{codim}\langle v, S\rangle = 1,\ S \in L(\mathcal{A})^{(2)}\}.$$



We call $\operatorname{mc}_v \mathcal{A}$ the *convolution* of $\mathcal{A}$ by v.

Remark 1.2. (i) When we fix a coordinate $x = (x_1, \dots, x_n)$ on $\mathbb{C}^n$, the i-th standard basis $e_i := (0, \dots, 0, \underset{\smile}{\overset{\mathrm{i}}{1}}, 0, \dots, 0)$ will occasionally be denoted by x_i for brevity.

(ii) Let $S = H \cap H' \in \mathcal{L}^{(2)}$ with H, $H' \in \mathcal{A}$. Suppose

$H = \{x \in \mathbb{C}^n \mid c_1x_1 + \cdots + c_nx_n + c = 0\}$ and $H' = \{x \in \mathbb{C}^n \mid c'_1x_1 + \cdots + c'_nx_n + c' = 0\}$.

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Then $H \in \mathcal{A}_{x_i}$ if and only if $c_i \neq 0$.

If $H,\, H' \in \mathcal{A}^c_{x_i}$, then $\langle x_i, S\rangle = S$.

If $H \in \mathcal{A}_{x_i}$ and $H' \in \mathcal{A}^c_{x_i}$, then $\langle x_i, S\rangle = H'$.

If $H,\, H' \in \mathcal{A}_{x_i}$, then

$$\langle x_i, S\rangle = \{x \in \mathbb{C}^n \mid c'_i(c_1x_1 + \cdots + c_nx_n + c) = c_i(c'_1x_1 + \cdots + c'_nx_n + c')\}.$$

Definition 1.3. A non-zero vector $v \in V$ is said to be parallel to $S \in L(\mathcal{A})$, or equivalently that S is called *v-closed* if and only if $\langle v, S\rangle = S$. If $\langle v, S\rangle \neq S$, v is transversal to S.

$\mathcal{A}$ is called v-closed if and only if $\langle v, S\rangle \in L(\mathcal{A})$ for every $S \in L(\mathcal{A})^{(2)}$.

If there exist n linearly independent vectors $v_1, \ldots, v_n \in \mathbb{C}^n$ such that $\mathcal{A}$ is v_i-closed for each i, we say that $\mathcal{A}$ is *stable*. Equivalently, in the coordinate system defined by $\{v_i\}$ we have

$$\mathrm{mc}_{x_i}\mathcal{A} = \mathcal{A} \qquad (i = 1, \ldots, n).$$

Example 1.4. (i) The braid arrangement

$$\mathcal{A} = \bigcup_{1\le i<j\le n} \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i = x_j\}$$

is a stable hyperplane arrangement, which corresponds to mirror hyperplanes of the reflection group of type A_{n-1}.

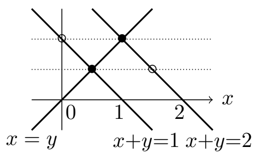
(ii) The arrangement of mirror hyperplanes of the reflection group of type D_n

$$\mathcal{A} = \{x_i = \pm x_j \mid 1 \le i < j \le n\} \quad (n \ge 4)$$

is not stable, which is contained in the stable hyperplane arrangement of type B_n

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \{x_i = 0 \mid 1 \le i \le n\}.$$

(iii) Under the coordinate system (x, y) of $\mathbb{C}^2$,
$\mathcal{A} = \big\{\{x = y\},\ \{x + y = 1\},\ \{x + y = 2\}\big\} \subset \mathbb{C}^2$,
$\mathcal{L}^{(2)} = \big\{(\frac{1}{2}, \frac{1}{2}),\ (1, 1)\big\}$,
$\mathrm{mc}_x\mathcal{A} = \mathcal{A} \cup \big\{\{y = \frac{1}{2}\},\ \{y = 1\}\big\}$.
There exists no stable arrangement $\tilde{\mathcal{A}}$ satisfying $\widetilde{\mathcal{A}} \supset \mathcal{A}$.



The purpose of this paper is to give a classification of stable hyperplane arrangements. Moreover, in §4, we determine the vectors v for which a stable hyperplane arrangement $\mathcal{A}$ is v-closed. We note that the definition of stable arrangements shares features with that of fibre-type arrangements (cf. [1]).

Lastly in this introduction, we explain the motivation for this paper.

A Pfaffian system with logarithmic singularities along hypersurface arrangement $\mathcal{A}$ is given by

$$\mathcal{M} : du = \Omega u,\ \Omega = \sum_{H\in\mathcal{A}} A_H d\log f_H \quad (\Omega \wedge \Omega = 0) \tag{1.1}$$

where A_H are constant square matrices of size N and u is a vector of N unknown functions Each A_H is called the *residue matrix* of $\mathcal{M}$ along H, and the condition $\Omega \wedge \Omega = 0$ is the integrability condition of $\mathcal{M}$. Then the convolution of $\mathcal{M}$ with respect to the variable x_i and a parameter $\mu \in \mathbb{C}$ produces a new Pfaffian system

$$\widehat{\mathcal{M}} = \widehat{\mathrm{mc}}_{x_i,\mu}\mathcal{M} : d\hat{u} = \hat{\Omega}\hat{u},\ \hat{\Omega} = \sum_{H\in\mathrm{mc}_{x_i}\mathcal{A}} \hat{A}_H d\log f_H. \tag{1.2}$$

The middle convolution $\mathrm{mc}_{x_i,\mu}\mathcal{M}$ of $\mathcal{M}$ is defined (see [3]) as an irreducible quotient of $\widehat{\mathcal{M}}$, and the corresponding transformation of solutions is realized by a Riemann-Liouville integral. Here the middle convolution of $\mathcal{M}$ generalizes the operation introduced for ordinary differential equations in [4, 2]. An *addition* of $\mathcal{M}$ is the transformation induced by the gauge change $u \mapsto \left(\prod_{H\in\mathcal{A}} f_H^{\lambda_H}\right)u$ with parameters $\lambda_H \in \mathbb{C}$. A wide class of systems can be generated from a given system by successive additions and middle convolutions. For example, any Fuchsian ordinary differential equation without an accessory parameter can be obtained from the trivial equation $u' = 0$ (cf. [4]). In particular, applying convolution to an addition of the trivial system (with $A_H = 0$), for generic parameters λ_H, v, and μ, yields an irreducible Pfaffian system whose singular locus equals $\mathrm{mc}_v\,\mathcal{A}$. If $\mathcal{A}$ is stable, these transformations may be analyzed while keeping the singular locus fixed. When $\mathcal{A}$ is the braid arrangement, the system is of KZ-type and the corresponding transformations were studied by [5]. The non-stable case will be treated in [6].

2. Classification of stable hyperplane arrangements

We first examine the property of being "v-closed," defined in the previous section, which will be used in [6].

Lemma 2.1. (i) *Let $S \in \mathcal{L}$ and $T, T' \in \mathcal{L}_{\subset S}^{(\mathrm{codim}\,S+1)}$ with $T \neq T'$. Then $\mathcal{A}_T \cap \mathcal{A}_{T'} = \mathcal{A}_S$.*

(ii) *The arrangement $\mathrm{mc}_v\,\mathcal{A}$ is v-closed.*

(iii) *For $S \in \mathcal{L}$, the subspace S is v-closed if and only if $\mathcal{A}_S \subset \mathcal{A}_v^{\mathrm{c}}$.*

(iv) *Let $H \in \mathcal{A}_v$ and $S \in \mathcal{L}$. If S is v-closed, then $H \cap S \neq \emptyset$, and*

$$\mathcal{A}_{H\cap S} \cap \mathcal{A}_v^c = \mathcal{A}_S. \tag{2.1}$$

(v) *Assume that $\mathcal{A}$ is v-closed and that $S \in \mathcal{L}$ is not v-closed. Let $H \in \mathcal{A}_S \cap \mathcal{A}_v^{\mathrm{c}}$. Then*

$$S = H \cap \langle v, S\rangle \quad \text{and} \quad \langle v, S\rangle = \bigcap_{H'\in\mathcal{A}_S\cap\mathcal{A}_v^c} H'. \tag{2.2}$$

In particular, the poset $\mathcal{L}$ is v-closed.

Proof. Let $T, T' \in \mathcal{L}_{\subset S}^{(\mathrm{codim}\,S+1)}$. If there exists $H \in (\mathcal{A}_T \cap \mathcal{A}_{T'}) \setminus \mathcal{A}_S$, then $T, T' \subset H \cap S \subsetneq S$, which implies $T = T' = H \cap S$. This proves (i).

Let $H, H' \in \mathrm{mc}_v\,\mathcal{A}$ with $\mathrm{codim}\langle v, H\cap H'\rangle = 1$. If $H \notin \mathcal{A}$, then $\langle v, H\cap H'\rangle = H$, which proves (ii).

Let $H \in \mathcal{A}$. For any $y \in H$, we have $v + y \in H$ if and only if $H \in \mathcal{A}_v^{\mathrm{c}}$. Hence $S \in \mathcal{L}$ is v-closed if and only if $\mathcal{A}_S \subset \mathcal{A}_v^{\mathrm{c}}$, which establishes (iii).

Suppose that $H \in \mathcal{A}_v$ and $S \in \mathcal{L}$ is v-closed. Since $\langle v, H\rangle = \mathbb{C}^n$ and $\langle v, S\rangle = S$, it follows that $H \cap S \neq \emptyset$. Let $H' \in (\mathcal{A}_{H\cap S} \cap \mathcal{A}_v^{\mathrm{c}}) \setminus \mathcal{A}_S$. Then $H' \cap S = H \cap S$ and $\mathcal{A}_{H'\cap S} \subset \mathcal{A}_v^c$, which contradicts the assumption that $H \in \mathcal{A}_{H\cap S}$. This completes the proof of (iv).

Under the assumption of (v), let $S \in \mathcal{L}^{(k)}$ with $k \geq 2$, and suppose $S = H \cap H_2 \cap \cdots \cap H_k$. Set $H_j' := \langle v, H\cap H_j\rangle \in \mathcal{A}_v^{\mathrm{c}}$. Then $H \cap H_j = H \cap H_j'$ for $j = 2, \ldots, k$. Hence $S = H \cap H_2' \cap \cdots \cap H_k'$ and $\langle v, S\rangle = H_2' \cap \cdots \cap H_k'$, which proves (v). □

Remark 2.2. Under the projection

$$\pi_i : \mathbb{C}^n \to \mathbb{C}^{n-1}, \quad (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

we have

$$L(\mathrm{mc}_{x_i}\,\mathcal{A}) = L(\mathcal{A}) \cup \bigcup_{S\in L(\mathcal{A})} \pi_i^{-1}\big(\pi_i(S)\big).$$

Hence $\mathcal{A}$ is stable if and only if

$$L(\mathcal{A}) = \pi_i^{-1}\big(\pi_i(L(\mathcal{A}))\big) \quad \text{for all } i = 1, \ldots, n.$$

We now study the condition under which $\mathcal{A}$ is x_i-closed. Throughout this section, a *coordinate transformation* of $(x_1,\dots,x_n)$ will usually mean a transformation of the form

$$x_j \mapsto a_j x_{\sigma(j)} + b_j \quad (j=1,\dots,n), \tag{2.3}$$

where a_j, $b_j \in \mathbb{C}$, $a_j \neq 0$, and σ is a permutation of the indices. Such transformations preserve stability.

Lemma 2.3. *Suppose that $\mathcal{A}$ is stable. Then there exists a decomposition of the set of indices*

$$\{1,\dots,n\} = \bigsqcup_{j=1}^{m} I_j \tag{2.4}$$

such that

$$\mathcal{A}_{x_i} \cap \mathcal{A}_{x_{i'}} \neq \emptyset \quad \textit{if and only if there exists } I_j \textit{ with } i,\, i' \in I_j.$$

Proof. Suppose that $\mathcal{A}_{x_1} \cap \mathcal{A}_{x_2} \neq \emptyset$, $\mathcal{A}_{x_2} \cap \mathcal{A}_{x_3} \neq \emptyset$, and $\mathcal{A}_{x_1} \cap \mathcal{A}_{x_2} \cap \mathcal{A}_{x_3} = \emptyset$. Then

$$\{x_2 = c_1x_1 + h_1(x_4,\dots,x_n)\}, \quad \{x_2 = c_3x_3 + h_3(x_4,\dots,x_n)\} \in \mathcal{A}$$

for some non-zero constants c_1, c_3 and suitable functions h_1 and h_3. Since $\mathcal{A}$ is x_2-closed, we have

$$\mathcal{A}_{x_1} \cap \mathcal{A}_{x_3} \ni \{\, c_1x_1 + h_1 = c_3x_3 + h_3 \,\} \in \mathcal{A}.$$

Thus, if $\mathcal{A}_{x_i} \cap \mathcal{A}_{x_j} \neq \emptyset$ and $\mathcal{A}_{x_j} \cap \mathcal{A}_{x_k} \neq \emptyset$, then $\mathcal{A}_{x_i} \cap \mathcal{A}_{x_k} \neq \emptyset$. This proves the lemma. □

According to the decomposition (2.4), the arrangement $\mathcal{A}$ can be written as

$$\begin{aligned} &\mathcal{A} = \bigcup_{1\le j\le m} \bigcup_{H\in\mathcal{A}_j} \pi_j^{-1}(H), \\ &\pi_j : \mathbb{C}^n \to \mathbb{C}^{\#I_j}, \quad (x_1,\dots,x_n) \mapsto (x_\nu)_{\nu\in I_j}, \end{aligned} \tag{2.5}$$

where each $\mathcal{A}_j$ is a stable hyperplane arrangement in $\mathbb{C}^{\#I_j}$. Conversely, for a given decomposition (2.4), the arrangement $\mathcal{A}$ defined by (2.5) is stable if and only if each $\mathcal{A}_j$ is a stable hyperplane arrangement in $\mathbb{C}^{\#I_j}$.

To classify stable arrangements, we may assume that $\mathcal{A}$ is *indecomposable*; namely,

$$\mathcal{A}_{x_i} \cap \mathcal{A}_{x_j} \neq \emptyset \qquad (1 \le i < j \le n). \tag{2.6}$$

We will give representatives of such arrangements under suitable coordinate transformations of the form (2.3).

Note that the following lemma is straightforward.

Lemma 2.4. *Let $a, b \in \mathbb{C}$ with $a \neq 0$, and define*

$$\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}, \quad (x_1,\dots,x_{n-2},x_{n-1},x_n) \mapsto (x_1,\dots,x_{n-2},ax_{n-1}+bx_n).$$

Then, for a hyperplane arrangement $\mathcal{A}'$ in $\mathbb{C}^{n-1}$, the inverse image $\pi^{-1}(\mathcal{A}')$ is a stable hyperplane arrangement in $\mathbb{C}^n$ if and only if $\mathcal{A}'$ is stable.

Definition 2.5. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^n$ is said to be *reducible* if $\mathcal{A} = \pi^{-1}(\mathcal{A}')$ for some arrangement $\mathcal{A}'$ in $\mathbb{C}^{n-1}$ as in Lemma 2.4, under a suitable coordinate system $(x_1,\dots,x_n)$. If $\mathcal{A}$ is not reducible, we say that $\mathcal{A}$ is *reduced*.

It is clear that an arrangement $\mathcal{A}$ given by the decomposition (2.5) is reduced if and only if each $\mathcal{A}_j$ is reduced.

Remark 2.6 (Trivial case). Assume that $\#\mathcal{L}^{(2)} = 1$. Let $\mathcal{L}^{(2)} = \{S\}$. Then it follows from the following Lemma 2.7 that

$$\widehat{\mathcal{A}} := \mathrm{mc}_{x_1}\,\mathrm{mc}_{x_2}\cdots\mathrm{mc}_{x_n}\,\mathcal{A} = \mathcal{A}\cup\{\langle x_i, S\rangle \mid \operatorname{codim}\langle x_i, S\rangle = 1,\quad i = 1,\dots,n\}$$

is stable, and moreover $L^{(2)}(\widehat{\mathcal{A}}) = \{S\}$.

Lemma 2.7. *Suppose that $\mathcal{L}^{(2)} = \{S\}$. Then $S \subset H$ for every $H \in \mathrm{mc}_v\,\mathcal{A}$ and every non-zero vector $v \in V$.*

Proof. Let $H \in \mathcal{A}$. Since $\mathcal{L}^{(2)} \neq \emptyset$, there exists $H' \in \mathcal{A}$ with $H \cap H' \neq \emptyset$, and so the assumption implies $S = H \cap H'$, and hence $S \subset \langle v, S\rangle \cap H$. □

Example 2.8. The hyperplane arragements

$$\begin{aligned}
\mathcal{A} &= \{x_i = \pm 1 \mid i = 1,\dots,4\} \cup \{x_i = x_{i+2} \mid i = 1, 2\} \subset \mathbb{C}^4,\\
\mathcal{A}' &= \{\{x_1 + x_2 + x_3 = 0\},\ \{x_1 = \pm 1\},\ \{x_2 + x_3 = \pm 1\}\} \subset \mathbb{C}^3,\\
\mathcal{A}'' &= \{\{x_1 + x_2 + x_3 = 0\},\ \{x_1 = x_2\},\ \{2x_1 + x_3 = 0\},\ \{2x_2 + x_3 = 0\}\} \subset \mathbb{C}^3
\end{aligned}$$

are stable. But $\mathcal{A}$ is decomposable, $\mathcal{A}'$ is reducible and $\#L^{(2)}(\mathcal{A}'') = 1$.

We now state the main result of this note.

Theorem 2.1. *Let $\mathcal{A}$ be a stable, reduced, and indecomposable hyperplane arrangement in $\mathbb{C}^n$. Assume that $\#\mathcal{L}^{(2)} > 1$, namely, $\mathcal{A}$ is non-trivial. Then, under a suitable coordinate system $(x_1,\dots,x_n)$, there exist a positive integer m, a non-negative integer r, and non-zero complex numbers $\alpha_1,\dots,\alpha_r$ such that $\mathcal{A}$ has the following form. Set*

$$\begin{aligned}
\Omega &:= \bigl\{e^{\frac{2\pi k\sqrt{-1}}{m}} \mid k = 1,\dots,m\bigr\},\\
\mathcal{A}_c &:= \bigl\{\{x_i = \omega\alpha_j\} \mid \omega \in \Omega,\ i = 1,\dots,n,\ j = 1,\dots,r\bigr\},\\
\mathcal{A}_0 &:= \bigl\{\{x_i = 0\} \mid 1 \le i \le n\bigr\}.
\end{aligned}$$

If $n = 2$, then $r \ge 1$ and

$$\mathcal{A} = \bigl\{\{x_1 = \omega x_2\} \mid \omega \in \Omega'\bigr\} \cup \mathcal{A}_c \cup \mathcal{A}_0, \quad \text{where } 1 \in \Omega' \subset \Omega.$$

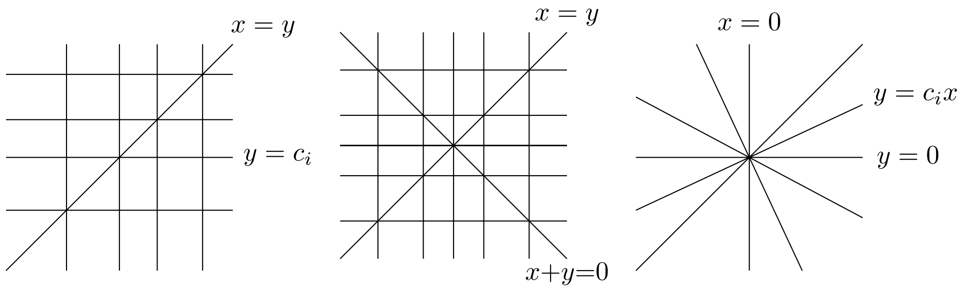
If $n \ge 3$, then

$$\begin{aligned}
&\mathcal{A} = \mathcal{A}' \quad \text{with } m = 1 \text{ and } n > 3, \text{ or } \quad \mathcal{A} = \mathcal{A}' \cup \mathcal{A}_c \cup \mathcal{A}_0,\\
&\mathcal{A}' := \bigl\{\{x_i = \omega x_j\} \mid \omega \in \Omega,\ 1 \le i < j \le n\bigr\}.
\end{aligned}$$

Remark 2.9. (i) When $n = 2$ in Theorem 2.1 and $\Omega' = \{\omega_1,\dots,\omega_L\}$, we may assume that

$$\Omega = \{\omega_1^{k_1}\cdots\omega_L^{k_L} \mid k_1,\dots,k_L \in \mathbb{Z}_{\ge 0}\}.$$

(ii) Examples of stable arrangements in $\mathbb{C}^2$ with coordinates (x, y):



3. Proof of Theorem 2.1

Proof of Theorem 2.1 for $n=2$. Choose $(a,b)\in\mathcal{L}^{(2)}$. Since $\mathcal{A}$ is x_i-closed, we have $\{x_1=a\},\ \{x_2=b\}\in\mathcal{A}$. By the assumption of the theorem, we may therefore assume that, under a suitable coordinate system (x,y) of $\mathbb{C}^2$,

$$\{x=0\},\ \{y=0\},\ \{x=1\},\ \{y=1\},\ \{x=y\}\in\mathcal{A}.$$

In this case, the condition $\{x=c\}\in\mathcal{A}$ equals $\{y=c\}\in\mathcal{A}$.

If $\#(\mathcal{A}_{x_1}\cap\mathcal{A}_{x_2})=1$, then the theorem follows immediately with $m=1$.

Hereafter we assume $\#(\mathcal{A}_{x_1}\cap\mathcal{A}_{x_2})>1$.

Now suppose $\{y=x+c\}\in\mathcal{A}$ for some $c\neq 0$. If $\{x=d\}\in\mathcal{A}$, then $\{y=c+d\}\in\mathcal{A}$ and hence $\{x=c+d\}\in\mathcal{A}$. By iteration, we obtain $\{x=nc+d\}\in\mathcal{A}$ for $n=0,1,2,\ldots$, which would imply $\#\mathcal{A}=\infty$. Thus, under a suitable coordinate transformation, we may instead assume that $\{y=\alpha_1x\}\in\mathcal{A}$ with $\alpha_1\neq 0,1$.

Hence we assume that

$$H_1=\{y=x\},\quad H_2=\{y=\alpha_1x\},\quad H_3=\{y=\alpha_2x+\alpha_3\},\quad H_4=\{x=1\}\in\mathcal{A}.$$

Here we do not necessarily assume $H_3\neq H_1$ or $H_3\neq H_2$.

If $\{x=z\}\in\mathcal{A}$ for some $z\in\mathbb{C}$, then $\{x=z\}\cap H_2=\{(z,\alpha_1z)\}\in\mathcal{L}$, so $\{y=\alpha_1z\}\in\mathcal{A}$ and therefore $\{x=\alpha_1z\}\in\mathcal{A}$. Similarly, $\{x=z\}\in\mathcal{A}$ implies $\{x=\alpha_2z+\alpha_3\}\in\mathcal{A}$. The desired conclusion then follows directly from Lemma 3.1. □

Lemma 3.1. *Let $\alpha_j\in\mathbb{C}$ $(j=1,2,3)$ satisfy $\alpha_1\alpha_2(\alpha_1-1)\neq 0$, and define*

$$T_1(z)=\alpha_1z,$$
$$T_2(z)=\alpha_2z+\alpha_3.$$

Suppose there exists a finite set $F\subset\mathbb{C}$ such that $0\neq z\in F$ and $T_1(F)=T_2(F)=F$. Then $\alpha_3=0$, and there exists an integer $m\geq 2$ such that

$$\alpha_1^m=\alpha_2^m=1\quad\text{and}\quad F\supset\bigl\{e^{\frac{2\pi k\sqrt{-1}}{m}}z\mid k=1,\ldots,m\bigr\}.\tag{3.1}$$

Proof. Let $a\in F$ be such that $|a|\geq|p|$ for all $p\in F$. Note that at least one of a or $T_1(a)$ is not a fixed point of T_2. Since $\#F<\infty$, there exists $m\in\mathbb{Z}_{>1}$ such that $\alpha_1^m=\alpha_2^m=1$. For $k=1,2,\ldots,m$, we then have $|\alpha_1^k\alpha_2a+\alpha_3|\leq|a|$, hence $|a+\alpha_1^{-k}\alpha_2^{-1}\alpha_3|\leq|a|$. Since $\alpha_1\neq 1$ and $\alpha_1^m=1$, this inequality holds only when $\alpha_3=0$, which proves (3.1). □

We next prepare a lemma that reduces the proof of the theorem to lower-dimensional cases.

Lemma 3.2. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$ with coordinates $(x_1,x_2,\ldots,x_n)$ and $n\geq 3$. Fix an integer m with $1<m<n$ and a point $p\in\mathbb{C}^{n-m}$. Define*

$$V':=\{(x_1,\ldots,x_n)\mid(x_{m+1},\ldots,x_n)=p\},$$

and call

$$\mathcal{A}':=\{H\cap V'\mid H\in\mathcal{A}\}\setminus\{\emptyset,V'\}$$

a specialization *of $\mathcal{A}$. For $i=1,\ldots,m$, $\mathcal{A}'$ is x_i-closed whenever $\mathcal{A}$ is x_i-closed. Moreover, if $\mathcal{A}$ is reduced, then so is $\mathcal{A}'$.*

Proof. For $H_1,H_2\in\mathcal{A}$, set $H'_j:=H_j\cap V'$ for $j=1,2$. Assume $H'_j\neq\emptyset$ and $H'_j\neq V'$ for $j=1,2$, and further that $H'_1\neq H'_2$ but $H'_1\cap H'_2\neq\emptyset$. Write

$$H_j=\{(x',x'')\in\mathbb{C}^n\mid f_j(x')=g_j(x'')\},\quad H'_j=\{x'\in\mathbb{C}^m\mid f_j(x')=g_j(p)\},$$

where $f_j(x')$ and $g_j(x'')$ are affine linear polynomials in $x'=(x_1,\ldots,x_m)$ and $x''=(x_{m+1},\ldots,x_n)$, respectively, satisfying $f_j(0)=0$. The assumptions imply that f_1 and f_2 are linearly independent over $\mathbb{C}$. It then follows that

$$\langle x_i,H'_1\cap H'_2\rangle=\langle x_i,H_1\cap H_2\rangle\cap V'\qquad(i=1,\ldots,m),$$

which proves the claim of the lemma. Note that $\dim H'_j = m-1$ holds precisely when $f_j \neq 0$ and $f_j(0)=0$. $\square$

Remark 3.3. A stable hyperplane arrangement $\mathcal{A}$ in Theorem 2.1 with $r>0$ and $n\geq 2$ can be viewed as a suitable specialization of a stable homogeneous hyperplane arrangement. For example, by setting $x_3=0$ and $x_4=1$ in $\big\{\{x_i=x_j\}\subset\mathbb{C}^4 \mid 1\leq i<j\leq 4\big\}$, we obtain the hyperplane arrangement $\big\{\{x_1=x_2\}, \{x_i=0\}, \{x_i=1\} \mid i=1,2\big\}$ in $\mathbb{C}^2$.

The following lemma is the key step in proving the theorem for the case $n\geq 3$.

Lemma 3.4. *Under the assumption of Theorem 2.1, we have*

$$\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}\cap\mathcal{A}_{x_k}=\emptyset \quad \textit{for } 1\leq i<j<k\leq n.$$

Proof of Theorem 2.1 for $n\geq 3$ assuming Lemma 3.4. We know that $\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}\neq\emptyset$ while $\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}\cap\mathcal{A}_{x_k}=\emptyset$ for distinct indices $i,j,k\in\{1,\dots,n\}$. Hence, we may assume that $\{x_i=x_{i+1}\}\in\mathcal{L}^{(2)}$ for $i=1,\dots,n-1$. It then follows that $\{x_i=x_j\}\in\mathcal{L}^{(2)}$ for all $1\leq i<j\leq n$. Applying Lemma 3.2 together with the case $n=2$, we obtain the theorem.

Indeed, if $\#(\mathcal{A}_{x_1}\cap\mathcal{A}_{x_2})=1$, then the result corresponds to the case $m=1$. If there exist two hyperplanes $H_1,H_2\in\mathcal{A}_{x_1}\cap\mathcal{A}_{x_2}$, we may assume $H_1\cap H_2\subset\{x_1=x_2=0\}$, and the theorem follows immediately. Note that if $\{x_1=a_2x_2\}$, $\{x_2=a_3x_3\}\in\mathcal{A}$, then $\{x_1=a_2a_3x_3\}$ and $\{x_1=a_2a_3x_2\}\in\mathcal{A}$, and therefore $\mathcal{A}\supset\mathcal{A}'$ with $m>1$. $\square$

Now we prove Lemma 3.4, starting with the case $n=3$.

Lemma 3.5. *Let $\mathcal{A}$ be a stable hyperplane arrangement in $\mathbb{C}^3$. Suppose $\mathcal{A}_{x_1}\cap\mathcal{A}_{x_2}\cap\mathcal{A}_{x_3}\neq\emptyset$.*

(i) *Then $\big\{\{x_1=c_1\},\ \{x_2=c_2\}\big\}\not\subset\mathcal{A}$ for any $c_1,\ c_2\in\mathbb{C}$.*

(ii) *Moreover, if $\{x_1=c_1\}\subset\mathcal{A}$ or $\#\mathcal{L}^{(2)}>1$, then $\mathcal{A}$ is reducible.*

Proof. We may assume $H:=\{x_1+x_2+x_3=0\}\in\mathcal{A}$. Suppose $\{x_1=c_1\},\{x_2=c_2\}\in\mathcal{A}$. Then, by translation, we may further assume $c_1=c_2=0$. Since $(x_1+x_2+x_3)-x_1=x_2+x_3$, we have $\{x_2+x_3=0\}\in\mathcal{A}$. If $\{nx_1=x_2\}\in\mathcal{A}$ (which holds for $n=0$), then the relations

$$(x_1+x_2+x_3)+(nx_1-x_2)=(n+1)x_1+x_3,\quad (n+1)x_1+x_3-(x_2+x_3)=(n+1)x_1-x_2,$$

imply $\{(n+1)x_1=x_2\}\in\mathcal{A}$, contradicting $\#\mathcal{A}<\infty$. Thus we have (i).

If $\{x_1=c_1\}\subset\mathcal{A}$, then $\{x_2+x_3=-c_1\}\subset\mathcal{A}$, and (i) implies that $\mathcal{A}$ is reducible. In fact, if $H':=\{a_1x_1+a_2x_2+a_3x_3=a_0\}\in\mathcal{A}$ with $a_2\neq a_3$, then $\langle x_1, H\cap H'\rangle=\{(a_2-a_1)x_2+(a_3-a_1)x_3=a_0\}\in\mathcal{A}$, and moreover $\{x_2=c_2\}$, $\{x_3=c_3\}\in\mathcal{A}$ with suitable c_2 and c_3.

Now suppose $\#\mathcal{L}^{(2)}\geq 2$. Since $\mathcal{A}$ is stable, we have $\#\mathcal{A}^c_{x_i}\geq 1$ for $i=1,2,3$. Assume $\#\mathcal{A}^c_{x_i}=1$ for $i=1,2,3$ and write $\mathcal{A}^c_{x_i}=\{\{H_i\}\}$. Then for $S\in\mathcal{L}^{(2)}$, we have $S\subset H_i$ for all $i=1,2,3$. Since $\mathcal{A}^c_{x_1}\cap\mathcal{A}^c_{x_2}\cap\mathcal{A}^c_{x_3}=\emptyset$, it follows that $S=H_1\cap H_2\cap H_3$, hence $\mathcal{L}^{(2)}=\{H_1\cap H_2\cap H_3\}$ and $\#\mathcal{L}^{(2)}=1$. Thus we may assume $\#\mathcal{A}^c_{x_3}\geq 2$.

Write

$$\mathcal{A}^c_{x_3}=\big\{\{x_1+ax_2=c_i\}\mid i=1,2,\dots\big\},$$

where $a\neq 0$ and $c_1\neq c_2$. Since $\{x_1+x_2+x_3=0\}\subset\mathcal{A}$, it follows from Lemma 3.2, Lemma 3.5 (i), and the theorem for $n=2$ that $a=1$, and hence $\mathcal{A}$ is reducible. $\square$

Lemma 3.6. *Let $\mathcal{A}$ be a reduced hyperplane arrangement in $\mathbb{C}^n$ with $n\geq 4$. Put $x=x_1,\ y=x_2,\ z=x_3,\ w=x_4,\ x'=(x_5,\dots,x_n)$. If*

$$\{x+y+z=h_0\},\quad \{w=h_1\},\quad \{y+az+w=h_2\}$$

belong to $\mathcal{A}$, then $\mathcal{A}$ is not stable. Here h_j are affine linear polynomials in x'.

Proof. If $a=0$, then it follows from Lemma 3.5 with the specialization $x_4=\cdots=x_n=0$ that $\mathcal{A}$ is not stable. If $a=1$, then $\{x-w=h_0-h_2\}\in\mathcal{A}$ because $\mathcal{A}$ is y-closed, and $\mathcal{A}$ is not stable as in the case $a=0$.

Thus we may assume $a\neq 0,1$. Then, from the relations

$$\begin{aligned}
ax+(a-1)y-w &= a(x+y+z)-(y+az+w) && (z\text{-closed}),\\
x+(1-a)z-w &= (x+y+z)-(y+az+w) && (y\text{-closed}),\\
x+(1-a)z &= \bigl(x+(1-a)z-w\bigr)+w && (w\text{-closed}),\\
ax+(a-1)y &= \bigl(ax+(a-1)y-w\bigr)+w && (w\text{-closed}),\\
(a-1)y+(a^2-a)z-w &= (ax+(a-1)y-w)-a(x+(1-a)z) && (x\text{-closed}),\\
ax+(a-a^2)z+w &= \bigl(ax+(a-1)y\bigr)-\bigl((a-1)y+(a^2-a)z-w\bigr) && (y\text{-closed}),\\
2ax+(a-1)y+(a-a^2)z &= \bigl(ax+(a-a^2)z+w\bigr)+\bigl(ax+(a-1)y-w\bigr) && (w\text{-closed}),\\
ax+(a-a^2)z+(1-a)w &= (1-a)(y+az+w)+\bigl(ax+(a-1)y\bigr) && (y\text{-closed}),\\
(1-a)(y+az+2w) &= 2\bigl(ax+(a-a^2)z+(1-a)w\bigr)\\
&\quad -\bigl(2ax+(a-1)y+(a-a^2)z\bigr) && (x\text{-closed}),
\end{aligned}$$

we obtain $\{y+az+2w=h_3\}\in\mathcal{A}$. Hence $\{y+az+2^nw=h_{n+2}\}\in\mathcal{A}$ for $n=0,1,2,\ldots$, where each $h_{n+2}(x')$ is an affine linear polynomial in x'. □

Proof of Lemma 3.4. Let m be the maximal integer such that there exist indices i_ν satisfying $1\le i_1<\cdots<i_m\le n$ and $\mathcal{A}_{x_{i_1}}\cap\cdots\cap\mathcal{A}_{x_{i_m}}\neq\emptyset$. We will show that $m=2$.

Assume to the contrary that $m>2$. Without loss of generality, we may assume $H:=\{x_1+\cdots+x_m=0\}\in\mathcal{A}$.

Suppose $m<n$. Choose $H_2\in\mathcal{A}_{x_2}\cap\mathcal{A}_{x_{m+1}}$. By the maximality of m, there exists i with $1\le i\le m$ such that $H_2\in\mathcal{A}^c_{x_i}$. We may assume $i=1$, and write

$$H_2=\{x_2+a_3x_3+\cdots+a_mx_m+x_{m+1}+\cdots+a_nx_n=a_0\}.$$

Suppose $\mathcal{A}^c_{x_1}\cap\cdots\cap\mathcal{A}^c_{x_m}\neq\emptyset$, and let $H_3\in\mathcal{A}^c_{x_1}\cap\cdots\cap\mathcal{A}^c_{x_m}$. We may assume $H_3\in\mathcal{A}_{x_{m+1}}$. By setting $x_4=\cdots=x_m=0$, Lemmas 3.2 and 3.6 imply that $\mathcal{A}$ is not stable, hence $\mathcal{A}^c_{x_1}\cap\cdots\cap\mathcal{A}^c_{x_m}=\emptyset$.

$$(\overset{x}{x_1},\overset{y}{x_2},\overset{z}{x_3},\ldots,x_m,\overset{w}{x_{m+1}},\ldots,x_n)$$

Thus, including the case $m=n$, we may assume

$$H_2=\{x_2+a_3x_3+\cdots+a_nx_n=a_0\}\in\mathcal{A}^c_{x_1}.$$

Since $\#\mathcal{L}^{(2)}>1$, there exists

$$H_3=\{b_1x_1+b_2x_2+b_3x_3+\cdots+b_nx_n=b_0\}\in\mathcal{A}$$

such that $H_3\not\supset H_1\cap H_2$.

If $H_3\cap H_1=\emptyset$, then by Lemma 3.4 and Lemma 3.2 (with the specialization $x_4=\cdots=x_n=0$), $\mathcal{A}$ is either reducible or not stable. Hence we must have $H_3\cap H_1\neq\emptyset$.

Since $\mathcal{A}$ is x_1-closed, we may assume $b_1=0$. Similarly, as $H_3\cap H_2\neq\emptyset$, the x_2-closedness of $\mathcal{A}$ implies that we may also assume $b_2=0$. Because $\mathcal{A}$ is non-trivial, there exists some $b_i\neq 0$ with $3\le i\le m$; we may take $b_3\neq 0$. Then, applying Lemma 3.5 to the restriction $x_4=\cdots=x_n=0$, we conclude that $\mathcal{A}$ is not stable. □

4. A related result

In this final section, we determine all vectors v for which a stable hyperplane arrangement $\mathcal{A}$ is v-closed.

Proposition 4.1. *Let $\mathcal{A}$ be the hyperplane arrangement described in Theorem 2.1, and assume that $\#\mathcal{L}^{(2)} > 1$. Let v be a non-zero vector in $\mathbb{C}^n$ such that $\mathcal{A}$ is v-closed.*

If $r > 0$ or $m > 1$, then v is a scalar multiple of one of the coordinate vectors x_i.

If $m = 1$ and $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}_0$, then v is a scalar multiple of either one of the x_i or of $(c, \dots, c) \in \mathbb{C}^n$ with $c \neq 0$.

If $m = 1$ and $\mathcal{A} = \mathcal{A}'$, then v is a scalar multiple of one of the x_i modulo $\mathbb{C}(1, \dots, 1)$.

Proof. We may assume $v = (1, c_2, c_3, \dots)$ with $c_2 \neq 0$.

Suppose $H_1 = \{x_1 = 0\}$ and $H_2 = \{x_2 = \alpha_1\}$ are in $\mathcal{A}$. Then

$$\langle v, H_1 \cap H_2 \rangle = \{x_2 = c_2 x_1 + \alpha_1\} \in \mathcal{A}.$$

Hence $r = 0$ and $n > 2$. Since $H_3 = \{x_2 = \omega x_3\} \in \mathcal{A}$, we also have

$$\langle v, H_1 \cap H_3 \rangle = \{x_2 - \omega x_3 = (c_2 - \omega c_3) x_1\} \in \mathcal{A},$$

which implies $c_2 = \omega c_3$, and therefore $m = 1$ and $c_2 = c_3$. By symmetry of the coordinates (cf. σ in (2.3)), we conclude that $m = 1$ and $v = (1, \dots, 1)$.

Thus we may assume

$$\mathcal{A} = \big\{\{x_i = x_j\} \mid 1 \leq i < j \leq n\big\} \quad \text{with } n \geq 4.$$

We may further assume $v = (1, 0, c_3, c_4, \dots)$. Then

$$\langle v, \{x_1 - x_2 = x_3 - x_4\} \rangle = \{x_2 - x_3 = (c_3 - c_4)(x_1 - x_2)\} \in \mathcal{A},$$
$$\langle v, \{x_1 - x_2 = x_2 - x_3\} \rangle = \{x_2 - x_3 = -c_3(x_1 - x_2)\} \in \mathcal{A}.$$

Hence $c_3 = c_4 = 0$ or 1. By symmetry, it follows that $v = (1, 0, 0, \dots, 0)$ or $(1, 0, 1, \dots, 1)$, which proves the claim. □

Remark 4.2. (i) For $m = 1$, the expression of the arrangement $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}_0$ in Theorem 2.1 remains the same under the coordinate system

$$(x_1 - x_n, \dots, x_{n-1} - x_n, -x_n)$$

of $\mathbb{C}^n$.

(ii) The arrangement $\mathcal{A} = \{x_i \pm x_j = 0 \mid 1 \leq i < j \leq 3\}$ is $(\epsilon_1, \epsilon_2, \epsilon_3)$-closed when $\epsilon_k \in \{1, -1\}$ for $k = 1, 2, 3$.

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