A Kac–Moody root system and linear ordinary differential equations

Toshio Oshima*

Abstract

We show that every irreducible linear differential equations on $\mathbb{P}^1$ with regular and/or unramified irregular singularities can be transformed into either the trivial equation $u' = 0$ or a Fuchsian system of E_8-fundamental spectral type by a sequence of invertible transformations consisting of confluences, unfoldings, Laplace transformations, gauge transformations and Möbius transformations. The E_8-spectral type is uniquely determined by the original equation. A Fuchsian system of E_8-fundamental spectral type has three singular points, 0, 1, and ∞. The degrees of the minimal polynomials of the residue matrices at 0 and 1 are two and three, respectively, and the sum of the maximal dimensions of eigenspaces of the residue matrices at 0, 1, and ∞ is not greater than the size of the matrices. The result follows from the correspondence between spectral types of Fuchsian systems and roots of a star-shaped Kac–Moody root system. We show that the E_8-fundamental spectral types form a complete set of representatives for the orbits of the corresponding transformations on the set of positive imaginary roots.

1 Introduction

Consider a linear differential equation

$$\frac{du}{dx} = A(x)u, \qquad A(x) \in M(n, \mathbb{C}(x)). \tag{1.1}$$

Here, $A(x)$ is a square matrix whose entries are rational functions. We write

$$A(x) = \sum_{j=1}^{p-1} \sum_{i=0}^{r_j} \frac{A_{j,i}}{(x - c_j)^{i+1}} + \sum_{i=1}^{r_p} A_{p,i-1} x^{i-1}, \qquad A_{j,i} \in M(n, \mathbb{C}). \tag{1.2}$$

The equation is called a Fuchsian system if $r_1 = \cdots = r_p = 0$. In this case, $A_j := A_{j,0}$ is called the residue matrix at $x = c_j$. The residue matrix at $x = \infty$ is given by $A_p := -A_1 - \cdots - A_{p-1}$.

Katz [10] introduced invertible operations, called middle convolutions and middle tensor operations, on local systems and proved that any rigid local system can be transformed into the trivial rank-one system. Dettweiler and Reiter [2] subsequently interpreted this result in terms of Fuchsian systems, introducing two operations on the set of residue matrices $A_1, \dots, A_p$: middle convolutions and additions. These operations correspond, respectively, to fractional derivatives and gauge transformations of solutions of the differential equations. This result was further interpreted by Oshima [14] for scalar linear differential equations of arbitrary order. These transformations have led to a number of fundamental results on linear ordinary differential equations; see, for example, [14].

The local structure of a Fuchsian system is determined by its generalized Riemann scheme, which specifies the conjugacy classes of its residue matrices. It is a table of the eigenvalues of the residue matrices together with their multiplicities. The table obtained by omitting the eigenvalues is called

*oshima@ms.u-tokyo.ac.jp

2020 Mathematics Subject Classification. Primary 34M03; Secondary 34M55

Key words and phrases. Harnad dual, confluence, middle convolution, Kac-Moody root system, Painlevé equation

the spectral type, which is a p-tuple of partitions of an integer n. The Riemann scheme is obtained by attaching eigenvalues to a spectral type so that the Fuchs relation

$$\operatorname{trace} A_1 + \cdots + \operatorname{trace} A_p = 0$$

is satisfied. A Fuchsian system is said to be rigid if it is uniquely determined by its Riemann scheme.

In general, a Fuchsian system corresponding to a generalized Riemann scheme has accessory parameters. The number of accessory parameters is determined by the index of rigidity, which in turn depends only on the spectral type. Katz [10] proved that an irreducible Fuchsian system is rigid if and only if its index of rigidity equals 2. In this case, the system can be transformed into the trivial equation $u' = 0$ by a sequence of Katz reductions. Here, a reduction of the rank of a differential equation by means of an addition and a middle convolution is called a Katz reduction.

We say that a Riemann scheme or a spectral type is irreducibly realizable if it is realized by an irreducible Fuchsian system. The problem of determining the conditions for such realizability is called the additive Deligne–Simpson problem (cf. [12]). Crawley-Boevey [1] solved this problem by establishing a correspondence between spectral types and elements of the root lattice of a star-shaped Kac–Moody root system. He proved that a spectral type is irreducibly realizable if and only if the corresponding element is a positive root. Under this correspondence, middle convolutions and additions correspond to the actions of the Weyl group, while the index of rigidity corresponds to the norm of the corresponding element of the root lattice. In particular, an irreducibly realizable rigid spectral type corresponds to a positive real root, whereas an irreducibly realizable non-rigid spectral type corresponds to a positive imaginary root. The latter can be reduced to a fundamental spectral type corresponding to a positive imaginary root in a fundamental domain of the Weyl group (cf. [9, Theorem 5.4]). There are only finitely many fundamental spectral types with a given index of rigidity (cf. [14]).

Hiroe [5] formulated and solved a Deligne–Simpson problem for certain systems of the form (1.1) without ramified irregular singularities. In [5, 6], he proved that a tuple of partitions with refinement structure is irreducibly realizable if and only if its unfolding is irreducibly realizable. In this paper, we consider systems allowing unramified irregular singularities.

Note that there exists a versal unfolding of an irreducible system allowing unramified irregular singularities, which realizes both confluences and unfoldings as a family of systems (1.1) whose singular points are local holomorphic parameters.

The finiteness of fundamental spectral types with a given index of rigidity also holds for systems allowing unramified irregular singularities (cf. [8]).

Isomonodromic deformations of Fuchsian systems with accessory parameters give rise to Painlevé-type equations, while their degenerations correspond to confluences of the systems. These equations are invariant under additions and middle convolutions. Thus, the classification of fundamental spectral types is closely related to the classification of Painlevé-type equations, as illustrated by the study of four-dimensional Painlevé-type equations in [7].

In a suitable enlargement of the space of Fuchsian systems, one can consider Laplace transformations, which are more fundamental than middle convolutions. For example, in [14], the author introduced operators corresponding to middle convolutions in terms of Laplace transformations and additions (cf. Remark 2.25 (2)).

In the space of systems allowing unramified irregular singularities, we say that two systems are equivalent if one is transformed into the other by a gauge transformation, a Laplace transformation, a confluence, an unfolding, or a Möbius transformation. We then consider the equivalence relation generated by these transformations.

It is therefore natural to study the equivalence classes of this relation. This problem was proposed by Kawakami at a conference on functional equations in a complex domain held at Tohoku University in December 2025. Our main purpose is to show that the trivial equation $u' = 0$ and the Fuchsian

systems of E_8-fundamental spectral types form a complete set of representatives of these equivalence classes. These systems are of the form

$$\frac{du}{dx} = \frac{A_1}{x}u + \frac{A_2}{x-1}u, \qquad A_1, A_2 \in M(n, \mathbb{C}),$$

where

$$(A_1 - \lambda_{11})(A_1 - \lambda_{12}) = (A_2 - \lambda_{21})(A_2 - \lambda_{22})(A_2 - \lambda_{23}) = 0 \qquad (\exists \lambda_{j\nu} \in \mathbb{C}),$$
$$\sum_{j=1}^{3} \max_{\lambda \in \mathbb{C}} \dim \ker(A_j - \lambda) \leq n, \qquad A_3 := -A_1 - A_2.$$

In Section 2, we explain the transformations introduced above and define the necessary notation. Most of the results in this section are known. We also define E_8-, E_7-, E_6-, and D_4-fundamental spectral types and formulate our problem precisely. In Section 2.7, we give examples of the equivalence classes.

In Section 3, we state our main result, Theorem 3.1. To prove the theorem, we define a map $\mathcal{U}$ from the space of irreducibly realizable spectral types to the space consisting of E_8-fundamental spectral types and the trivial partition. The map $\mathcal{U}$ is a composition of additions, Möbius transformations, confluences, unfoldings, Laplace transformations, and middle convolutions. Theorem 3.1 is reduced to Lemma 3.6, which describes a key property of the map $\mathcal{U}$. We prove the lemma in this section, except for the claim that the representative has the highest order among fundamental spectral types in every equivalence class.

In Section 4.1, we compute the restriction of the map $\mathcal{U}$ to the space of fundamental spectral types. The result is given in Theorem 4.1, and the proof of Lemma 3.6 is completed by Proposition 4.12, which follows from this theorem. Moreover, for a T-fundamental spectral type $\mathbf{m}$ with $T = E_7$, E_6, or D_4, we give an explicit expression for $\mathcal{U}\mathbf{m}$, which clarifies the T-fundamental spectral types contained in each equivalence class.

2 Preliminary results

2.1 Spectral types of Fuchsian differential equations

First we consider a linear differential equation

$$\mathcal{M} : \frac{du}{dx} = \sum_{j=1}^{p-1} \frac{A_j}{x - c_j}u, \qquad A_j \in M(n, \mathbb{C}), \tag{2.1}$$

which we call a *Fuchsian system*. A linear ordinary differential equation whose coefficients are polynomials or rational functions, or equivalently, a linear differential equation on the Riemann sphere, can be presented by a Fuchsian system if it is irreducible and all its singular points are regular. Here, the irreducibility is understood either in the sense of $\mathcal{D}$-module or in terms of the monodromy representation of the solution space. We say that the Fuchsian system is *irreducible* if the matrices $A_1, \ldots, A_p$. have no non-trivial common invariant subspace. The matrix A_j is called the *residue matrix* at the singular point c_j. The residue matrix at the infinite point is denoted by $A_p = A_\infty$ and is given by

$$A_p = -(A_1 + \cdots + A_{p-1}). \tag{2.2}$$

The conjugacy class of A_j is determined by generalized eigenvalues $\lambda_{j,\nu}$ with multiplicities $m_{j,\nu}$ written by $[A_j] = \{[\lambda_{j,\nu}]_{m_{j,\nu}} \mid \nu = 1, \ldots, n_j\}$. If the multiplicities are *ordered*, namely,

$$m_{j,1} \geq m_{j,2} \geq \cdots \geq m_{j,n_j} \quad (j = 1, \ldots, p),$$

which can be assumed by arranging the order of indices, the condition imposed on A_j is

$$\operatorname{rank} \prod_{\nu=1}^{k} (A_j - \lambda_{j,\nu}) = n - m_{j,1} - \cdots - m_{j,k} \quad (k = 1, \ldots, n_j).$$

The (generalized) eigenvalues of A_j are called *characteristic exponents* and their multiplicities determine the spectral type

$$n = m_{j,1} + \cdots + m_{j,n_j} \quad (j = 1, \ldots, p). \tag{2.3}$$

The (generalized) *Riemann scheme* (cf. [14])

$$\left\{ \begin{array}{cc} x = c_j \ (j = 1, \ldots, p-1) & x = \infty \\ [\lambda_{j,1}]_{m_{j,1}} & [\lambda_{p,1}]_{m_{p,1}} \\ \vdots & \vdots \\ [\lambda_{j,n_j}]_{m_{j,n_j}} & [\lambda_{p,n_p}]_{m_{p,n_p}} \end{array} \right\}$$

describes the local structure of the equation at the singular points. In particular, if the characteristic exponents are generic, the local solutions have the asymptotic forms

$$\begin{aligned} u &\sim C_{j,\nu,i}(x - c_j)^{\lambda_{j,\nu}} \quad (x \to c_j, \ \ i = 1, \ldots, m_{j,\nu}, \ \nu = 1, \ldots, n_j), \\ &\sim C_{p,\nu,i}(\tfrac{1}{x})^{\lambda_{p,\nu}} \qquad (x \to \infty, \ \ i = 1, \ldots, m_{p,\nu}, \ \nu = 1, \ldots, n_p). \end{aligned}$$

Here $C_{j,\nu,1}, \ldots, C_{j,\nu,m_{j,\nu}}$ are linearly independent vectors.

Definition 2.1. A *spectral type* is a tuple of partitions (2.3) of an integer n. We express it in several equivalent ways:

$$\begin{aligned} \mathbf{m} &:= \big(m_{j,\nu}\big)_{\substack{1 \le \nu \le n_j \\ 1 \le j \le p}} \\ &= m_{1,1} \cdots m_{1,n_1}, \ \ldots, \ m_{p-1,1} \cdots m_{p-1,n_{p-1}}, \ m_{p,1} \cdots m_{p,n_p} \\ &= \big[[m_{1,1}, \ldots, m_{1,n_1}], \ldots, [m_{p-1,1}, \ldots, m_{p-1,n_{p-1}}], [m_{p,1}, \ldots, m_{p,n_p}]\big]. \end{aligned} \tag{2.4}$$

The integer n is called the *rank* or *order* of $\mathbf{m}$ and is denoted by $\operatorname{ord} \mathbf{m}$. The spectral type $\mathbf{m}$ determined by the residue matrices of a Fuchsian system $\mathcal{M}$ is called the spectral type of $\mathcal{M}$. A spectral type realized by an irreducible Fuchsian system is called *irreducibly realizable*.

Example 2.2. The generalized hypergeometric series ${}_nF_{n-1}(x)$ satisfies a linear differential equation of order n having an $(n-1)$-dimensional space of local holomorphic solutions at $x = 1$. Its spectral type is

$$\begin{aligned} {}_nF_{n-1} : n = \overbrace{1 + \cdots + 1}^{n} &= (n-1) + 1 = \overbrace{1 + \cdots + 1}^{n}, \\ 1 \cdots 1, (n-1)1, 1 \cdots 1 &= 1^n, (n-1)1, 1^n \\ \big[[1, \ldots, 1], [n-1, 1], [1, \ldots, 1]\big]&. \end{aligned}$$

We give some examples of differential equations characterized by their spectral types:

Gauss : $11, 11, 11$ $\quad (\operatorname{ord} = 2, \ \text{singularities} : \{0, 1, \infty\})$,
${}_3F_2$: $\ 111, 21, 111$ $\ (\operatorname{ord} = 3, \ \text{singularities} : \{0, 1, \infty\}$, 2-dimensional holomorphic solutions at $x = 1)$,
Heun: $\ 11, 11, 11, 11$ $(\operatorname{ord} = 2, \ \text{singularities} : \{0, 1, y, \infty\})$
Monodromy preserving deformation with respect to y gives Painlevé VI equation,
Jordan-Pochhammer : $21, 21, 21, 21$ $\ (\operatorname{ord} = 3, \ \text{singularities} : \{0, 1, y, \infty\})$
Regarding y as a new variable, we obtain Appell's F_1.

2.2 Middle convolutions

Riemann-Liouville transformations

$$u(x) \mapsto \frac{1}{\Gamma(\mu)} \int_{c_j}^{x} u(t)(x-t)^{\mu-1} dt$$

and gauge transformations

$$u(x) \mapsto (x-c)^{\lambda} u(x)$$

of solutions $u(x)$ of the system $\mathcal{M}$ induce invertible transformations of $\mathcal{M}$ that preserve irreducibility. They also induce transformations of the spectral type $\mathbf{m}$ of $\mathcal{M}$, which we call middle convolutions of $\mathbf{m}$.

Definition 2.3 (middle convolution, [2], [10])**.** For $\sigma \in \mathbb{Z}_{\geq 0}^{p}$, the middle convolution $\mathrm{mc}_\sigma \mathbf{m}$ of $\mathbf{m}$ is defined by

$$\mathrm{mc}_\sigma \mathbf{m} := \Big(m_{j,\nu} - \delta_{\nu,\sigma_j} d_\sigma(\mathbf{m})\Big)_{\substack{1 \leq \nu \leq n_j \\ 1 \leq j \leq p}},$$
$$d_\sigma(\mathbf{m}) := m_{1,\sigma_1} + \cdots + m_{p,\sigma_p} - (p-2) \cdot \mathrm{ord}\, \mathbf{m} \quad (m_{j,\nu} = 0 \text{ if } \nu \leq 0 \text{ or } \nu > n_j).$$

The relation $\mathbf{m}' = \mathrm{mc}_\sigma \mathbf{m}$ with $\sigma = (\sigma_1, \ldots, \sigma_p)$ and $D = d_\sigma(\mathbf{m})$ is denoted by

$$\mathbf{m} \xrightarrow[\sigma_1,\ldots,\sigma_p]{D} \mathbf{m}'. \tag{2.5}$$

Theorem 2.1 ([10])**.** *A spectral type* $\mathbf{m}$ *is irreducibly realizable if and only if* $\mathrm{mc}_\sigma \mathbf{m}$ *is irreducibly realizable or* $\mathrm{ord}\, \mathbf{m} = 1$.

If $\mathrm{ord}\, \mathrm{mc}_\sigma \mathbf{m} < \mathrm{ord}\, \mathbf{m}$, namely, $d_\sigma(\mathbf{m}) > 0$ (cf. (2.6)), we call $\mathrm{mc}_\sigma \mathbf{m}$ a *Katz reduction* of $\mathbf{m}$. In particular, if σ is chosen so that $d_\sigma(\mathbf{m})$ is maximal among positive integers, $\mathrm{mc}_\sigma \mathbf{m}$ is called a *maximal Katz reduction* of $\mathbf{m}$.

Definition 2.4 (index of rigidity, [10])**.** The *index of rigidity* of $\mathbf{m}$ is the even integer defined by

$$\mathrm{idx}\, \mathbf{m} := \sum_{\substack{1 \leq \nu \leq n_j \\ 1 \leq j \leq p}} m_{j,\nu}^2 - (p-2)(\mathrm{ord}\, \mathbf{m})^2.$$

Remark 2.5. (1) m_{j,σ_j} $(1 \leq j < p)$ corresponds to the multiplicity of the eigenvalue 0 of the resulting residue matrix under a gauge transformation, while m_{p,σ_p} corresponds to the multiplicity of the eigenvalue μ of the resulting residue matrix at $x = \infty$ (cf. [2]).

Usually we assume $m_{j,\nu} > 0$. However, in defining $\mathrm{mc}_\sigma \mathbf{m}$, we allow some multiplicities $m_{j,\nu} = 0$. If a spectral type $\mathbf{m}$ contains zero multiplicities, we identify it with the spectral type defined by omitting them.

(2) We have the following relations.

$$\mathrm{ord}\, \mathrm{mc}_\sigma \mathbf{m} = \mathrm{ord}\, \mathbf{m} - d_\sigma(\mathbf{m}), \tag{2.6}$$
$$\mathrm{mc}_\sigma^2 = \mathrm{id}, \tag{2.7}$$
$$\mathrm{idx}\, \mathrm{mc}_\sigma \mathbf{m} = \mathrm{idx}\, \mathbf{m}. \tag{2.8}$$

We show some typical examples of middle convolutions of spectral types.

Example 2.6. $\underline{1}11, \underline{2}1, \underline{1}11 \xrightarrow[1,1,1]{1+2+1-1\cdot 3=1} 011, 11, 011 \simeq \underline{1}1, \underline{1}1, \underline{1}1 \xrightarrow[1,1,1]{3-2} 1, 1, 1 \simeq \mathbf{1}$

$\underline{2}11, \underline{2}11, \underline{1}111 \xrightarrow[1,1,1]{5-4=1} \underline{1}11, \underline{1}11, 0\underline{1}11 \simeq \underline{1}11, \underline{1}11, \underline{1}11 \xrightarrow[1,1,1]{3-3=0} 111, 111, 111$

$\underline{6}511, \underline{6}511, \underline{6}1111111 \xrightarrow[1,1,1]{18-13=5} 1\underline{5}11, 1\underline{5}11, \underline{1}1111111 \xrightarrow[2,2,1]{11-8=3} \times$

Here $\mathbf{1}$ denotes the trivial partition $1 = 1$, which correspond to the trivial equation $u' = 0$.

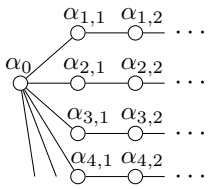
In the last example, $\mathrm{mc}_{(2,2,1)}$ is not defined as a spectral type, because it contains a negative multiplicity. Thus the spectral type $6511, 6511, 61111111$ is not irreducibly realizable.

2.3 A star-shaped Kac-Moody root system

We explain the correspondence between spectral types and a Kac–Moody root system introduced by Crawley-Boevey [1].

Definition 2.7. We introduce the following Kac–Moody root system (Π, W).

$$
\begin{aligned}
(\alpha|\alpha) &= 2 \quad (\alpha \in \Pi := \{\alpha_0,\, \alpha_{j,\nu} \mid j \geq 1,\, \nu \geq 1\}) \\
(\alpha_0|\alpha_{j,\nu}) &= -\delta_{\nu,1} \\
(\alpha_{i,\mu}|\alpha_{j,\nu}) &= \begin{cases} 0 & (i \neq j \quad \text{or} \quad |\mu-\nu| > 1) \\ -1 & (i = j \quad \text{and} \quad |\mu-\nu| = 1) \end{cases} \\
s_\alpha &: x \mapsto x - (x|\alpha)\alpha \quad (\alpha \in \Pi), \quad W := \langle s_\alpha | \alpha \in \Pi \rangle
\end{aligned}
$$

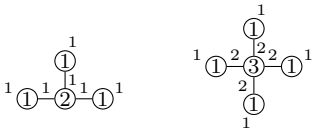


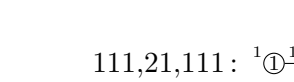
Definition 2.8. We associate each spectral type $\mathbf{m}$ with an element $\alpha_{\mathbf{m}}$ of this root lattice:

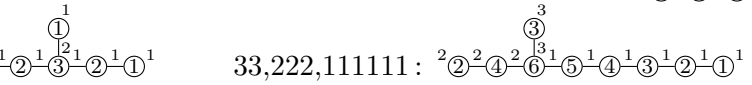
$$
\begin{aligned}
\alpha_{\mathbf{m}} &:= n\alpha_0 + \sum_{j\geq 1}\sum_{i\geq 1}\Big(\sum_{\nu>i} m_{j,\nu}\Big)\alpha_{j,i} \qquad (n = \operatorname{ord}\mathbf{m}) \\
&= n\alpha_0 + \sum_{j\geq 1}\sum_{\nu\geq 1} n_{j,\nu}\alpha_{j,\nu}, \\
m_{j,\nu} &= n_{j,\nu-1} - n_{j,\nu} \quad (j = 1, 2, \ldots,\ \nu = 1, 2, \ldots,\ \ n_{j,0} = n).
\end{aligned}
$$

Example 2.9. Correspondence $\mathbf{m}$ and $\alpha_{\mathbf{m}}$: Here, the number inside a circle is the coefficient $n_{j,\nu}$ of the simple root $\alpha_{j,\nu}$ in the Dynkin diagram, while $m_{j,\nu}$ is the difference between two consecutive coefficients. Simple roots with zero coefficients are usually omitted from the diagram.

$$
\begin{aligned}
\texttt{11,11,11} &\leftrightarrow \alpha = 2\alpha_0 + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} \\
\texttt{21,21,21,21} &\leftrightarrow \alpha = 3\alpha_0 + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1}
\end{aligned}
$$



111,21,111 : 

33,222,111111 : 

Proposition 2.10. *Under Definition 2.8, we have*

$$
\begin{aligned}
\alpha_{\mathrm{mc}_{(1,\ldots,1)}\mathbf{m}} &= s_{\alpha_0}\alpha_{\mathbf{m}}, & \alpha_{\mathbf{m}'} &= s_{\alpha_{j,\nu}}\alpha_{\mathbf{m}} \quad (\mathbf{m}' = \cdots \overbrace{m_{j,\nu+1} m_{j,\nu}} \cdots). \\
\operatorname{idx}\mathbf{m} &= (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}), & d_{1,\ldots,1}\mathbf{m} &= (\alpha_0|\alpha_{\mathbf{m}}).
\end{aligned}
$$

In particular, the transformations of spectral types induced by middle convolutions mc_σ *correspond to the transformations of the root lattice by the Weyl group.*

2.4 Deligne-Simpson problem

The problem of determining the conditions for the existence of an irreducible Fuchsian system with a given generalized Riemann scheme is called the additive *Deligne-Simpson problem*. It is solved as follows.

Theorem 2.2 ([12, 10, 1, 2, 14])**.** *The spectral type* $\mathbf{m}$ *is irreducibly realizable if and only if* $\alpha_{\mathbf{m}}$ *is a root of the star-shaped Kac–Moody root system (more precisely, when* $\operatorname{idx}\mathbf{m} = 1$, $\mathrm{GCD}\{m_{j,\nu}\} = 1$ *should be satisfied). Moreover, if the spectral type of* $(\bar{A}_1, \dots \bar{A}_p) \in M(n,\mathbb{C})^p$ *is irreducible realizable, there exists an irreducible tuple* $(A_1, \dots, A_p) \in M(n,\mathbb{C})^p$ *such that*

$$A_1 + \cdots + A_p \in \mathbb{C} \quad \textit{and} \quad A_j \sim \bar{A}_j.$$

Here $(A_1, \dots, A_p)$ *is called irreducible if the common invariant subspace of* $A_1, \dots, A_p$ *is trivial.*

Remark 2.11. (1) Kostov [12] studied the problem in some cases using implicit function theorem. Katz [10] introduced middle convolutions and solved the problem for rigid spectral types. Dettweiler and Reiter [2] interpreted the results of [10] in a language of linear algebra. Crawley-Boevey [1] solved the problem by establishing a relation with a Kac–Moody root system. The corresponding problem for single differential equations of arbitrary order was solved by [14].

(2) It follows from (2.2) that the residue matrices A_j of a Fuchsian system $\mathcal{M}$ satisfy $\operatorname{trace} A_1 + \cdots + \operatorname{trace} A_p = 0$, which corresponds to the Fuchs relation for the Riemann scheme. If the sum of the traces of $\bar{A}_j$ is zero in Theorem 2.2, $A_1, \dots, A_p$ can be realized as the residue matrices of a Fuchsian system.

(3) When $\mathbf{m}$ is rigid, a realized system $\mathcal{M}$ is uniquely determined by the Riemann scheme (cf. Theorem 2.3) but it may be reducible if the eigenvalues of $\bar{A}_j$ take special values (cf. [1, 14]).

Theorem 2.3 ([10])**.** *Let* $\mathbf{m}$ *be the spectral type of an irreducible Fuchsian system given by* (2.1)*. Then*

$$\dim\{(\tilde{A}_1, \dots, \tilde{A}_p) \in M(n,\mathbb{C})^p \mid \tilde{A}_j \sim A_j,\ \tilde{A}_1 + \cdots + \tilde{A}_p = 0\}/GL(n,\mathbb{C}) = 2 - \operatorname{idx}\mathbf{m}$$

This theorem says that the system has $2 - \operatorname{idx}\mathbf{m}$ parameters that are not determined by the Riemann scheme. They are called *accessory parameters*.

Definition 2.12. A spectral type $\mathbf{m}$ is *rigid* if $\mathbf{m}$ is irreducibly realizable and $\operatorname{idx}\mathbf{m} = 2$. This is equivalent to the condition that $\alpha_{\mathbf{m}}$ is a real root satisfying $\operatorname{ord}\mathbf{m} > 0$.

Remark 2.13. (1) The definition of $\operatorname{idx}\mathbf{m}$ and $d_\sigma(\mathbf{m})$ gives

$$\sum_{j=1}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}(m_{j,1} - m_{j,\nu}) = d_{1,\cdots,1}(\mathbf{m}) \cdot \operatorname{ord}\mathbf{m} - \operatorname{idx}\mathbf{m}.$$

(2) Suppose $\operatorname{idx}\mathbf{m} > 0$. We may assume that $\mathbf{m}$ is ordered. Then, $d_{1,\cdots,1}(\mathbf{m}) > 0$, and if $\operatorname{ord}\mathbf{m} \neq 1$, a Katz reduction of $\mathbf{m}$ is possible. Hence, Theorem 2.2 follows from Theorem 2.1.

An irreducible system $\mathcal{M}$ with a spectral type $\mathbf{m}$ satisfying $\operatorname{idx}\mathbf{m} > 0$ is called a rigid system. Then $\operatorname{idx}\mathbf{m} = 2$ and $\alpha_{\mathbf{m}}$ is a positive real root. In this case, the system $\mathcal{M}$ is constructed by a finite sequence of suitable additions and middle convolutions starting from the trivial equation $u' = 0$. This result is obtained in [10] and corresponds to the fact that the real positive root $\alpha_{\mathbf{m}}$ belongs to the orbit $W\alpha_0$. From this point of view, equations with rigid spectral types can be analyzed; for example, integral representations and series expressions of solutions, conditions for irreducibility, contiguous relations, and connection problems are studied in [14].

(3) Suppose $\operatorname{idx}\mathbf{m} \leq 0$, which will be the main case considered in this paper. We may moreover assume $\mathbf{m}$ is ordered. Note that if $\mathbf{m}' := \mathrm{mc}_{(1,\dots,1)}\mathbf{m}$ is not defined, $\mathbf{m}$ is not irreducibly realizable (cf. the last example in Example 2.6). If $d_{1,\cdots,1}(\mathbf{m}) > 0$, the condition that $\mathbf{m}$ is irreducibly realizable is reduced to corresponding condition for $\mathbf{m}'$, which satisfies $\operatorname{ord}\mathbf{m}' < \operatorname{ord}\mathbf{m}$. Hence it suffices to consider the case $d_{1,\cdots,1}(\mathbf{m}) \leq 0$. Crawley-Boevey [1] proved that every $\mathbf{m}$ satisfying $d_{1,\cdots,1}(\mathbf{m}) \leq 0$ is irreducibly realizable by applying a result on representations of quivers, thereby proving Theorem 2.2. In the case $\operatorname{idx}\mathbf{m} = 0$, the only exception occurs when $\mathrm{GCD}\{m_{j,\nu}\} > 1$.

By an iteration of additions and middle convolutions, any irreducible system $\mathcal{M}$ with accessory parameters, namely, a non-rigid irreducible system $\mathcal{M}$ is transformed into a system with a *fundamental spectral type* whose order cannot be reduced by a Katz reduction.

A spectral type $\mathbf{m} = (m_{j,\nu})_{\substack{j=1,\dots,p\\ \nu=1,\dots,n_j}}$ is called *fundamental* if and only if

$$d_\sigma(\mathbf{m}) \leq 0 \quad (\forall \sigma \in \mathbb{Z}^p_{\geq 0}) \tag{2.9}$$

and moreover $\mathrm{GCD}\{m_{j,\nu}\} = 1$ when $\operatorname{idx}\mathbf{m} = 0$. If $\mathbf{m}$ is ordered, the condition (2.9) is equivalent to

$$m_{1,1} + \cdots + m_{p,1} \leq (p-2)\operatorname{ord}\mathbf{m}.$$

This condition is equivalent to

$$(\alpha_{\mathbf{m}}|\alpha) \leq 0 \quad (\forall \alpha \in \Pi)$$

and also to the condition that $\alpha_{\mathbf{m}}$ is a positive imaginary root in a fundamental domain of the Weyl group (cf. [9, Theorem 5.4]).

Remark 2.14. We will study the classification of irreducibly realizable spectral types. Two spectral types will be identified if one can be obtained from the other by permuting the tuple of partitions, adding or deleting trivial partitions, or adding or deleting terms equal to 0. For example,

$$21, 111, 111 \simeq 12, 111, 111 \simeq 111, 21, 111 \simeq 021, 111, 111 \simeq 3, 21, 111, 111.$$

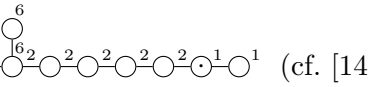
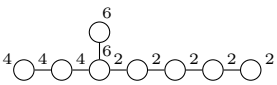
A spectral type $\mathbf{m} = (m_{j,\nu})_{\substack{j=1,\dots,p\\ \nu=1,\dots,n_j}}$ is uniquely expressed in a *normal form*

$$\begin{cases} m_{j,1} \geq m_{j,2} \geq \cdots \geq m_{j,n_j} > 0, \quad n_j > 1 & (1 \leq j \leq p),\\ m_{j,\nu} = m_{j+1,\nu}\ (1 \leq \nu < k_j), \text{ and } m_{j,k_j} > m_{j+1,k_j} \text{ or } k_j = n_j{+}1 & (\exists k_j,\ 1 \leq j < p). \end{cases}$$

Remark 2.15. If $d_\sigma(\mathbf{m}) = 0$, then $\mathrm{mc}_\sigma(\mathbf{m}) = \mathbf{m}$, although the middle convolution mc_σ induces a nontrivial transformation of Fuchsian systems of spectral type $\mathbf{m}$. When $\mathbf{m}$ is fundamental and ordered and $d_{1,\dots,1}(\mathbf{m}) = 0$, the Fuchsian systems of spectral type $\mathbf{m}$ admit an automorphism group. This group is isomorphic to the Weyl group of the root system whose simple roots are

$$\{\alpha_0\} \cup \{\alpha_{j,\nu} \mid m_{j,\nu+1} = m_{j,1},\ j = 1, \dots, p\}.$$

When $\operatorname{idx}\mathbf{m} < 0$, this root system is of classical type (cf. [14]). For example, for $\mathbf{m} = 6^2, 4^3, 2^5 11$, denoted by $E_8^{(2)}$ in [14], the root system is of type E_8:

$E_8^{(2)}$:  (cf. [14, §13.1.5]) $\Rightarrow$ E_8 : 

Definition 2.16. For a spectral type $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=1,\dots,p\\ \nu=1,\dots,n_j}}$, set

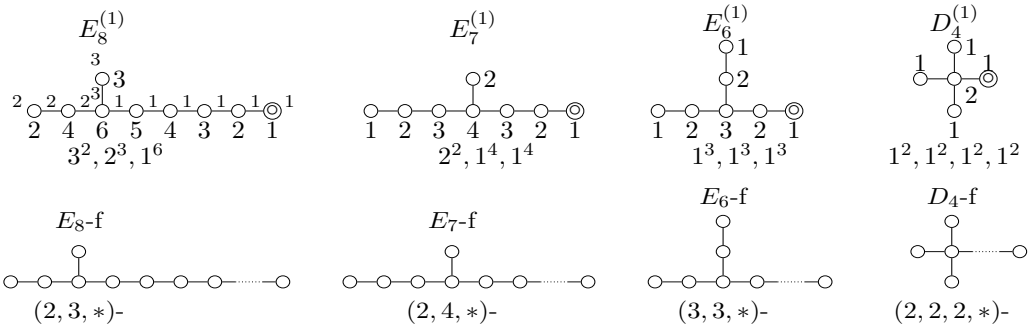
$$\mathrm{N_p}(\mathbf{m}) := |\{j \in \{1, \dots, p\} \mid m_{j,\nu} \neq n \text{ for } \nu = 1, \dots, n_j\}|.$$

We say that $\mathbf{m}$ is of $(n_1, \dots, n_p)$-type if $m_{j,\nu} \neq 0, n$ and $n_j > 1$ in the expression (2.3).

We say that $\mathbf{m}$ is of $(n_1, n_2, *)$-type if $\mathbf{m}$ is of (n_1, n_2, n_3)-type for some integer n_3.

We say that $\mathbf{m}$ is E_8-fundamental E_7-fundamental, E_6-fundamental or D_4-fundamental if $\mathbf{m}$ is fundamental and $(2, 3, *)$-type, $(2, 4, *)$-type, $(3, 3, *)$-type or $(2, 2, 2, *)$-type, respectively.

Dynkin diagrams of roots corresponding to some spectral types are as follows:



Definition 2.17. For the classification of irreducibly realizable spectral types, we use the following notation under the identification given in Remark 2.14:

$$\begin{aligned}
\mathcal{P} &: \text{ the set of spectral types of irreducible Fuchsian systems,}\\
\mathcal{F} &: \text{ the set of fundamental spectral types,}\\
\mathcal{F}_\ell &:= \{\mathbf{m} \in \mathcal{F} \mid \operatorname{idx} \mathbf{m} = -\ell\},\\
\mathcal{F}^T &:= \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m} : T\text{-fundamental}\} \quad (T = E_8,\ E_7,\ E_6 \text{ or } D_4).
\end{aligned}$$

For spectral types in $\mathcal{P}$, we define an equivalence relation $\underset{\text{mc}}{\sim}$ by declaring that $\mathbf{m} \underset{\text{mc}}{\sim} \mathbf{m}'$ if $\mathbf{m}'$ is obtained from $\mathbf{m}$ by a finite sequence of middle convolutions mc_σ.

Note that

$$\mathcal{P}/\underset{\text{mc}}{\sim} \simeq \mathcal{F} \cup \{\mathbf{1}\}, \quad \mathcal{F} = \bigsqcup_{k=0}^{\infty} \mathcal{F}_{2k}. \tag{2.10}$$

The spectral types in $\mathcal{F}_\ell$ for small $|\ell|$ are classified and listed in [14, 18]. We have the following result.

Theorem 2.4 ([12])**.** $\mathcal{F}_0 = \{33, 222, 111111 \ \ 22, 1111, 1111 \ \ 111, 111, 111 \ \ 11, 11, 11, 11\}$.

Theorem 2.5 ([14])**.** $|\mathcal{F}_\ell| < \infty \quad (\ell = 0, 2, 4, 6, \ldots)$.

We describe the elements of $\mathcal{F}^{E_8}$ that play an important role in this paper.

Proposition 2.18. *Let*

$$\mathbf{m} = m_{11}m_{12},\ m_{21}m_{22}m_{23},\ m_{31}m_{32}\cdots m_{3,n_3}$$

be an ordered $(2, 3, *)$-spectral type. Then,

$$m_{12} \geq m_{21} + m_{31} \tag{2.11}$$

is a necessary and sufficient condition so that $\mathbf{m}$ is fundamental.

If this condition is satisfied, then $n_3 \geq 6$ and

$$m_{11} \geq m_{12} > m_{21} \geq m_{22} \geq m_{23} > m_{31} \geq m_{32} \geq \cdots \geq m_{3,n_3}.$$

Conversely, any $\mathbf{m} \in \mathcal{F}^{E_8}$ with $\operatorname{ord} \mathbf{m} = n$ is obtained as follows. First, choose a pair of integers (m_{21}, m_{12}) satisfying

$$\tfrac{n}{3} \leq m_{21} < m_{12} \leq \tfrac{n}{2} \tag{2.12}$$

and a positive integer m_{31} satisfying (2.11). Next, choose m_{22} satisfying $\frac{n-m_{21}}{2} \leq m_{22} \leq m_{21}$. Then set $m_{11} = n - m_{12}$ and $m_{23} = n - m_{21} - m_{22}$. Finally, choose a non-increasing sequence $m_{31}, m_{32}, \ldots$ whose sum is n.

Note that $\mathbf{m} \in \mathcal{F}^{E_8}$ exists if $n = 6$ or $n > 7$. Moreover when $n = 6$, $\mathbf{m} = E_8^{(1)} = 3^2, 2^3, 1^6 \in \mathcal{F}_0$.

Proof. The ordered $(2,3,*)$-spectral type $\mathbf{m}$ satisfies

$$\begin{aligned} n &= m_{11}+m_{12} = m_{21}+m_{22}+m_{23} = m_{31}+\cdots+m_{3,n_3}, \\ m_{11} &\geq m_{12},\ m_{21}\geq m_{22}\geq m_{23},\ m_{31}\geq m_{32}\geq\cdots\geq m_{3,n_3}. \end{aligned} \tag{2.13}$$

The condition that $\mathbf{m}$ is fundamental is equivalent to

$$d_{1,1,1}(\mathbf{m}) = m_{11}+m_{21}+m_{31}-n = m_{21}+m_{31}-m_{12}\leq 0. \tag{2.14}$$

Thus, the first assertion follows.

It follows from (2.13) that $m_{12}\leq \frac{n}{2}$ and $m_{21}\geq \frac{n}{3}$ and (2.14) proves $m_{21}\leq m_{12}-m_{31}<m_{12}$. Hence (2.12) follows.

There exist integers m_{21} and m_{12} satisfying (2.12) when $n=6$ or $n>7$. If $n=6$, $(m_{21},m_{12}) = (2,3)$. Finally, note that

$$\begin{aligned} m_{31} &\leq n-m_{11}-m_{21}\leq n-\tfrac{n}{2}-\tfrac{n}{3}\leq \tfrac{n}{6}, \\ m_{23}-m_{31} &\geq (n-m_{21}-m_{22})-(n-m_{11}-m_{21}) = m_{11}-m_{22}\geq m_{12}-m_{21}>0. \end{aligned}$$

The proposition now follows. □

2.5 Confluences and unfoldings

The Riemann scheme of a differential equation on $\mathbb{P}^1$ with regular singularities at finite points and an irregular singularity of Poincaré rank 1 at infinity is

$$\left\{\begin{array}{cc} x=\infty & x=c_j\ (j=2,\dots,p-1) \\ [\lambda_1^{(1)}x+\lambda_{1,1}^{(0)}]_{m_{1,1}^{(0)}} & [\lambda_{j,1}]_{m_{j,1}} \\ \vdots & \vdots \\ [\lambda_{n_1}^{(1)}x+\lambda_{n_1,R_{n_1}}^{(0)}]_{m_{n_0,R_{n_0}}^{(0)}} & [\lambda_{j,n_j}]_{m_{j,n_j}} \end{array}\right\} \tag{2.15}$$

This implies the existence of local solutions with asymptotic expansions

$$u\sim C_{\nu,k,\ell}(\tfrac{1}{x})^{\lambda_{\nu,k}^{(0)}}e^{-\lambda_\nu^{(1)}x}\quad (x\to\infty,\ 1\leq\ell\leq m_{\nu,k}^{(0)},\ 1\leq k\leq R_\nu).$$

In this case, the spectral type at ∞ is defined by

$$\big\{[\lambda_\nu^{(1)}x+\lambda_{\nu,k}^{(0)}]_{m_{\nu,k}^{(0)}}\mid 1\leq k\leq R_\nu,\ 1\leq\nu\leq n_1\big\}.$$

We consider the multiplicity

$$m_\nu^{(1)} := m_{\nu,1}^{(0)}+\cdots+m_{\nu,R_\nu}^{(0)}$$

of the exponential term $e^{-\lambda_\nu^{(1)}x}$. Then the partition $\{m_{\nu,k}^{(0)}\}$ of n is a refinement of the partition $n=m_1^{(1)}+\cdots+m_{n_1}^{(1)}$. In general, characteristic exponents at an unramified singular point with Poincaré rank r are polynomials of degree r and the spectral type at the point is a refinement sequence of $r+1$ tuples of partitions of n (cf. [16]).

By inserting parentheses or separating the entries by $|$, this spectral type can be expressed as follows (cf. [7, 16]).

$$\begin{aligned} \hat{\mathbf{m}} &= (m_{1,1}^{(0)}\cdots m_{1,R_1}^{(0)})\cdots(m_{n_1,1}^{(0)}\cdots m_{n_1,R_{n_1}}^{(0)}), m_{2,1}\cdots,\cdots m_{p-1,n_{p-1}} \\ &= m_{1,1}^{(0)}m_{1,2}^{(0)}\cdots m_{n_1,R_{n_1}}^{(0)}\,|\,m_1^{(1)}\cdots m_{n_1}^{(1)}, m_{2,1}\cdots,\cdots,m_{p-1,1}\cdots m_{p-1,n_{p-1}} \end{aligned}$$

Unfolding of $\hat{\mathbf{m}}$: $m_{1,1}^{(0)}m_{1,2}^{(0)}\cdots m_{n_1,R_{n_1}}^{(0)}$, $m_{1,1}\cdots m_{1,n_1}, m_{2,1}\cdots,\cdots,m_{p-1,1}\cdots m_{p-1,n_{p-1}}$

Definition 2.19. A *confluence* is an operation that adds a refinement structure to a spectral type, while an *unfolding* is its inverse, namely, an operation that removes a refinement structure from a spectral type. When a spectral type with a refinement structure is expressed using `|`, unfolding is the operation of replacing "`|`" by "`,`".

Example 2.20. In general, there are several refinement structures of a spectral type:

$33, 222, 111111 \xrightarrow{\text{cf}}$
$111111|222, 33 = (11)(11)(11), 33 = 11 \cdot 11 \cdot 11|222, 33 \qquad 111111|33, 222 = (111)(111), 222$

$42, 21111, 21111 \xrightarrow{\text{cf}}$
$11112|42, 21111 = (1111)(2), 21111 \qquad 21111|42, 21111 = (211)(11), 21111$
$21111|21111, 42 = (2)(1)(1)(1), 42 \qquad 11112|11112|42 = ((1)(1)(1)(1))((2))$
$21111|21111|42 = ((2)(1)(1))((1)(1)) = 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1|211 \cdot 11|42$

Remark 2.21. Middle convolutions and indices of rigidity can be defined similarly for spectral types of systems allowing unramified irregular singularities, and they are compatible with operations such as confluences and unfoldings. For a fixed index of rigidity, there are only finitely many such spectral types that cannot be reduced in order by middle convolutions. This finiteness extends Theorem 2.5 and follows from the finiteness of the "types" of Weyl group orbits in a general symmetric Kac–Moody root space (cf. [8]). Here, if the Dynkin diagram of a root contains a subdiagram of the form $\overset{m}{\circ}\!\!-\!\!\overset{m}{\circ}\!\!-\cdots-\!\!\overset{m}{\circ}$ we may extend its length without changing the corresponding spectral type. Hence, we regard such roots as belonging to the same "type."

The Deligne-Simpson problem on systems allowing unramified irregular singularities was formulated in [5] and solved as follows.

Theorem 2.6 ([5, 6])**.** *Let* $\mathbf{m}$ *be a spectral type with a refinement structure. Then* $\mathbf{m}$ *is irreducibly realizable by systems* (1.1) *satisfying a suitable condition on* (1.2) *if and only if the unfolding of* $\mathbf{m}$ *is irreducibly realizable.*

There is also an analytic realization of unfolding and confluence as follows.

Definition 2.22 ([15, 16])**.** Let $\mathcal{N}$ be a differential equation with unramified irregular singularities. A versal unfolding $\widetilde{\mathcal{N}}$ of $\mathcal{N}$ is a family of differential equations which holomorphically depend on some singular points as local holomorphic parameters, such that the index of rigidity is independent of the parameters and $\mathcal{N}$ is contained in the family as a confluence of singular points of Fuchsian systems.

For example, a versal unfolding of a system $\mathcal{N}$ having an irregular singularity of Poincaré rank 1 at ∞ is of the form

$$\mathcal{M}_b : \frac{du}{dx} = \frac{A_1}{1-bx}u + \sum_{j=2}^{p-1} \frac{A_j}{x-c_j}u \quad (|b| \ll 1).$$

Here, the matrices A_j depend holomorphically on the parameter b, $\operatorname{idx}\mathcal{M}_b = \operatorname{idx}\mathcal{N}$, and $\mathcal{N} = \mathcal{M}_0$.

Theorem 2.7 ([15, 11, 6])**.** *An irreducible system* $\mathcal{N}$ *of spectral type* $\mathbf{m}$ *with a refinement structure admits a versal unfolding.*

Defining middle convolutions and additions of versal unfoldings, this theorem is proved in the rigid case and conjectured in general by [15]. When the Poincaré rank of every irregular singularity of $\mathcal{N}$ is equal to one, a versal unfolding of $\mathcal{N}$ was explicitly constructed by Kawakami [11]. The general existence theorem above is proved by Hiroe [6].

Definition 2.23. We denote by $\hat{\mathcal{P}}$ the set of spectral types of irreducible systems allowing unramified irregular singularities; namely, the set of irreducibly realizable spectral types with refinement structures.

2.6 Harnad duality (Laplace transformations)

The Laplace transformation

$$u(x) \mapsto \int_c^\infty u(t)e^{-xt}dt$$

of $u(x)$ induces the transformation $(x, \frac{d}{dx}) \mapsto (-\frac{d}{dx}, x)$ of differential operators and hence Harnad duality (cf. [4]) for differential equations. Combining this transformation with gauge transformations gives a bijection between irreducible systems $\mathcal{N}$ having an irregular singularity of Poincaré rank 1 at ∞ and regular singularities at finite points. Consequently, it induces a transformation Hd_σ between their spectral types. Here, σ corresponds to the characteristic exponents at regular singular points, as in the case of mc_σ.

Each regular singular point is transformed into an irregular singularity at ∞, and its position determines the exponential factor, namely, the coefficient of the degree-1 term of the characteristic exponents, while the constant terms are the original characteristic exponents other than 0. The characteristic exponent 0 corresponds to local holomorphic solutions at finite singular points and is encoded by σ. Conversely, the irregular singularity at ∞ is transformed into singularities at finite points. The multiplicities $m^{(0)}_{k,0}$ of the characteristic exponent 0 in Definition 2.24 are determined by the order of the resulting spectral type, as specified in Definition 2.24.

Suppose that the Riemann scheme of the equation $\mathcal{N}$ is given by (2.15) and, moreover, that $\lambda_{j,1} = 0$ for $j = 1, \dots, n_j$. The Laplace transformation of $\mathcal{N}$, which is the Harnad dual of $\mathcal{N}$, has the following Riemann scheme:

$$\left\{ \begin{array}{ll} x = \infty & x = -\lambda^{(0)}_i \\ [c_j x + \lambda_{j,\nu}]_{m_{j,\nu}} \ (1 \le j \le p,\ 2 \le \nu \le n_j) & [0]_{\hat{n} - m^{(1)}_i} \\ & [\lambda^{(1)}_{i,\nu}]_{m^{(0)}_{i,\nu}} \ \ (1 \le \nu \le R_i) \end{array} \right\},$$

$$\hat{n} := pn - (m_{1,1} + \cdots + m_{p,1}).$$

When considering the Harnad duality, we choose ∞ as the first singularity in the Riemann scheme and the corresponding spectral type.

Definition 2.24. The Harnad dual $\mathrm{Hd}\,\mathbf{m}$ of a system with an irregular singularity of Poincaré rank 1 at infinity and regular singularities at finite points is defined by

$$\begin{aligned} \mathbf{m} &= \big(m^{(0)}_{1,1} \cdots m^{(0)}_{1,R_1}\big) \cdots \big(m^{(0)}_{n_1,1} \cdots m^{(0)}_{n_1,R_{n_1}}\big), \underline{m_{2,1}} m_{2,2} \cdots m_{2,n_2}, \ \dots \ , \underline{m_{q,1}} m_{q,2} \cdots m_{q,n_q} \\ &= m^{(0)}_{1,1} \cdots m^{(0)}_{1,R_1} \cdots m^{(0)}_{n_1,1} \cdots m^{(0)}_{n_1,R_{n_1}} \,|\, m^{(1)}_1 \cdots m^{(1)}_{n_1}, \underline{m_{2,1}} \cdots m_{2,n_2}, \ \dots \ , \underline{m_{q,1}} \cdots m_{q,n_q}, \\ \mathrm{Hd}\,\mathbf{m} &= \big(m_{2,2} \cdots m_{2,n_2}\big) \cdots \big(m_{q,2} \cdots m_{q,n_q}\big), \underline{m^{(0)}_{1,0}} m^{(0)}_{1,1} \cdots m^{(0)}_{1,R_1}, \ \dots, \ \underline{m^{(0)}_{n_1,0}} m^{(0)}_{n_1,1} \cdots m^{(0)}_{n_1,R_{n_1}}, \\ & \qquad m^{(0)}_{k,0} \text{ are determined by } \sum_{j=2}^{q} \sum_{\nu=2}^{n_j} m_{j,\nu} = \sum_{\nu=0}^{R_k} m^{(0)}_{k,\nu} \ \ (k = 1, \dots, n_1). \end{aligned}$$

Here, $\mathrm{Hd}_{1,\dots,1}$ is simply denoted by Hd.

Remark 2.25. (1) Hd_σ defines a map between irreducibly realizable spectral types.

(2) $\mathrm{idx}\,\mathrm{Hd}_\sigma\,\mathbf{m} = \mathrm{idx}\,\mathbf{m}$ and $\mathrm{Hd}^2 = \mathrm{id}$.

(3) Let $\mathcal{M}$ be a Fuchsian system with spectral type $\mathbf{m}$. Applying a generic linear fractional transformation L to $\mathcal{M}$, we obtain a system that is non-singular at infinity. The spectral type obtained by

applying Hd_2 to the Harnad dual $\mathrm{Hd}_{1,\dots,1}$ of this system gives the middle convolution $\mathrm{mc}_{(1,\dots,1)}\,\mathbf{m}$:

$$\begin{aligned}
\mathbf{m} &= m_{1,1}\cdots m_{1,n_1},\ \dots,\ m_{p,1}\cdots m_{p,n_p} \overset{L}{\simeq} n,\ \underline{m_{1,1}}m_{1,2}\cdots m_{1,n_1},\ \dots,\ \underline{m_{p,1}}m_{1,2}\cdots m_{p,n_p}\\
&\xrightarrow{\mathrm{Hd}} (m_{1,2}\cdots m_{1,n_1})\cdots(m_{p,2}\cdots m_{p,n_p}),\ n'\underline{n} \qquad (n' = (p-1)n - (m_{1,1}+\cdots+m_{p,1}))\\
&\xrightarrow{\mathrm{Hd}_2} n', m'_{1,1}m_{1,2}\cdots m_{1,n_1},\ \dots,\ m'_{p,1}m_{p,2}\cdots m_{p,n_p} \quad (m'_{j,1} = n' - (n - m_{j,1}) = m_{j,1} - d_{1,\dots,1}(\mathbf{m}))\\
&\simeq m'_{1,1}m_{1,2}\cdots m_{1,n_1},\ \dots,\ m'_{p,1}m_{p,2}\cdots m_{p,n_p} = \mathrm{mc}_{(1,\dots,1)}\,\mathbf{m}.
\end{aligned}$$

(4) Let $\mathbf{m}^{(0)}$ be a Fuchsian spectral type. Consider a confluence $\mathbf{m}^{(1)}$ of $\mathbf{m}^{(0)}$ at infinity, the Harnad dual $\mathbf{m}^{(2)}$ of $\mathbf{m}^{(1)}$, and the unfolding $\mathbf{m}^{(3)}$ of $\mathbf{m}^{(2)}$. Then we obtain the following transformation between Fuchsian spectral types:

$$\mathbf{m}^{(0)} \xrightarrow{\mathrm{cf}} \mathbf{m}^{(1)} \xrightarrow{\mathrm{Hd}} \mathbf{m}^{(2)} \xrightarrow{\mathrm{uf}} \mathbf{m}^{(3)}.$$

Example 2.26. We give some examples of Remark 2.25 (4) with abbreviated expressions.

$33,222,111111 \xrightarrow{\mathrm{cf}} (11)(11)(11),\underline{3}3 \xrightarrow{\mathrm{Hd}} (3),\underline{1}11,\underline{1}11,\underline{1}11 \xrightarrow{\mathrm{uf}} 111,111,111$

: $33,222,111111 \xrightarrow{(1^2)(1^2)(1^2)} 111,111,111$

The above second line is a short expression of the first line showing the spectral type at ∞.

$33,222,111111 \xrightarrow{\mathrm{cf}} (111)(111),\underline{2}22 \xrightarrow{\mathrm{Hd}} (22),\underline{1}111,\underline{1}111 \xrightarrow{\mathrm{uf}} 22,1111,1111$

: $33,222,111111 \xrightarrow{(1^3)(1^3)} 22,1111,1111$

$21,21,111,111 \xrightarrow{\mathrm{cf}} (1)(1)(1),\underline{2}1,\underline{2}1 \xrightarrow{\mathrm{Hd}} (1)(1),\underline{1}1,\underline{1}1,\underline{1}1 \xrightarrow{\mathrm{uf}} 11,11,11,11,11$

: $21,21,111,111 \xrightarrow{(1)(1)(1)} 11,11,11,11,11$

2.7 Examples

As shown in Example 2.26, the transformations considered above give rise to correspondences between different fundamental spectral types. These correspondences induce isometric transformations of the star-shaped Kac–Moody root space and map roots to roots. Hence, these correspondences can be described within a fixed index of rigidity. We describe these correspondences for the cases $\mathcal{F}_0$, $\mathcal{F}_2$, and $\mathcal{F}_4$. The cases $\mathcal{F}_6$ and $\mathcal{F}_8$ are given in [18].

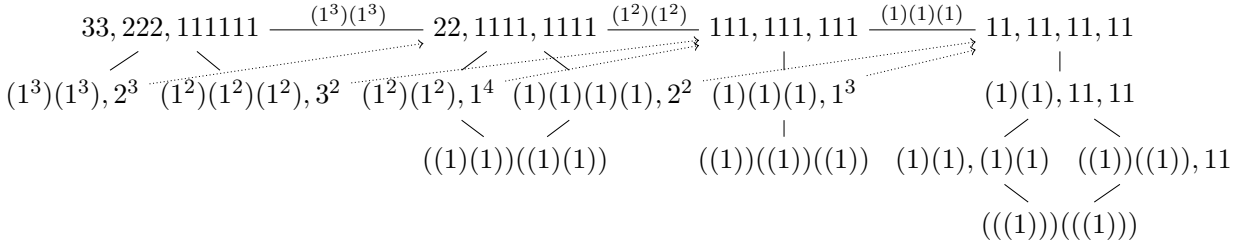
2.7.1 $\mathcal{F}_0$ contains 4 fundamental spectral types

idx=0 : 4 fundamental spectral types, 1 class

6: 33,222,111111 $E_8^{(1)}$	4: 22,1111,1111 $E_7^{(1)}$	3: 111,111,111 $E_6^{(1)}$	2: 11,11,11,11 $D_4^{(1)}$

Here, the order of each spectral type is indicated at the head.

In the following figure, some correspondences between spectral types (cf. Example 2.26) and unramified confluences of fundamental spectral types are shown.



2.7.2 Examples of fundamental spectral types with $\mathrm{idx} = 2 - 2m$ $(m \geq 1)$

$$Gar_m : \overbrace{11, 11, \ldots, 11}^{m+3} = [11]_{m+3} \qquad [11]^T_{m+3} : (m+2)^2, (m+1)^2 2, 1^{2m+4}$$
$$FS_m : m1, m1, 1^{m+1}, 1^{m+1} \qquad Sa_m : m^2, m^2, (2m-1)1, 1^{2m}$$
$$D_4^{(m)} : m^2, m^2, m^2, m(m-1)1 \qquad E_6^{(m)} : m^3, m^3, m^2(m-1)1$$
$$E_7^{(m)} : (2m)^2, m^4, m^3(m-1)1 \qquad E_8^{(m)} : (3m)^2, (2m)^3, m^5(m-1)1$$

Here, we use the notation such as $[11]_3 = 11, 11, 11$ and $[2^2]_2, [21^2]_2 = 22, 22, 211, 211$ etc.

We remark that $D_4^{(1)} = [11]_4 = Gar_1 = FS_1 = Sa_1$ when $m = 1$.

The spectral types $D_4^{(m)}$, $E_6^{(m)}$, $E_7^{(m)}$ and $E_8^{(m)}$ are introduced by [14]. The spectral types Gar_m, FS_m, Sa_m and $D_4^{(m)}$ are related to higher order Painlevé-type equations called Garnier systems, Fuji-Suzuki systems [3, 21], Sasano systems [19] and matrix Painlevé systems, respectively.

Note that the inequalities

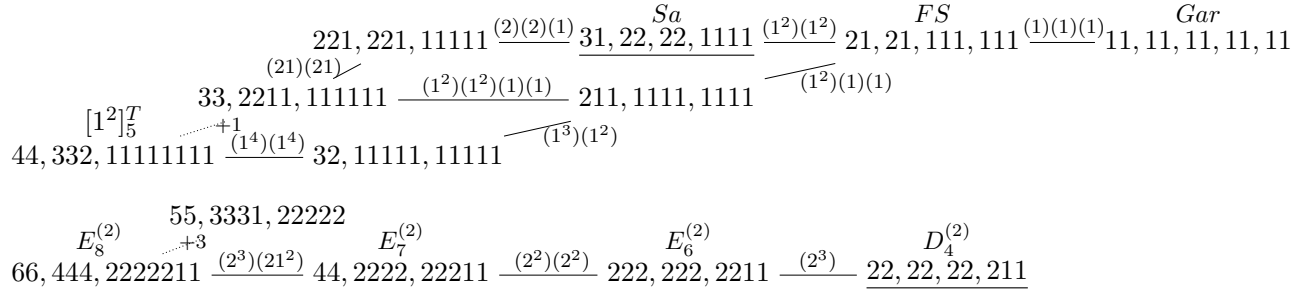
$$\mathrm{ord}\,\mathbf{m} \leq \mathrm{ord}\,E_8^{(m)} = 6m \quad \text{and} \quad \mathrm{Np}(\mathbf{m}) \leq \mathrm{Np}([11]_{m+3}) = m+3 \quad (\forall \mathbf{m} \in \mathcal{F}_{2m-2})$$

are proved by [14], which assures $|\mathcal{F}_{2m-2}| < \infty$ for $m \in \mathbb{Z}_{>0}$.

2.7.3 $\mathcal{F}_2$ contains 13 fundamental spectral types

$\mathrm{idx} = -2$: 13 fundamental spectral types, 2 classes (cf. [14, 7, 11])

8: 44,332,11111111 $[11]^T_5$	6: 33,2211,111111	5: 32,11111,11111	5: 221,221,11111
4: 211,1111,1111	4: 31,22,22,1111 Sa	3: 21,21,111,111 FS	2: 11,11,11,11,11 $[11]_5$
12: 66,444,2222211 $E_8^{(2)}$	10: 55,3331,22222	8: 44,2222,22211 $E_7^{(2)}$	6: 222,222,2211 $E_6^{(2)}$
4: 22,22,22,211 $D_4^{(2)}$			



The above dotted line with $+1$ or $+3$ represents the following relations.

$44, 332, 11111111 \xleftarrow[1]{\mathrm{mc}} \underline{5}4, 33\underline{3}, \underline{2}1111111 \xrightarrow{\mathrm{cf}} 11111 \cdot 211 | 54, \underline{3}33 \xrightarrow{\mathrm{Hd}} 33, \underline{1}11111, \underline{2}211$

$66, 444, 2222211 \xleftarrow[1]{\mathrm{mc}} \underline{7}6, \underline{5}44, 2222\underline{2}1 \xleftarrow[1]{\mathrm{mc}} 7\underline{7}, 5\underline{5}4, 3\underline{2}22221 \xleftarrow[1]{\mathrm{mc}} \underline{8}7, 55\underline{5}, 3\underline{3}22221 \xrightarrow{\mathrm{cf}} 2^4 \cdot 3^2 1 | 87, 5^3$
$\xrightarrow{\mathrm{Hd}} 5^2, \underline{2}2^4, \underline{3}3^2 1 \simeq 55, 3331, 22222$

Here, D_4-fundamental spectral types are indicated by underlines.

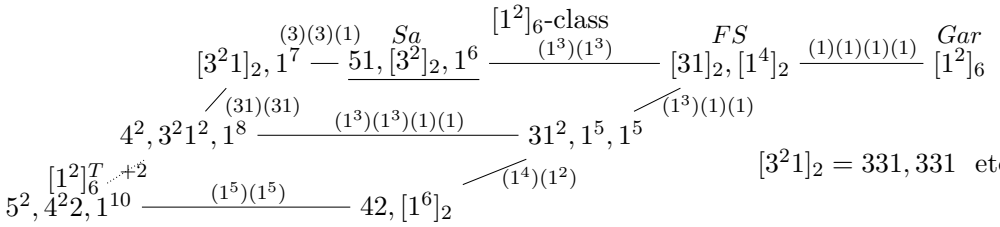
Note that $(1^5)(21^2), 3^3$ is fundamental (cf. [8]) but its unfolding $21^7, 54, 3^3$ is not.

The systems of the spectral types in $\mathcal{F}_2$ and their confluences are studied by [7, 11] related to 4-dimensional Painlevé-type equations.

2.7.4 $\mathcal{F}_4$ contains 37 fundamental spectral types

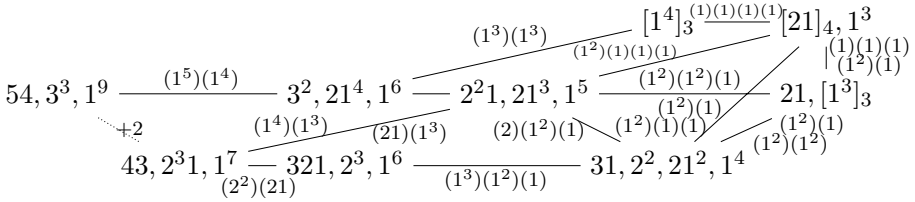
idx = −4 : 37 fundamental spectral types, 5 classes

10: 55,442,1111111111 $[11]_6^T$	8: 44,3311,11111111	7: 331,331,1111111	6: 42,111111,111111
6: 51,33,33,111111 Sa	5: 311,11111,11111	4: 31,31,1111,1111 FS	2: 11,11,11,11,11,11 $[11]_6$
18: 99,666,3333321 $E_8^{(3)}$	12: 66,3333,33321 $E_7^{(3)}$	9: 333,333,3321 $E_6^{(3)}$	6: 33,33,33,321 $D_4^{(3)}$
9: 54,333,111111111	7: 43,2221,11111111	6: 321,222,111111	6: 33,21111,11111
5: 221,2111,11111	4: 1111,1111,1111	4: 31,22,211,1111	3: 21,111,111,111
3: 21,21,21,21,111			
12: 66,444,22221111 $E_8^{(2,2)}$	10: 55,3331,222211	9: 441,333,22221	8: 44,22211,22211
8: 44,2222,221111 $E_7^{(2,2)}$	7: 331,2221,2221	6: 222,2211,2211	6: 222,222,21111 $E_6^{(2,2)}$
6: 51,33,222,222	4: 22,22,211,211	4: 31,31,22,22,22	4: 22,22,22,1111 $D_4^{(2,2)}$
14: 77,554,2222222	10: 55,3322,22222	8: 332,332,2222	6: 42,33,33,222

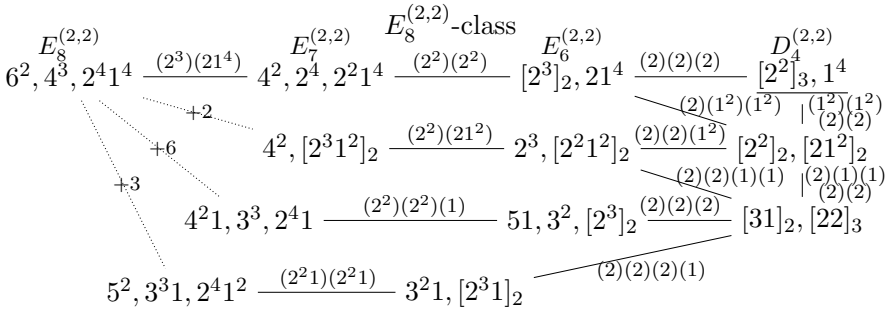


$$55, 442, 1111111111 \xleftarrow[2]{\text{mc}} \cdot \xrightarrow{\text{cf}} 1111111 \cdot 311 | 75, \underline{4}44 \xrightarrow{\text{Hd}} 44, \underline{1}1111111, \underline{3}311$$

Here, $\xleftarrow[d]{\text{mc}}$ represents the map $\mathcal{R}$ and d is the difference of the orders thorough the map.



$$54, 333, 111111111 \xleftarrow[2]{\text{mc}} \cdot \xrightarrow{\text{cf}} 1^6 \cdot 221 | 65, \underline{4}43 \xrightarrow{\text{Hd}} 43, \underline{1}111111, \underline{2}221$$



$$66, 444, 22221111 \xleftarrow[2]{\text{mc}} \cdot \xrightarrow{\text{cf}} 2221 \cdot 2221 | 77, \underline{6}44 \xrightarrow{\text{Hd}} 44, \underline{1}2221, \underline{1}2221$$

$$66, 444, 22221111 \xleftarrow[6]{\text{mc}} \cdot \xrightarrow{\text{cf}} 2221 \cdot 33 \cdot 41 | 765, \underline{9}9 \xrightarrow{\text{Hd}} \underline{2}2221, \underline{3}33, \underline{4}41$$

$$66, 444, 22221111 \xleftarrow[3]{\text{mc}} \cdot \xrightarrow{\text{cf}} 22211 \cdot 331 | 87, \underline{5}55 \xrightarrow{\text{Hd}} 55, \underline{2}22211, \underline{3}331$$

We have another class:

$$77, 554, 2222222 \xleftarrow[1]{\text{mc}} \cdot \xrightarrow{\text{cf}} 2^4 \cdot 32^2 | 87, 5^3 \xrightarrow{\text{Hd}} 5^2, \underline{2}2^4, \underline{3}32^2 = 55, 3322, 22222$$

$$\xrightarrow{\text{cf}} 32 \cdot 32 | 55, \underline{2}2222 \xrightarrow{\text{Hd}} 2222, \underline{3}32, \underline{3}32 \xrightarrow{\text{cf}} 3 \cdot 3 \cdot 2 | 332, \underline{2}222 \xrightarrow{\text{Hd}} 222, \underline{3}3, \underline{3}3, \underline{4}2$$

Putting $m=3$ in the following, we have the case when $\operatorname{idx}\mathbf{m}=2-2m=-4$.

$$
\begin{array}{ccccccc}
 & & & & [1^2]_{m+3}\text{-class } (m\geq 2) & & \\
 & & & & \overset{(m)(m)(1)}{\longrightarrow}\ \underline{(2m-1)1,[m^2]_2,1^{2m}}\ \overset{(1^m)(1^m)}{\longrightarrow} & & \\
[m^2 1]_2, 1^{2m+1} & \overset{Sa}{\longrightarrow} & & & & \overset{FS}{\longrightarrow} & [m1]_2,[1^{m+1}]_2 \overset{(1)\cdots(1)}{\longrightarrow} [1^2]_{m+3}\ \overset{Gar}{} \\
 & \swarrow^{(m1)(m1)} & & & & \swarrow^{(1^m)(1)(1)} & \\
(m+1)^2, m^2 1^2, 1^{2m+2} & & \xrightarrow{\quad(1^m)(1^m)(1)(1)\quad} & & & & m1^2, 1^{m+2}, 1^{m+2} \\
 & {}^{[1^2]^T_{m+3}}\!\!\searrow^{+(m-1)} & & & & \swarrow^{(1^{m+1})(1^2)} & \\
(m+2)^2, (m+1)^2 2, 1^{2m+4} & & \xrightarrow{\quad(1^{m+2})(1^{m+2})\quad} & & & & (m+1)2, [1^{m+3}]_2
\end{array}
$$

$$
1^{2m+4}, (m+2)^2, (m+1)^2 2 \xleftarrow[m-1]{\text{mc}} 1^{2m+1}\cdot m1^2 | (2m+1)(m+2), (m+1)^3 \xrightarrow{\text{Hd}} (m+1)^2, \underline{1}1^{2m+1}, \underline{m}m1^2
$$

$$
\begin{array}{ccccccc}
 & & & & D_4^{(m)}\text{-}E_8^{(m)}\text{-class } (m\neq 2) & & \\
 & E_8^{(m)} & & & E_7^{(m)} & & E_6^{(m)} \\
(3m)^2, (2m)^3, m^5(m-1)1 & \overset{(m^3)(m^2m-11)}{\longrightarrow} & (2m)^2, m^4, m^3(m-1)1 & \overset{(m^2)(m^2)}{\longrightarrow} & [m^3]_2, m^2(m-1)1 & \overset{(m^3)}{\longrightarrow} & \underline{[m^2]_3, m(m-1)1}\ \overset{D_4^{(m)}}{}
\end{array}
$$

Suzuki [20] studied Fuchsian systems of the spectral types in $\mathcal{F}_4$ to get 6-dimensional Painlevé systems.

3 Main results

3.1 A main theorem

In this section, we state our main theorem on the following equivalence classes.

Definition 3.1 ([11])**.** If a spectral type $\mathbf{m}$ of an irreducible differential equation $\mathcal{M}$ is transformed to a spectral type $\mathbf{m}'$ by a finite sequence of gauge transformations, confluences, unfoldings, Harnad duality and fractional linear transformations of $\mathcal{M}$, we say that $\mathbf{m}$ and $\mathbf{m}'$ are equivalent and write $\mathbf{m}\sim\mathbf{m}'$. Moreover, we set

$$\mathcal{F}_{[\mathbf{n}]}:=\{\mathbf{m}\in\mathcal{F}\mid \mathbf{m}\sim\mathbf{n}\}.$$

Remark 3.2. Since $\operatorname{idx}\mathbf{m}=\operatorname{idx}\mathbf{n}$ for $\mathbf{m}\in\mathcal{F}_{[\mathbf{n}]}$, we have $|\mathcal{F}_{[\mathbf{n}]}|<\infty$.

Under the notation in Definition 2.17, set

$$\mathcal{F}_\ell^T:=\mathcal{F}^T\cap\mathcal{F}_\ell=\{\mathbf{m}\in\mathcal{F}^T\mid \operatorname{idx}\mathbf{m}=-\ell\}\quad(\ell=0,2,4,\ldots). \tag{3.1}$$

The following table shows the numbers of certain elements of $\mathcal{F}_\ell$.

Equivalence classes and E_8, E_7, E_6, D_4-fundamental spectral types in $\mathcal{F}_\ell$

ℓ	$\lvert\mathcal{F}_\ell\rvert$	$\mathrm{N_p}$=3	$\mathrm{N_p}$=4	ord $\leq$	$\lvert\mathcal{F}_{[\mathbf{n}]}\rvert\leq$	$\lvert\mathcal{F}_{[\mathbf{n}]}\rvert\geq$	$\lvert\mathcal{F}_\ell^{E_8}\rvert$	$\lvert\mathcal{F}_\ell^{E_7}\rvert$	$\lvert\mathcal{F}_\ell^{E_6}\rvert$	$\lvert\mathcal{F}_\ell^{D_4}\rvert$
0	4	3	1	6	4	4	1	1	1	1
2	13	9	3	12	8	5	2	3	2	2
4	37	25	9	18	9	4	5	6	6	4
6	69	46	17	24	12	4	7	12	10	6
8	113	73	29	30	23	4	10	15	12	8
10	198	127	50	36	36	3	15	25	20	9
20	1186	680	345	66	104	4	40	80	67	25
50	40617	18878	12953	156	757	4	351	931	701	137
80	419160	165023	136352	246	3173	3	1456	4386	3251	414
100	1461132	522509	471378	306	7613	3	3181	10389	7500	743

For example, the column headed "$|\mathcal{F}_{[\mathbf{n}]}|\leq$" gives the maximum cardinality of $\mathcal{F}_{[\mathbf{n}]}$ among $\mathbf{n}\in\mathcal{F}_\ell$. In particular, $\max_{\mathbf{n}\in\mathcal{F}_{100}}|\mathcal{F}_{[\mathbf{n}]}|=7613$. Moreover, $|\{\mathbf{m}\in\mathcal{F}_{100}\mid \mathrm{N_p}(\mathbf{m})=4\}|=471378$ etc.

Theorem 3.1. *$\mathcal{F}^{E_8}$ is a set of complete representatives of $\mathcal{F}/\sim$. Namely,*

$$\mathcal{F} = \bigsqcup_{\mathbf{n}\in\mathcal{F}^{E_8}} \mathcal{F}_{[\mathbf{n}]}.$$

Moreover, every $\mathbf{n} \in \mathcal{F}^{E_8}$ satisfies

$$\operatorname{ord}\mathbf{n} > \operatorname{ord}\mathbf{m} \quad (\forall \mathbf{m} \in \mathcal{F}_{[\mathbf{n}]} \setminus \{\mathbf{n}\}).$$

Remark 3.3. Since $\mathcal{P}/\underset{\text{mc}}{\sim} \simeq \mathcal{F} \cup \{\mathbf{1}\}$ and $\hat{\mathcal{P}}/\sim \simeq \mathcal{P}/\sim$, we have

$$\hat{\mathcal{P}}/\sim \simeq \mathcal{P}/\sim \simeq \mathcal{F}^{E_8} \cup \{\mathbf{1}\}.$$

Here, recall that $\hat{\mathcal{P}}$ and $\mathcal{P}$ are the set of irreducibly realizable spectral types of systems allowing unramified irregular singularities (cf. Definition 2.23) and that of Fuchsian systems, respectively.

3.2 Proof of the main theorem

By defining a map $\mathcal{U}$ of $\mathcal{P}$ to $\mathcal{F}^{E_8} \cup \{\mathbf{1}\}$, the proof of Theorem 3.1 is reduced to Lemma 3.6.

Definition 3.4. Let $\mathbf{m} = m_{11}\cdots m_{1n_1}, m_{21}\cdots m_{2n_2}, \ldots, m_{p1}\cdots m_{pn_p} \in \mathcal{P}$.

(1) We define $\mathcal{R}\mathbf{m} \in \mathcal{F} \cup \{\mathbf{1}\}$ by the condition $\mathcal{R}\mathbf{m} \underset{\text{mc}}{\sim} \mathbf{m}$ (cf. Definition 2.17 and (2.10)).

(2) Let $\mathcal{M}$ be a Fuchsian system of the spectral type $\mathbf{m}$ and let $L\mathbf{m}$ be the spectral type of the system obtained by applying a generic linear fractional transformation on $\mathbb{P}^1$ to $\mathcal{M}$. We denote by $\mathcal{S}\mathbf{m}$ the unfolding of the spectral type obtained by applying the Harnad dual $\mathrm{Hd}_{0,\ldots,0}$ to $L\mathbf{m}$:

$$\begin{aligned} \mathcal{S}\mathbf{m} &:= \big[[(p-1)n, n],\ [\overbrace{n,\ldots,n}^{p}],\ [\{m_{j\nu} \mid j = 1,\ldots,p,\ \nu = 1,\ldots,n_j\}]\big], \\ n &:= \operatorname{ord}\mathbf{m}. \end{aligned} \tag{3.2}$$

Then, we define

$$\mathcal{U} : \mathcal{P} \to \mathcal{F}^{E_8} \cup \{\mathbf{1}\}, \quad \mathcal{U}\mathbf{m} := \mathcal{R}\mathcal{S}^2\mathbf{m}.$$

Remark 3.5. (1) Note that $\mathcal{R}\mathbf{m}$ is obtained from $\mathbf{m}$ by successively applying middle convolutions according to the Katz reduction procedure, until no further reduction is possible.

Moreover, (3.2) is obtained by

$$\begin{aligned} \mathbf{m} &\simeq n,\ \underline{0}m_{11}\cdots m_{1n_1},\ \cdots,\ \underline{0}m_{p1}\cdots m_{p,n_p} \\ &\xrightarrow{\text{Hd}}\ (m_{11}\cdots m_{1n_1})\cdots(m_{p1}\cdots m_{p,n_p}),\ ((p-1)n)n \\ &\xrightarrow{\text{uf}}\ \mathcal{S}\mathbf{m} = m_{11}\cdots m_{1n_1}\cdots m_{p1}\cdots m_{p,n_p},\ \overbrace{n\cdots n}^{p},\ ((p-1)n)n \end{aligned}$$

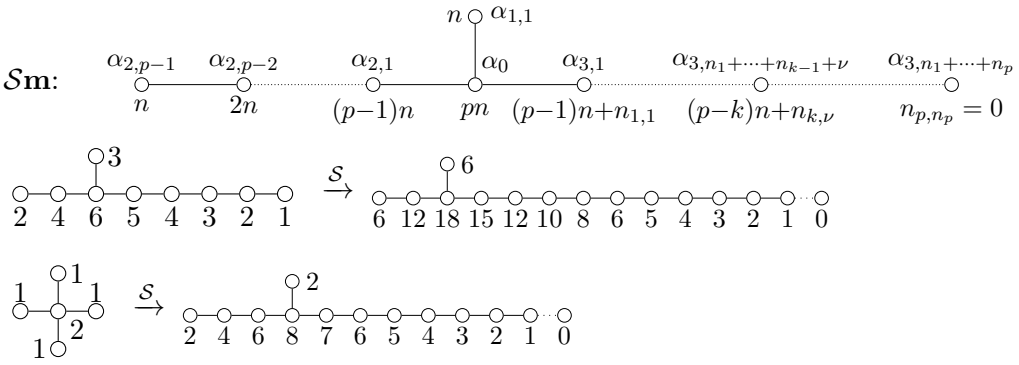
and $\mathcal{S}\mathbf{m}$ is $(2, p, *)$-type. Then,

$$\mathcal{S}^2\mathbf{m} = \big[[2pn, pn],\ [pn, pn, pn],\ [(p-1)n, \overbrace{n,\ldots,n}^{p+1}, \{m_{j,\nu}\}]\big].$$

(2) It follows from the above definition that if $\mathbf{m} = (m_{j,\nu})$, $\mathbf{m}' = (m'_{j,\nu}) \in \mathcal{P}$ satisfies $\{m_{j,\nu}\} = \{m'_{j,\nu}\}$ and $\mathrm{Np}(\mathbf{m}) = \mathrm{Np}(\mathbf{m}')$, then $\mathcal{S}\mathbf{m} \simeq \mathcal{S}\mathbf{m}'$ and $\mathbf{m} \sim \mathbf{m}'$. For example,

$$44, 2211, 2211 \sim 44, 222, 21111 \in \mathcal{F}_4.$$

The root corresponding to $\mathcal{S}\mathbf{m}$ for $\alpha_{\mathbf{m}}$ in Definition 2.8 is as follows:



Lemma 3.6. (1) *If* $\mathbf{m}$ *is of* $(2,3,*)$*-type, then* $\mathcal{S}\mathbf{m} \underset{\text{mc}}{\sim} \mathbf{m}$.

(2) *If* $N_p(\mathbf{m}) = 3$*, then* $\mathcal{S}^2\mathbf{m} \underset{\text{mc}}{\sim} \mathcal{S}\mathbf{m}$.

(3) *If* $\mathbf{m}'$ *is the unfolding of a Harnad dual of a confluence of* $\mathbf{m}$*, then* $\mathcal{S}^2\mathbf{m}' \underset{\text{mc}}{\sim} \mathcal{S}^2\mathbf{m}$.

(4) $\operatorname{ord}\mathcal{R}\mathcal{S}\mathbf{m} > \operatorname{ord}\mathbf{m} \quad (\forall \mathbf{m} \in \mathcal{F} \setminus \mathcal{F}^{E_8})$

The proof of Lemma 3.6 (1), (2) and (3) is given in this section. In the next section, we show a second main theorem which calculates $\mathcal{R}\mathcal{S}\mathbf{m}$ for $\mathbf{m} \in \mathcal{F}$ and Proposition 4.12 says Lemma 3.6 (4).

Let

$$\mathbf{m} = m_{11}m_{12},\ m_{21}m_{22}m_{23},\ m_{31}m_{32}\cdots m_{3n_3}$$

be an ordered $(2,3,*)$ spectral type. Then Katz reductions of $\mathcal{S}\mathbf{m}$ are given by

$$\begin{aligned}
\mathcal{S}\mathbf{m} &= \underline{(2n)}n,\ \underline{n}nn,\ \underline{m_{11}}m_{12}m_{21}m_{22}m_{23}m_{31}m_{32}\cdots \\
&\qquad (d_{1,1,1}(\mathbf{m}) = n + m_{11} - n = m_{11}) \\
&\xrightarrow[1,1,1]{m_{11}} \underline{(n+m_{12})}n,\ m_{12}\underline{n}n,\ 0\underline{m_{12}}m_{21}m_{22}m_{23}m_{31}m_{32}\cdots \\
&\xrightarrow[1,2,2]{m_{12}} \underline{n}n,\ m_{12}m_{11}\underline{n},\ 00\underline{m_{21}}m_{22}m_{23}m_{31}m_{32}\cdots \\
&\xrightarrow[1,3,3]{m_{21}} (m_{22}+m_{23})\underline{n},\ m_{12}m_{11}\underline{(m_{22}+m_{23})},\ 000\underline{m_{22}}m_{23}m_{31}\cdots \\
&\xrightarrow[2,3,4]{m_{22}} (m_{22}+m_{23})\underline{(m_{21}+m_{23})},\ m_{12}\underline{m_{11}}m_{23},\ 0000\underline{m_{23}}m_{31}m_{32}\cdots \\
&\xrightarrow[2,2,5]{m_{11}-m_{22}} \underline{(m_{22}+m_{23})}m_{12},\ \underline{m_{12}}m_{22}m_{23},\ 0000\underline{(m_{12}-m_{21})}m_{31}m_{32}\cdots \\
&\xrightarrow[1,1,5]{m_{12}-m_{21}} m_{11}m_{12},\ m_{21}m_{22}m_{23}, 00000\, m_{31}m_{32}\cdots m_{3,n_3} \simeq \mathbf{m},
\end{aligned}$$

which proves Lemma 3.6 (1). Here, the underlined multiplicities such as $\underline{(2n)}$ indicate the σ in mc_σ to be considered. They are mostly maximal multiplicities in partitions.

Considering the following Katz reductions

$$\begin{aligned}
\mathcal{S}^2\mathbf{m} &= \underline{(2pn)}(pn), (pn)(pn)\underline{(pn)}, \underline{((p-1)n)}\overbrace{n\cdots n}^{p+1}m_{11}\cdots m_{p1}\cdots \\
&\xrightarrow[1,3,1]{(p-1)n} (\underline{(p+1)n})(pn), (pn)(\underline{pn})n, 0\overbrace{\underline{n}\cdots n}^{p+1}m_{11}\cdots m_{p1}\cdots
\end{aligned}$$

$$\xrightarrow[1,2,2]{n} (pn)\underline{(pn)}, \underline{(pn)}((p-1)n)n, 00\overbrace{\underline{n}\cdots n}^{p} m_{11}\cdots m_{p1}\cdots$$
$$\cdots\cdots \quad (\nu = 1, 2, \ldots)$$
$$\xrightarrow[2,1,2\nu+1]{n} \underline{((p-\nu+1)n)}((p-\nu)n), ((p-\nu)n)\underline{((p-\nu)n)}n, \overbrace{0\cdots 0}^{2\nu+1}\overbrace{\underline{n}\cdots n}^{p-2\nu+1} m_{11}\cdots$$
$$\xrightarrow[1,2,2\nu+2]{n} ((p-\nu)n)\underline{((p-\nu)n)}, \underline{((p-\nu)n)}((p-\nu-1)n)n, \overbrace{0\cdots 0}^{2\nu+2}\overbrace{\underline{n}\cdots n}^{p-2\nu} m_{11}\cdots,$$

we make the following definition.

Definition 3.7. Set

$$\mathcal{T}\,\mathbf{m} := \begin{cases} \big[[kn, (k-1)n], [(k-1)n, (k-1)n, n], [m_{11}, \cdots, m_{pn_p}]\big] & (p = 2k-1), \\ \big[[kn, kn], [kn, (k-1)n, n], [m_{11}, \cdots, m_{pn_p}]\big] & (p = 2k). \end{cases}$$

By the above reductions, we have

$$\mathcal{T}\,\mathbf{m} \underset{\mathrm{mc}}{\sim} \mathcal{S}^2\,\mathbf{m},$$

which proves Lemma 3.6 (2).

Proof of Lemma 3.6 (3). Let $\mathbf{m} \in \mathcal{P}$ be written as in (2.4). Suppose that $m_{p-1,1}\cdots m_{p-1,n_{p-1}}$ is a refinement of $m_{p,1}\cdots m_{p,n_p}$. Write

$$\{m_{p-1,\nu} \mid 1 \le \nu \le n_{p-1}\} = \{x_{j,\nu} \mid 1 \le j \le n_p,\ 1 \le \nu \le R_j\},$$
$$n_{p-1} = R_1 + \cdots + R_{n_p},\ m_{p,j} = x_{j,1} + \cdots + x_{j,R_j} \quad (j = 1, \ldots, n_p).$$

Moreover, set

$$k := n_p, \qquad a_\nu := m_{p,\nu} \quad (1 \le \nu \le n_p),$$
$$\ell := p-2, \quad b_j := m_{j,1}, \quad b_{j,\nu} := m_{j,\nu} \quad (j = 1, \ldots, \ell,\ 2 \le \nu \le n_j),$$
$$\bar{a} = a_1 + \cdots + a_k = \operatorname{ord}\mathbf{m}, \quad \bar{b} = b_1 + \cdots + b_\ell.$$

We consider the confluence of the points c_{p-1} and $c_p = \infty$ corresponding to the refinement and denote by $\operatorname{Hu}\mathbf{m}$ the unfolding of the Harnad dual of this confluence. We have

$$\begin{aligned}
\mathbf{m} &= m_{1,1}\cdots m_{1,n_1},\ \ldots,\ m_{p-2,1}\cdots,\ m_{p-1,1}\cdots m_{p-1,n_{p-1}},\ m_{p,1}\cdots m_{p,n_p} \\
&= b_1 b_{1,2}\cdots b_{1,n_1},\ \ldots,\ b_\ell b_{\ell,2}\cdots b_{\ell,n_\ell},\ \overbrace{x_{1,1}\cdots x_{1,R_1}}^{a_1}\cdots\overbrace{x_{k,1}\cdots x_{k,R_k}}^{a_k},\ a_1\cdots a_k \\
&\xrightarrow{\mathrm{cf}} (x_{1,1}\cdots x_{1,R_1})\cdots(x_{k,1}\cdots x_{k,R_k}),\ \underline{b_1}b_{1,2}\cdots b_{1,n_1},\ \ldots,\ \underline{b_\ell}b_{\ell,2}\cdots b_{\ell,n_\ell} \\
&\xrightarrow{\mathrm{Hd}} (b_{1,2}\cdots b_{1,n_1})\cdots(b_{\ell,2}\ldots b_{\ell,n_\ell}),\ (\ell\bar{a}-\bar{b}-a_1)x_{1,1}\cdots x_{1,R_1},\ \ldots,\ (\ell\bar{a}-\bar{b}-a_k)x_{k,1}\cdots x_{k,R_k} \\
&\xrightarrow{\mathrm{uf}} b_{1,2}\cdots b_{1,n_1}\cdots b_{\ell,2}\cdots b_{\ell,n_\ell},\ (\bar{a}-b_1)\cdots(\bar{a}-b_\ell),\ (\ell\bar{a}-\bar{b}-a_1)x_{1,1}\cdots x_{1,R_1}, \\
&\qquad \ldots,\ (\ell\bar{a}-\bar{b}-a_k)x_{k,1}\cdots x_{k,R_k}.
\end{aligned}$$

Here we have

$$\operatorname{ord}\operatorname{Hu}\mathbf{m} = (b_{1,2}+\cdots+b_{1,n_1})+\cdots+(b_{\ell,2}+\cdots+b_{\ell,n_\ell}) = \ell\bar{a}-\bar{b}.$$

Putting

$$x_i = \{x_{i,1}, \dots, x_{i.R_i}\},\ y_j = \{b_{j,2}, \dots y_{j,n_j}\},\ x = \{x_1 \dots x_k, y_1 \dots y_\ell\},$$

we have

$$\begin{aligned}\mathbf{m} &= \big[[b_1, y_1], \dots, [b_\ell, y_\ell], [x_1, \dots, x_k], [a_1 \dots, a_k]\big] \\ &\overset{\text{Hu}}{\to} \big[\bar a - b_1, \dots, \bar a - b_\ell], [\ell\bar a - \bar b - a_1, x_1], \dots, [\ell\bar a - \bar b - a_k, x_k], [y_1, \dots, y_\ell]\big].\end{aligned}$$

Suppose k is divisible by 4. Set $k+2 = 2q$. Then

$$\begin{aligned}&\mathcal{T}\,\mathrm{Hu}\,\mathbf{m} \\ &= \big[[\underline{q(\ell\bar a - \bar b)},\ q(\ell\bar a - \bar b)], \\ &\quad [\underline{q(\ell\bar a - \bar b)},\ (q-1)(\ell\bar a - \bar b),\ \ell\bar a - \bar b], \\ &\quad\quad [\bar a - b_1, \dots, \bar a - b_\ell, \ell\bar a - \bar b - a_1, \cdots, \underline{\ell\bar a - \bar b - a_k}, x]\big] \\ &\xrightarrow[1,1,k+\ell]{\ell\bar a - \bar b - a_k} \\ &\big[[(q-1)(\ell\bar a - \bar b) + a_k,\ \underline{q(\ell\bar a - \bar b)}], \\ &\quad [\underline{(q-1)(\ell\bar a - \bar b) + a_k},\ (q-1)(\ell\bar a - \bar b),\ \ell\bar a - \bar b], \\ &\quad\quad [\bar a - b_1, \dots, \bar a - b_\ell, \ell\bar a - \bar b - a_1, \cdots, \underline{\ell\bar a - \bar b - a_{k-1}}, 0, x] \\ &\xrightarrow[2,1,k+\ell-1]{\ell\bar a - \bar b - a_{k-1}} \\ &\big[[(q-1)(\ell\bar a - \bar b) + a_k,\ \underline{(q-1)(\ell\bar a - \bar b) + a_{k-1}}], \\ &\quad [(q-2)(\ell\bar a - \bar b) + a_{k-1} + a_k,\ \underline{(q-1)(\ell\bar a - \bar b)},\ \ell\bar a - \bar b], \\ &\quad\quad [[\bar a - b_1, \dots, \bar a - b_\ell, \ell\bar a - \bar b - a_1, \dots, \underline{\ell\bar a - \bar b - a_{k-2}}, 0, 0, x]] \\ &\xrightarrow[2,2,k+\ell-2]{\ell\bar a - \bar b - a_{k-2} - a_k} \\ &\big[[\underline{(q-1)(\ell\bar a - \bar b) + a_k}, (q-2)(\ell\bar a - \bar b) + a_{k-2} + a_{k-1} + a_k], \\ &\quad [(q-2)(\ell\bar a - \bar b) + a_{k-1} + a_k,\ \underline{(q-2)(\ell\bar a - \bar b) + a_{k-2} + a_k},\ \ell\bar a - \bar b], \\ &\quad\quad \big[[\bar a - b_1, \dots, \bar a - b_\ell, \ell\bar a - \bar b - a_1, \dots, \underline{\ell\bar a - \bar b - a_{k-3}}, a_k, 0, 0, x]\big] \\ &\xrightarrow[1,2,k+\ell-3]{\ell\bar a - \bar b - a_{k-3} - a_{k-1}} \\ &\big[[\underline{(q-2)(\ell\bar a - \bar b) + a_{k-3} + a_{k-1} + a_k},\ (q-2)(\ell\bar a - \bar b) + a_{k-2} + a_{k-1} + a_k], \\ &\quad [\underline{(q-2)(\ell\bar a - \bar b) + a_{k-1} + a_k},\ (q-3)(\ell\bar a - \bar b) + a_{k-3} + \cdots + a_k,\ \ell\bar a - \bar b], \\ &\quad\quad \big[[\bar a - b_1, \dots, \bar a - b_\ell, \ell\bar a - \bar b - a_1, \dots, \underline{\ell\bar a - \bar b - a_{k-4}}, a_{k-1}, a_k, 0, 0, x]\big] \\ &\xrightarrow[1,1,k+\ell-4]{\ell\bar a - \bar b - a_{k-4} - a_{k-2}} \\ &\cdots \\ &\xrightarrow[1,2,\ell+1]{\ell\bar a - \bar b - a_1 - a_3} \\ &\big[[\ell\bar a - \bar b + a_1 + a_3 + \cdots + a_k,\ \ell\bar a - \bar b + a_2 + a_3 + \cdots + a_k], \\ &\quad [\ell\bar a - \bar b + a_3 + \cdots + a_k,\ a_1 + a_2 + \cdots + a_k,\ \ell\bar a - \bar b], \\ &\quad\quad [\bar a - b_1, \dots, \bar a - b_\ell, a_3, a_4, \dots, a_k, 0, 0, x]\big]\end{aligned}$$

$$
\begin{aligned}
&= \big[[\underline{(\ell+1)\bar{a}-\bar{b}-a_2}, (\ell+1)\bar{a}-\bar{b}-a_1],\\
&\quad [\underline{(\ell+1)\bar{a}-\bar{b}-a_1-a_2},\ \bar{a},\ \ell\bar{a}-\bar{b}],\\
&\quad\quad [\bar{a}-b_1,\dots,\underline{\bar{a}-b_\ell}, a_3, a_4, \dots, a_k, 0, 0, x]\big]\\
&\xrightarrow[1,1,\ell]{\bar{a}-a_2-b_\ell}\\
&\big[[\ell\bar{a}-\bar{b}+b_\ell,\ \underline{(\ell+1)\bar{a}-\bar{b}-a_1}],\\
&\quad [\ell\bar{a}-\bar{b}+b_\ell-a_1,\ \bar{a},\ \underline{\ell\bar{a}-\bar{b}}],\\
&\quad\quad [\bar{a}-b_1,\dots,\underline{\bar{a}-b_{\ell-1}}, a_2, a_3, a_4, \dots, a_k, 0, 0, x]\big]\\
&\xrightarrow[2,3,\ell-1]{\bar{a}-b_{\ell-1}-b_\ell}\\
&\big[[\underline{\ell\bar{a}-\bar{b}+b_\ell},\ \ell\bar{a}-\bar{b}-a_1+b_{\ell-1}+b_\ell],\\
&\quad [\underline{\ell\bar{a}-\bar{b}+b_\ell-a_1},\ \bar{a},\ (\ell-1)\bar{a}-\bar{b}+b_{\ell-1}+b_\ell],\\
&\quad\quad [\bar{a}-b_1,\dots,\underline{\bar{a}-b_{\ell-2}}, b_\ell, a_2, a_3, a_4, \dots, a_k, 0, 0, x]\big]\\
&\xrightarrow[1,1,\ell-2]{\bar{a}-b_{\ell-2}-b_{\ell-1}}\\
&\big[[(\ell-1)\bar{a}-\bar{b}+b_{\ell-2}+b_{\ell-1}+b_\ell,\ \underline{\ell\bar{a}-\bar{b}-a_1+b_{\ell-1}+b_\ell}],\\
&\quad [(\ell-1)\bar{a}-\bar{b}+b_{\ell-2}+b_{\ell-1}+b_\ell-a_1,\ \bar{a},\ \underline{(\ell-1)\bar{a}-\bar{b}+b_{\ell-1}+b_\ell}],\\
&\quad\quad [\bar{a}-b_1,\dots,\underline{\bar{a}-b_{\ell-3}}, b_{\ell-1}, b_\ell, a_2, a_3, a_4, \dots, a_k, 0, 0, x]\big]\\
&\xrightarrow[2,3,\ell-3]{\bar{a}-b_{\ell-3}-b_{\ell-2}}\\
&\big[[\underline{(\ell-1)\bar{a}-\bar{b}+b_{\ell-2}+b_{\ell-1}+b_\ell},\ (\ell-1)\bar{a}-\bar{b}-a_1+b_{\ell-3}+\cdots+b_\ell],\\
&\quad [\underline{(\ell-1)\bar{a}-\bar{b}+b_{\ell-2}+b_{\ell-1}+b_\ell-a_1},\ \bar{a},\ (\ell-2)\bar{a}-\bar{b}+b_{\ell-3}+\cdots+b_\ell],\\
&\quad\quad [\bar{a}-b_1,\dots,\underline{\bar{a}-b_{\ell-4}}, b_{\ell-2}, b_{\ell-1}, b_\ell, a_2, a_3, a_4, \dots, a_k, 0, 0, x]\big]\\
&\dots
\end{aligned}
$$

When $\ell = 2r-2$ with a positive integer r, we have $p = 2r$ and eventually

$$
\begin{aligned}
&\xrightarrow[2,3,1]{\bar{a}-b_1-b_2}\\
&\big[[r\bar{a}-\bar{b}+b_2+\cdots+b_\ell,\ r\bar{a}-\bar{b}-a_1+b_1+\cdots+b_\ell],\\
&\quad [r\bar{a}-\bar{b}+b_2+\cdots+b_\ell-a_1,\ \bar{a},\ (r-1)\bar{a}-\bar{b}+b_1+\cdots+b_\ell],\\
&\quad\quad [b_2, b_3, \dots, b_\ell, a_2, a_3, a_4, \dots, a_k, 0, 0, x]\big]\\
&= \big[[r\bar{a}-b_1, r\bar{a}-a_1], [r\bar{a}-a_1-b_1, \bar{a}, (r-1)\bar{a}], [b_2, b_3, \dots, b_\ell, a_2, a_3, a_4, \dots, a_k, 0, 0, x]\big]\\
&\xrightarrow[1,1,k+\ell-1]{-b_1} \xrightarrow[2,1,k+\ell]{-a_1}\\
&\big[[r\bar{a},\ r\bar{a}],\ [r\bar{a},\ \bar{a},\ (r-1)\bar{a}],\ [b_2, b_3, \dots, b_\ell, a_2, a_3, a_4, \dots, a_k, b_1, a_1, x]\big].
\end{aligned}
$$

When $\ell = 2r-1$ with a positive integer r, we have $p = 2r+1$ and

$$
\begin{aligned}
&\xrightarrow[1,1,1]{\bar{a}-b_1-b_2}\\
&\big[[r\bar{a},\ (r+1)\bar{a}-a_1-b_1], [r\bar{a}-a_1, \bar{a}, r\bar{a}-b_1],\ [b_2, b_3, \dots, b_\ell, a_2, a_3, \dots, a_k, 0, 0, x]\big]\\
&\xrightarrow[2,3,k+\ell-1]{-b_1} \xrightarrow[2,1,k+\ell]{-a_1}\\
&\big[[r\bar{a},\ (r+1)\bar{a}],\ [r\bar{a},\ \bar{a},\ r\bar{a}],\ [b_2, b_3, \dots, b_\ell, a_2, a_3, \dots, a_k, b_1, a_1, x]\big].
\end{aligned}
$$

Hence

$$\mathcal{T}\,\mathrm{Hu}\,\mathbf{m} \underset{\mathrm{mc}}{\sim} \mathcal{T}\,\mathbf{m}. \tag{3.3}$$

For a general k, we replace $m_{p-1,1}\cdots m_{p-1,n_{p-1}}$ and $a_1\cdots a_k$ by $m_{p-1,1}\cdots m_{p-1,n_{p-1}}\overbrace{0\cdots 0}^{k'}$ and $a_1\cdots a_k\overbrace{0\cdots 0}^{k'}$, respectively, where k' is chosen so that $k+k'$ is divisible by 4. Thus, (3.3) is proved for arbitrary k. This proves Lemma 3.6 (3). □

3.3 Relation to middle convolutions

Since mc_σ is a composition of two Harnad duals, as is shown in Remark 2.25 (3), $\mathcal{T}\,\mathrm{mc}_\sigma\,\mathbf{m} \underset{\mathrm{mc}}{\sim} \mathcal{T}\mathbf{m}$. We investigate the direct relation between $\mathcal{T}\,\mathrm{mc}_\sigma\,\mathbf{m}$ and $\mathcal{T}\mathbf{m}$.

For a positive integer p, we define the $(2,3,*)$-spectral type

$$\mathbf{n}_p := \begin{cases} \left[[k,k],[k,k-1,1],\overbrace{[1,\dots,1]}^{p}\right] & (p=2k),\\ \left[[k,k-1],[k-1,k-1,1],\overbrace{[1,\dots,1]}^{p}\right] & (p=2k-1). \end{cases}$$

Applying successive the middle convolutions to $\mathbf{n}_p$, we obtain

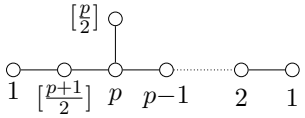
$$\mathbf{n}_{2k} = k\underline{k},\ \underline{k}(k-1)1,\ \overbrace{1\cdots 1\underline{1}}^{2k} \xrightarrow[2,1,2k]{1} \underline{k}(k-1),\ (k-1)(\underline{k-1})1,\ \overbrace{1\cdots 1\underline{1}}^{2k-1}0 \simeq \mathbf{n}_{2k-1}$$

$$\xrightarrow[1,2,2k-1]{1} (k-1)(\underline{k-1}),\ (\underline{k-1})(k-2)1,\ \overbrace{1\cdots 1\underline{1}}^{2k-2}00 \simeq \mathbf{n}_{2k-2} \xrightarrow[2,1,2k-3]{1} \cdots.$$

Hence, $\mathbf{n}_p \underset{\mathrm{mc}}{\sim} \mathbf{n}_1$, and $\mathbf{n}_p$ is a rigid spectral type.

The corresponding real root is

$$\beta_p := p\alpha_0 + [\tfrac{p+1}{2}]\alpha_{11} + [\tfrac{p}{2}]\alpha_{21} + \alpha_{22} + \sum_{\nu=1}^{p-1}(p-\nu)\alpha_{3\nu},$$



Here, we note that

$$\begin{aligned} &(\beta_{2k}|\alpha_0)=1, && &(\beta_{2k-1}|\alpha_0)=1,\\ &(\beta_{2k}|\alpha_{11})=0,\quad (\beta_{2k}|\alpha_{12})=-k, && &(\beta_{2k-1}|\alpha_{11})=-1,\quad (\beta_{2k-1}|\alpha_{12})=1-k,\\ &(\beta_{2k}|\alpha_{21})=-1,\quad (\beta_{2k}|\alpha_{22})=2-k, && &(\beta_{2k-1}|\alpha_{21})=0,\quad (\beta_{2k-1}|\alpha_{22})=2-k,\\ &(\beta_p|\alpha_{23})=-1, && &(\beta_p|\alpha_{3j}) = \begin{cases} 0 & (j\geq 1,\ j\neq p),\\ -1 & (j=p). \end{cases} \end{aligned}$$

Set

$$\begin{aligned} \mathbf{m}_3 &= [m_{11},m_{21},\dots,m_{p1},m_{12},\dots,m_{1n_1},\ \dots\ ,m_{p2},\dots,m_{pn_p}],\\ \mathcal{T}'\mathbf{m} &:= \begin{cases} \left[[kn,(k-1)n],[(k-1)n,(k-1)n,n],\mathbf{m}_3\right] & (p=2k-1),\\ \left[[kn,kn],[kn,(k-1)n,n],\mathbf{m}_3\right] & (p=2k),\end{cases}\\ \alpha_{\mathcal{T}'\mathbf{m}} &= n'_0\alpha_0 + n'_{11}\alpha_{11} + n'_{21}\alpha_{21} + n'_{22}\alpha_{22} + \sum_{\nu\geq 1} n'_{3\nu}\alpha_{3\nu} \end{aligned}$$

for $\mathbf{m}$ given by (2.4). We have $\mathcal{T}\mathbf{m} \simeq \mathcal{T}'\mathbf{m}$ and

$$n'_0 = pn, \quad n'_{11} = [\tfrac{p}{2}]n, \quad n'_{21} = [\tfrac{p+1}{2}]n, \quad n'_{22} = n,$$
$$n'_{3\ell} = pn - \sum_{j=1}^{\ell} m_{j1} = pn - (m_{11} + \cdots + m_{\ell 1}) \qquad (1 \le \ell \le p).$$

When $p = 2k$, we have

$$\begin{aligned}(\beta_p|\alpha_{\mathcal{T}'\mathbf{m}}) &= (\beta_p|pn\alpha_0 + kn\alpha_{11} + kn\alpha_{21} + n\alpha_{22} + \sum_{\nu\ge 1} n'_{3\nu}\alpha_{3\nu}) \\ &= pn + 0 - kn + (2-k)n - (pn - (m_{11} + \cdots + m_{p1})) \\ &= m_{11} + \cdots + m_{p1} - (p-2)n \\ &= d_{1,\ldots,1}(\mathbf{m}).\end{aligned}$$

Since $\mathrm{mc}_{(1,\ldots,1)}$ maps $m_{j\nu}$ to $m_{j\nu} - d_{1,\ldots,1}(\mathbf{m})\delta_{\nu,1}$ with $D := d_{1,\ldots,1}(\mathbf{m})$, we have

$$\begin{aligned}\alpha_{\mathcal{T}'\,\mathrm{mc}_{(1,\ldots,1)}\,\mathbf{m}} &= (n'_0 - pD)\alpha_0 + (n'_{11} - kD)\alpha_{11} + (n'_{21} - D)\alpha_{21} \\ &\quad + \sum_{\nu=1}^{p-1}\big((p-\nu)D + n'_{3\nu}\big)\alpha_{3\nu} + \sum_{\nu\ge p} n'_{3\nu}\alpha_{3\nu} \\ &= \alpha_{\mathcal{T}'\mathbf{m}} - D\beta_p = s_{\beta_p}\alpha_{\mathcal{T}'\mathbf{m}}.\end{aligned}$$

When $p = 2k-1$, owing to the relation

$$\begin{aligned}(\beta_p|\alpha_{\mathcal{T}'\mathbf{m}}) &= (\beta_p|pn\alpha_0 + (k-1)n\alpha_{11} + kn\alpha_{21} + n\alpha_{22} + \sum_{\nu\ge 1} n'_{3\nu}\alpha_{3\nu}) \\ &= pn + (1-k)n + 0 + (2-k)n - (pn - (m_{11} + \cdots + m_{p1})) \\ &= d_{1,\ldots,1}(\mathbf{m}),\end{aligned}$$

we have the same result as in the case $p = 2k$. Thus, the following theorem is proved.

Theorem 3.2. *Under the notation above, we have*

$$\alpha_{\mathcal{T}'\,\mathrm{mc}_{(1,\ldots,1)}\,\mathbf{m}} = s_{\beta_p}\alpha_{\mathcal{T}'\,\mathbf{m}}.$$

We note that the reflection s_{β_p} with respect to the real root β_p is realized by a sequence of reflections with respect to simple roots.

4 Correspondences to complete representatives

4.1 Another main theorem

For a given $\mathbf{m} = m_{11}\cdots m_{1,n_1}, \ldots, m_{p,1}\cdots m_{p,n_p} \in \mathcal{F}$, we calculate $\mathcal{RS}\mathbf{m} \in \mathcal{F}$. Note that $\mathbf{n} \in \mathcal{F}^{E_8}$ satisfying $\mathbf{n} \sim \mathbf{m}$ is given by $\mathcal{RS}(\mathcal{RS}\mathbf{m})$. As mentioned in Remark 3.5 (2), $\mathcal{S}\mathbf{m}$ is determined by the set $\{m_{j,\nu}\}$. Hence, we arrange the elements $m_{j,\nu}$ in nonincreasing order and set

$$\begin{aligned}\{m_{j,\nu} \mid j = 1,\ldots,p,\ \nu = 1,\ldots,n_j\} &= \{a_1, \ldots a_{\tilde{n}}\}, \\ a_1 \ge a_2 \ge \cdots \ge a_{\tilde{n}},\ \tilde{n} &= n_1 + \cdots + n_p.\end{aligned} \tag{4.1}$$

We introduce the following notation in order to state the second main theorem of this paper.

Definition 4.1. Let $a_1, a_2, \dots, a_{\tilde{n}}$ be the sequence of integers (4.1) for a given $\mathbf{m} \in \mathcal{P}$. Set

$$a_{[k,\ell]} := \begin{cases} a_k + a_{k+1} + \cdots + a_\ell & (k \le \ell), \\ 0 & (k > \ell), \end{cases}$$
$$D_p := \sum_{\nu=1}^{p-2} (n - a_\nu) = (p-2)n - a_{[1,p-2]},$$
$$n := \operatorname{ord} \mathbf{m}.$$

Then we denote $N_{\mathbf{m}}$ the maximal positive integer satisfying

$$a_{p+N-2} + a_{p+N} > D_p. \tag{$D_p : N$}$$

If $(D_p : 1)$ does not hold, we set $N_{\mathbf{m}} = 0$.

Remark 4.2. If $(D_p : N)$ holds and $0 < N' < N$, then $(D_p : N')$ also holds.

Theorem 4.1. *We write a relation $m'' = \mathrm{mc}_{(\sigma_1,\sigma_2,\dots)} \mathbf{m}'$ with $c = d_{\sigma_1,\sigma_2,\dots}(\mathbf{m}')$ in the form*

$$\mathbf{m}' \xrightarrow[\sigma_1,\sigma_2,\dots]{c} \mathbf{m}'' \quad \text{or} \quad \mathbf{m}' \xrightarrow[\sigma_1,\sigma_2,\dots]{f>g} \mathbf{m}'' \quad \text{with } c = f - g,$$

which represents a Katz reduction if $c > 0$ or $f > g$. Then a sequence of Katz reductions of $\mathcal{S}\mathbf{m}$ is given by

$$\mathcal{S}\mathbf{m} \xrightarrow[1,1,1]{a_1} \bullet \xrightarrow[1,2,2]{a_2} \bullet \to \cdots \to \bullet \xrightarrow[1,p,p]{a_p} \mathbf{m}^{(0)} \xrightarrow[2,p,p+1]{a_{[1,p-1]}+a_{p+1}>(p-2)n} \mathbf{m}^{(1)}$$
$$\xrightarrow[2,p-1,p+2]{a_{[1,p-2]}+a_p+a_{p+2}>(p-2)n} \mathbf{m}^{(2)} \begin{cases} \xrightarrow[1,p-2,p+2]{a_{[1,p-3]}+a_{p-1}+a_{p+1}+a_{p+2}>(p-2)n} \mathbf{m}^{(3,1)}, \\ \xrightarrow[2,p-2,p+3]{a_{[1,p-3]}+a_{p-1}+a_p+a_{p+3}>(p-2)n} \mathbf{m}^{(3,2)}, \\ \xrightarrow[1,p-1,p+3]{a_{[1,p-2]}+a_{p+1}+a_{p+3}>(p-2)n} \mathbf{m}^{(3)} \to \end{cases}$$
$$\to \overset{k=1,2,\dots}{\cdots} \to \mathbf{m}^{(4k-1)}$$
$$\xrightarrow[1,p,p+4k]{a_{[1,p-2]}+a_{p+4k-2}+a_{p+4k}>(p-2)n} \mathbf{m}^{(4k)} \xrightarrow[2,p,p+4k+1]{a_{[1,p-2]}+a_{p+4k+1}>(p-2)n} \mathbf{m}^{(4k+1)}$$
$$\xrightarrow[2,p-1,p+4k+2]{a_{[1,p-2]}+a_{p+4k}+a_{p+4k+2}>(p-2)n} \mathbf{m}^{(4k+2)} \xrightarrow[1,p-1,p+4k+3]{a_{[1,p-2]}+a_{p+4k+1}+a_{p+4k+3}>(p-2)n} \mathbf{m}^{(4k+3)} \xrightarrow{k\mapsto k+1}$$

and

$$\mathcal{RS}\mathbf{m} = \begin{cases} \mathbf{m}^{(N_{\mathbf{m}})} & (N_{\mathbf{m}} \neq 2), \\ \mathbf{m}^{(2)} & (N_{\mathbf{m}} = 2 \text{ and } (4.2)), \\ \mathbf{m}^{(3,1)} & (N_{\mathbf{m}} = 2 \text{ and } (\mathrm{C}{:}3.1)), \\ \mathbf{m}^{(3,2)} & (N_{\mathbf{m}} = 2 \text{ and } (\mathrm{C}{:}3.2)). \end{cases}$$

Here,

$$a_{[1,p-3]} + a_{p-1} + \max\{a_{p+1} + a_{p+2}, a_p + a_{p+3}\} \le (p-2)n, \tag{4.2}$$
$$a_{[1,p-3]} + a_{p-1} + a_{p+1} + a_{p+2} > (p-2)n, \tag{C:3.1}$$
$$a_{[1,p-3]} + a_{p-1} + a_p + a_{p+3} > (p-2)n \tag{C:3.2}$$

and for $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
\mathbf{m}^{(4k)} &= \big[[(2k+1)D_p + n - a_{[p-1,p+4k]}, 2kD_p + n - a_{[p-1,p+4k-2]}], \\
&\quad [n-a_1, \dots, n-a_{p-2}, 2kD_p + n - a_{[p-1,p+4k-1]}, 2kD_p + n - a_{[p-1,p+4k-2]} - a_{p+4k}], \\
&\quad [0^p, D_p - a_{p-1}, \dots, D_p - a_{p+4k-2}, a_{p+4k+1}, a_{p+4k+2}, \dots, a_{\tilde{n}}]\big], \\
\mathbf{m}^{(4k+1)} &= \big[[(2k+1)D_p + n - a_{[p-1,p+4k]}, (2k+1)D_p + n - a_{[p-1,p+4k-1]} - a_{p+4k+1}], \\
&\quad [n-a_1, \dots, n-a_{p-2}, 2kD_p + n - a_{[p-1,p+4k-1]}, (2k+1)D_p + n - a_{[p-1,p+4k+1]}], \\
&\quad [0^p, D_p - a_{p-1}, \dots, D_p - a_{p+4k-1}, a_{p+4k+2}, a_{p+4k+3}, \dots, a_{\tilde{n}}]\big], \\
\mathbf{m}^{(4k+2)} &= \big[[(2k+1)D_p + n - a_{[p-1,p+4k]}, (2k+2)D_p + n - a_{[p-1,p+4k+2]}], \\
&\quad [n-a_1, \dots, n-a_{p-2}, (2k+1)D_p + n - a_{[p-1,\dots,p+4k]} - a_{p+4k+2}, \\
&\quad (2k+1)D_p + n - a_{[p-1,p+4k+1]}], \\
&\quad [0^p, D_p - a_{p-1}, \dots, D_p - a_{p+4k}, a_{p+4k+3}, a_{p+4k+4}, \dots]\big], \\
\mathbf{m}^{(4k+3)} &= \big[[(2k+2)D_p + n - a_{[p-1,p+4k+1]} - a_{p+4k+3}, (2k+2)D_p + n - a_{[p-1,p+4k+2]}], \\
&\quad [n-a_1, \dots, n-a_{p-2}, (2k+2)D_p + n - a_{[p-1,p+4k+3]}, \\
&\quad (2k+1)D_p + n - a_{[p-1,p+4k+1]}], \\
&\quad [0^p, D_p - a_{p-1}, \dots, D_p - a_{p+4k+1}, a_{p+4k+4}, a_{p+4k+5}, \dots, a_{\tilde{n}}]\big]. \\
\mathbf{m}^{(3,1)} &= \big[[2D_p + n + a_{p-2} - a_{p-1} - a_{[p-1,p+2]}, 2D_p + n - a_{[p-1,p+2]}], \\
&\quad [n-a_1, \dots, n-a_{p-3}, D_p + n - a_{p-1} - a_{p+1} - a_{p+2}, \\
&\quad D_p + n - a_{p-1} - a_p - a_{p+2}, D_p + n - a_{p-1} - a_p - a_{p+1}] \\
&\quad [0^p, D_p - a_{p-1}, 2D_p + a_{p-2} - a_{[p-1,p+2]}, a_{p+3}, a_{p+4}, \dots\dots, a_{\tilde{n}}]\big]. \\
\mathbf{m}^{(3,2)} &= \big[[D_p + n - a_{p-1} - a_p, 3D_p + n + a_{p-2} - 2a_{p-1} - 2a_p - a_{p+1} - a_{p+2} - a_{p+3}], \\
&\quad [n-a_1, \dots, n-a_{p-3}, D_p + n - a_{p-1} - a_p - a_{p+3}, \\
&\quad D_p + n - a_{p-1} - a_p - a_{p+2}, D_p + n - a_{p-1} - a_p - a_{p+1}], \\
&\quad [0^p, D_p - a_{p-1}, D_p - a_p, D_p + a_{p-2} - a_{p-1} - a_p, a_{p+4}, a_{p+5}, \dots, a_{\tilde{n}}]\big].
\end{aligned}$$

Moreover, if

$$a_{p+N-2} + a_{p+N} = D_p$$

for a positive integer N*, then the value of* $\mathcal{RS}\mathbf{m}$ *is unchanged whether* $N_{\mathbf{m}}$ *is taken to be* $N-1$ *or* N*.*

Remark 4.3. The last claim of the theorem follows from the fact that $\mathrm{mc}_\sigma(\mathbf{m}) = \mathbf{m}$ if $d_\sigma(\mathbf{m}) = 0$ (cf. Remark 2.15). It also follows from the fact that, for example, by replacing $a_1, \dots, a_{\tilde{n}}$ with $a_1, \dots, a_{N-1}, a_N - \epsilon, \dots, a_{\tilde{n}} - \epsilon, (\tilde{n} - N + 1)\epsilon$, where $0 \leq \epsilon \ll 1$. Note that Remark 4.6 justifies this argument.

Remark 4.4. Under the notation in Theorem 4.1, $\mathbf{m} = (2n-3)3, 2^n, 1^{2n} \in \mathcal{F}$ $(n \geq 3)$ satisfies $\mathbf{m}^{(3,1)} \in \mathcal{F}$ when $n = 3$ and $\mathbf{m}^{(n)} \in \mathcal{F}$ when $n > 3$. Moreover, $\mathbf{m} = 1^3, 1^3, 1^3$ satisfies $\mathbf{m}^{(0)} \in \mathcal{F}$ and $\mathbf{m} = 9^2, 64^3, 3^6$ satisfies $\mathbf{m}^{(3,2)} \in \mathcal{F}$.

The following table gives the numbers $|\{\mathbf{m} \in \mathcal{F}_{50} \mid \mathrm{Np}(\mathbf{m}) = 3,\ \mathbf{m}^{(*)} \in \mathcal{F}^{E_8}\}|$ for $(*) = (0), (1), \dots$.

$\mathbf{m} \in \mathcal{F}_{50},\ N_p(\mathbf{m}) = 3$

$\mathbf{m}^{(0)}$	$\mathbf{m}^{(1)}$	$\mathbf{m}^{(2)}$	$\mathbf{m}^{(3)}$	$\mathbf{m}^{(3,1)}$ $\mathcal{F}_{50}^{E_8}$	$\mathbf{m}^{(3,2)}$	$\mathbf{m}^{(4)}$	$\mathbf{m}^{(5)}$	$\mathbf{m}^{(6)}$	$\mathbf{m}^{(7)}$	$\mathbf{m}^{(8)}$
4772	8870	2617	276	2312 (351)	34	32	9	1	3	2

Example 4.5. Let $\mathbf{m} = (2n)21, 2^{n+1}1, 1^{2n+3}$ with $n \geq 1$. Then

$$\mathcal{S}\mathbf{m} = \big[[4n+6, 2n+3], [2n+3, 2n+3, 2n+3], [2n, \overbrace{2, \dots, 2}^{n+2}, \overbrace{1 \dots, 1}^{2n+5}]\big]$$

and $\mathcal{RS}\mathbf{m}$ is obtained by the following sequence of Katz reductions:

$$\begin{aligned}
\mathcal{S}\mathbf{m} &\xrightarrow[1,1,1]{2n} \big[[\underline{2n+6}, 2n+3], [3, \underline{2n+3}, 2n+3], [0, \overbrace{\underline{2}, \dots, 2}^{n+2}, \overbrace{1 \dots, 1}^{2n+5}]\big] \\
&\xrightarrow[1,2,2]{2} \big[[\underline{2n+4}, 2n+3], [3, 2n+1, \underline{2n+3}], [0, 0, \overbrace{\underline{2}, \dots, 2}^{n+1}, \overbrace{1 \dots, 1}^{2n+5}]\big] \\
&\xrightarrow[1,3,3]{2} \big[[2n+2, \underline{2n+3}], [3, 2n+1, \underline{2n+1}], [0, 0, 0, \overbrace{\underline{2}, \dots, 2}^{n}, \overbrace{1 \dots, 1}^{2n+5}]\big] \\
&\xrightarrow[2,3,4]{1} \big[[2n+2, \underline{2n+2}], [3, \underline{2n+1}, 2n], [0, 0, 0, 1, \overbrace{\underline{2}, \dots, 2}^{n-1}, \overbrace{1 \dots, 1}^{2n+5}]\big] \ \cdots\cdots \\
&(k = 0, 1, 2, \dots) \\
&\xrightarrow[2,2,5+4k]{1} \big[[\underline{2n+2-2k}, 2n+1-2k], [3, \underline{2n-2k}, 2n-2k], [0, 0, 0, \overbrace{1, \cdots, 1}^{4k+2}, \overbrace{\underline{2}, \dots, 2}^{n-2-4k}, \overbrace{1 \dots, 1}^{2n+5}]\big] \\
&\xrightarrow[1,2,6+4k]{1} \big[[\underline{2n+1-2k}, 2n+1-2k], [3, 2n-1-2k, \underline{2n-2k}], [0, 0, 0, \overbrace{1, \cdots, 1}^{4k+3}, \overbrace{\underline{2}, \dots, 2}^{n-3-4k}, \overbrace{1 \dots, 1}^{2n+5}]\big] \\
&\xrightarrow[1,3,7+4k]{1} \big[[2n-2k, \underline{2n+1-2k}], [3, 2n-1-2k, \underline{2n-1-2k}], [0, 0, 0, \overbrace{1, \cdots, 1}^{4k+4}, \overbrace{\underline{2}, \dots, 2}^{n-4-4k}, \overbrace{1 \dots, 1}^{2n+5}]\big] \\
&\xrightarrow[2,3,8+4k]{1} \big[[2n-2k, \underline{2n-2k}], [3, \underline{2n-1-2k}, 2n-2-2k], [0, 0, 0, \overbrace{1, \cdots, 1}^{4k+5}, \overbrace{\underline{2}, \dots, 2}^{n-5-4k}, \overbrace{1 \dots, 1}^{2n+5}]\big] \\
&\cdots\cdots
\end{aligned}$$

The above reductions terminate when the term 2 disappears from the third partition of the spectral type.

Note that this sequence of reductions is the one given in the theorem but it is not unique. For example, $\xrightarrow[1,3,7+4k]{1} \xrightarrow[2,3,8+4k]{1}$ can be replaced by $\xrightarrow[2,3,7+4k]{1} \xrightarrow[1,3,8+4k]{1}$ for some k. Moreover, Remark 4.6 yields many variations of this example.

Remark 4.6. Since Hd_σ and mc_σ define linear maps on the root space, the arguments and results such as Theorem 4.1 for a spectral type $\mathbf{m} \in \mathcal{P}$ with $\operatorname{idx}\mathbf{m} < 0$ also apply to $r\mathbf{m}$ for every positive integer r. Here, $\operatorname{idx} r\mathbf{m} = r^2 \operatorname{idx}\mathbf{m}$. Hence, we may extend the multiplicities $m_{j,\nu}$ to nonnegative rational numbers, since a rational spectral type can be realized by multiplying $\mathbf{m}$ by a suitable positive integer.

Set

$$\mathbf{m}_\epsilon = \big[[n-\epsilon, \epsilon],\ [m_{11}, \cdots, m_{1,n_1}], \dots, [m_{p1}, \cdots, m_{pn_p}]\big]$$

for a positive rational number ϵ with $0 \leq \epsilon \ll 1$. Then $\mathbf{m}_\epsilon \in \mathcal{F}$ and $N_{\mathbf{m}_\epsilon}$ does not depend on ϵ.

The "spectral type"

$$\begin{aligned}
\tilde{\mathbf{m}}_\epsilon = \big[&[m_{11}-\epsilon_{11}, \dots, m_{1n_1}-\epsilon_{1n_1}, \epsilon_{11}+\cdots+\epsilon_{1n_1}], \dots, [m_{p1}-\epsilon_{p1}, \dots, m_{pn_p}-\epsilon_{pn_p}, \epsilon_{p1}+\cdots+\epsilon_{pn_p}]\big] \\
&\text{with } \ 0 \ll \epsilon_{11} \ll \cdots \ll \epsilon_{1n_1} \ll \epsilon_{21} \ll \cdots \ll \epsilon_{pn_p} \ll 1
\end{aligned}$$

is useful to obtain a unique algorithm using maximal Katz reductions as is given in Theorem 4.1.

Furthermore, applying these deformations to spectral types such as $3^2, 2^3, 1^6$ and $(2n)21, 2^{n+1}1, 1^{2n+3}$, we obtain many further examples.

To prove the theorem, we redefine (4.1) as follows.

Definition 4.7. Fix a bijective map

$$\{1,\dots,\tilde{n}\} \ni k \mapsto I(k) \in \mathcal{I} := \{(j,\nu) \mid j = 1,\dots,p,\ \nu = 1,\dots,n_j\} \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$$

with $\tilde{n} := n_1 + \cdots + n_p$ so that

$$1 \le k < \ell \le \tilde{n} \ \Rightarrow\ m_{I(k)} \ge m_{I(\ell)}.$$

Moreover, we set

$$a_k = \begin{cases} m_{I(k)} & (1 \le k \le \tilde{n}), \\ 0 & (k > \tilde{n}), \end{cases}$$
$$I_{k_1,\dots,k_m} := \{j_{k_i} \mid I(k_i) = (j_{k_i}, \nu_{k_i}),\ 1 \le i \le m\} \subset \{1,\dots,p\},$$
$$I_{[k,\ell]} := \begin{cases} I_{k,k+1,\dots,\ell} & (k \le \ell), \\ \emptyset & (k > \ell). \end{cases}$$

The following lemma is useful for considering Katz reductions.

Lemma 4.8. *Let* $\mathbf{m} = m_{11}\cdots m_{1n_1}, \dots, m_{p1}\cdots m_{p,n_p} \in \mathcal{P}$ *and* $\sigma = (\sigma_1,\dots,\sigma_p) \in \mathbb{Z}_{>0}^p$. *Put*

$$\sigma' = (\sigma_1,\dots,\sigma_{k-1},\sigma_k',\sigma_{k+1},\dots,\sigma_p),$$

where $\sigma_k' \in \mathbb{Z}_{>0} \setminus \{\sigma_k\}$. *Then*

$$d_\sigma(\mathrm{mc}_\sigma\,\mathbf{m}) = -d_\sigma(\mathbf{m}), \tag{4.3}$$
$$d_{\sigma'}(\mathrm{mc}_\sigma\,\mathbf{m}) = m_{k,\sigma_k'} - m_{k,\sigma_k}. \tag{4.4}$$

In particular, if $m_{k,\sigma_k'} \le m_{k,\sigma_k}$, *we have* $d_{\sigma'}(\mathrm{mc}_\sigma\,\mathbf{m}) \le 0$.

Proof. Equation (4.3) follows from (2.6) and (2.7). Moreover, by Definition 2.3, we have

$$d_{\sigma'}(\mathrm{mc}_\sigma\,\mathbf{m}) - d_\sigma(\mathrm{mc}_\sigma\,\mathbf{m}) = m_{k,\sigma_k'} - \big(m_{k,\sigma_k} - d_\sigma(\mathbf{m})\big).$$

Hence (4.4) follows. □

The following lemma will be used in the proof of Theorem 4.1.

Lemma 4.9. (1) *Suppose that there exist indices* $i_1,\dots,i_p$ *such that* $i_1 < i_2 < \cdots < i_p$ *and*

$$a_{i_1} + \cdots + a_{i_p} > (p-2)n. \tag{4.5}$$

Then

$$|I_{i_1,\dots,i_p}| = p-1$$

and

$$a_{j_1} + \cdots + a_{j_p} < (p - \tfrac{3}{2})n. \tag{4.6}$$

(2) *Suppose* $i_{p-1} \ge p$ *in* (1). *Then*

$$|I_{[1,i_p]}| = p-1, \qquad |I_{[1,p-2]}| = p-2, \qquad a_{[1,i_p]} \le (p-1)n.$$

Hence, after relabeling, we may assume

$$a_j = m_{j,1} \quad (j = 1, \dots, p-2), \qquad I_{p-1,p} \ni p-1, \qquad I_{[1,i_p]} = \{1, \dots, p-1\}. \tag{4.7}$$

If there exists k such that

$$I_k \neq \{p-1\} \qquad \text{and} \qquad p-1 \leq k \leq i_{p-1},$$

then $I_j = \{p-1\}$ for all j satisfying $p-1 \leq j \leq i_p$ and $j \neq k$.

Moreover, if $m \geq 2$, we have

$$a_{[p-1,i_p]} \leq a_{p-2} + mD_p. \tag{4.8}$$

Let j_1 and j_2 be positive integers such that $p-1 \leq j_1 < j_2 \leq i_p$ and $j_1 \leq i_{p-1}$. Put $J = \{p-1, \dots, i_p\} \setminus \{j_1, j_2\}$. Then, if $m \geq 1$, we have

$$\sum_{\nu \in J} a_\nu < a_{p-2} + mD_p. \tag{4.9}$$

Remark 4.10. (1) The inequality (4.6) is best possible because of the example

$$\mathbf{m} = \big[[m,m], [m-1, \overbrace{1, \dots, 1}^{m+1}], [\overbrace{1, \dots, 1}^{2m}]\big] \qquad (m \gg 1).$$

(2) Under the notation in Theorem 4.1, Lemma 4.9 (2) is applied to the following cases.

Suppose $(D_p : N)$ holds with an integer $N \geq 2$. Then

$$a_{[1,p-2]} + a_{p+N-2} + a_{p+N} > (p-2)n.$$

Hence, it follows from (4.7) that if $\mathbf{m}$ is of $(2, n_2, *)$-type, then $N < n_2$.

Moreover, if $\mathbf{m}^{(N)}_{3,p+N} < \mathbf{m}^{(N)}_{3,p+N+1}$ for $\mathbf{m}^{(N)}$ in Theorem 4.1, then

$$a_{[1,p-2]} + a_{p+N-2} + a_{p+N+1} > (p-2)n.$$

(3) Set $\mathbf{m} = (16)66, 7777, 4^7 \in \mathcal{F}$. Then $p = 3$, $D_p = 12$, $a_{p-2} + D_p = 28$, $a_{p+2} + a_{p+4} = 13$, $N_{\mathbf{m}} = 4$ and $a_{p-1} + \cdots + a_{p+2} = a_{p-2} + D_p$. Hence, the condition $j_1 \leq i_{p-1}$ is necessary for (4.9).

Proof of Lemma 4.9. Suppose that (4.5) holds. Since $\mathbf{m}$ is fundamental, (4.5) implies $|I_{i_1,\dots,i_p}| < p$. Since $a_{i_1} + \cdots + a_{i_p} \leq |I_{i_1,\dots,i_p}|n$, we conclude $|I_{i_1,\dots,i_p}| = p-1$. Hence, there exist $1 \leq k < \ell \leq p$ satisfying $|I_{i_k,i_\ell}| = 1$. Then

$$a_{i_1} + \cdots + a_{i_p} = (a_{i_1} + \cdots + a_{i_p} - a_{i_\ell}) + a_{i_\ell} < (p-2)n + \tfrac{n}{2}.$$

Thus, (4.6) follows.

If

$$0 < k_1 < k_2 < \cdots < k_p \text{ and } k_\nu \leq i_\nu \quad (1 \leq \nu \leq p),$$

then $a_{k_1} + \cdots + a_{k_p} > (p-2)n$ and $|I_{k_1,\dots,k_p}| = p-1$.

If $|I_{[1,p-2]}| \leq p-3$, then $a_{p-2} \leq \frac{n}{2}$ and $a_{[1,p]} = a_{[1,p-2]} + a_{p-1} + a_p \leq (p-3)n + \frac{n}{2} + \frac{n}{2} \leq (p-2)n$. Hence, $|I_{[1,p-2]}| = p-2$ and we may assume $a_j = m_{j,1}$ for $j = 1, \dots, p-2$ and $a_{j_0} = m_{p-1,1}$ with a suitable $j_0 \in \{p-1, p\}$.

Suppose $j_{p-1} \geq p$. Since $|I_{1,\dots,p-2,p-1,p+1}| = |I_{1,\dots,p-2,p,p+1}| = p-1$, we have $I_{[1,p+1]} = \{1, \dots, p-1\}$. Hence, $I_{1,\dots,p-2,j_0,j} = \{1, \dots, p-1\}$ for $p+1 \leq j \leq i_p$ and therefore, $I_{[1,i_p]} = \{1, \dots, p-1\}$.

Moreover, if there exists k such that $I_k \neq \{p-1\}$ and $p-1 \leq k \leq i_{p-1}$, then $p-1 \in I_{1,\dots,p-2,k,j}$ and $I_j = \{p-1\}$ for j satisfying $p-1 \leq j \leq i_p$ and $j \neq k$.

When $m \geq 2$,

$$\begin{aligned} a_{[p-1,i_p]} - a_{p-2} + mD_p &= a_{[p-1,i_p]} - a_{p-2} + ma_{[1,p-2]} - m(p-2)n \\ &= a_{[1,i_p]} + \sum_{\nu=1}^{p-2}(m-2)(a_\nu - n) + a_{[1,p-3]} - 2(p-2)n \\ &\leq (p-1)n + (p-3)n - 2(p-2)n = 0. \end{aligned}$$

Let j_1 and j_2 be indices in Lemma 4.9 (2). Then $a_{j_1} + a_{j_2} \geq D_p$, namely,

$$a_{j_1} + a_{j_2} + a_{[1,p-2]} > (p-2)n.$$

Suppose $\sum_{j\in J} a_j \geq a_{p-2} + D_p$, namely,

$$\sum_{j\in J} a_j + a_{[1,p-3]} \geq (p-2)n.$$

Then

$$\begin{aligned} (2p-4)n &< \sum_{j\in J} a_j + a_{[1,p-3]} + a_{[1,p-2]} + a_{j_1} + a_{j_2} \\ &= a_{[1,i_p]} + a_{[1,p-3]} \leq (p-1)n + (p-3)n, \end{aligned}$$

which is a contradiction. Since $D_p \geq 0$, we have the lemma. □

The following lemma justifies the case classification of the reductions of $\mathbf{m}^{(2)}$ given in the theorem.

Lemma 4.11. *If one of three conditions* (C:3.1), (C:3.2) *and* $(D_p : 3)$ *holds, then the other two conditions do not hold, whereas* $(D_p : 2)$ *holds.*

Proof. Note first that each of the these conditions implies $(D_p : 2)$.

Suppose (C:3.2) or $(D_p : 3)$ holds. Then Lemma 4.9 implies $|I_{1,\dots,p+3}| = p - 1$.

If $(D_p : 3)$ holds, then

$$\begin{aligned} &(a_{[1,p-3]} + a_{p-2} + a_{p+1} + a_{p+3}) + (a_{[1,p-3]} + a_{p-1} + a_{p+1} + a_{p+2}) \\ &= a_{[1,p+3]} + a_{[1,p-3]} + (a_{p+1} - a_p) \\ &\leq (p-1)n + (p-3)n = 2(p-2)n, \end{aligned}$$

which shows that (C:3.1) does not hold.

Similarly, if (C:3.2) holds, then neither (C:3.1) nor $(D_p : 3)$ holds, since

$$\begin{aligned} &(a_{[1,p-3]} + a_{p-1} + a_p + a_{p+3}) + (a_{[1,p-3]} + a_{p-1} + a_{p+1} + a_{p+2}) \\ &= a_{[1,p+3]} + a_{[1,p-3]} + (a_{p-1} - a_{p-2}) \leq 2(p-2)n, \\ &(a_{[1,p-3]} + a_{p-1} + a_p + a_{p+3}) + (a_{[1,p-3]} + a_{p-2} + a_{p+1} + a_{p+3}) \\ &= a_{[1,p+3]} + a_{[1,p-3]} + (a_{p+3} - a_{p+2}) \leq 2(p-2)n. \end{aligned}$$

Thus we have the lemma. □

Proof of Theorem 4.1. Under the notation in the theorem,

$$\begin{aligned}
\mathcal{S}\mathbf{m} &= \mathcal{S}m_{1,1}\cdots m_{1,n_1},\, m_{2,1}\cdots m_{2,n_p},\, \ldots,\, m_{p,1}\cdots m_{p,n_p} \\
&= \big[[(p-1)n, n], [\overbrace{n,\ldots,n}^{p}], [a_1, a_2, \ldots, a_p, a_{p+1}, \ldots]\big] \\
&\xrightarrow[1,1,1]{a_1} \xrightarrow[1,2,2]{a_2} \cdots \xrightarrow[1,p,p]{a_p} \\
\mathbf{m}^{(0)} &= \big[[(p-1)n - a_1 - \cdots - a_p, n], \\
&\qquad [n-a_1, n-a_2, \ldots, \underline{n-a_p}], [\overbrace{0,\ldots,0}^{p}, \underline{a_{p+1}}, a_{p+2}, \ldots]\big].
\end{aligned}$$

Then we have $d_{1,p,p+1}(\mathbf{m}^{(0)}) \le 0$ by Lemma 4.8 with $a_p \ge a_{p+1}$. Since

$$d_{2,p,p+1}(\mathbf{m}^{(0)}) = a_1 + \cdots + a_{p-1} + a_{p+1} - (p-2)n = a_{p-1} + a_{p+1} - D_p,$$

the spectral type $\mathbf{m}^{(0)}$ is fundamental if $(D_p : 1)$ does not hold.

Hereafter we assume $N_{\mathbf{m}} \ge 1$. Then

$$\begin{aligned}
&\mathbf{m}^{(0)} \xrightarrow[2,p,p+1]{a_{p-1}+a_{p+1}-D_p} \\
&\mathbf{m}^{(1)} = \big[[D_p + n - a_{p-1} - a_p, \underline{D_p + n - a_{p-1} - a_{p+1}}], \\
&\qquad [n - a_1 \ldots, n - a_{p-1}, D_p + n - a_{p-1} - a_p - a_{p+1}], \\
&\qquad\quad [0^p, D_p - a_{p-1}, a_{p+2}, a_{p+3}, \cdots]\big].
\end{aligned}$$

Lemma 4.8 shows $d_{2,p,p+1}(\mathbf{m}^{(1)}) < 0$, $d_{2,p-1,p+1}(\mathbf{m}^{(1)}) \le 0$ and $d_{2,p,p+2}(\mathbf{m}^{(1)}) \le 0$. Since

$$d_{2,p-1,p+2}(\mathbf{m}^{(1)}) = a_p + a_{p+2} - D_p,$$

the tuple $\mathbf{m}^{(1)}$ is fundamental if $(D_p : 2)$ does not hold.

Hereafter we assume $N_{\mathbf{m}} \ge 2$. Then

$$\begin{aligned}
&\mathbf{m}^{(1)} \xrightarrow[2,p-1,p+2]{a_p+a_{p+2}-D_p} \\
&\mathbf{m}^{(2)} = \big[[D_p + n - a_{p-1} - a_p, 2D_p + n - a_{p-1} - a_p - a_{p+1} - a_{p+2}], \\
&\qquad [n - a_1 \ldots, n - a_{p-2}, D_p + n - a_{p-1} - a_p - a_{p+2}, D_p + n - a_{p-1} - a_p - a_{p+1}], \\
&\qquad\quad [0^p, D_p - a_{p-1}, D_p - a_p, a_{p+3}, a_{p+4}, a_{p+5}, \ldots]\big].
\end{aligned}$$

It follows from Lemma 4.8 that

$$d_{2,p-1,p+2}(\mathbf{m}^{(2)}) < 0,\ d_{1,p-1,p+2}(\mathbf{m}^{(2)}) \le 0,\ d_{2,p-2,p+2}(\mathbf{m}^{(2)}) \le 0 \ \text{ and } \ d_{2,p-1,p+3}(\mathbf{m}^{(2)}) \le 0.$$

Moreover,

$$\begin{aligned}
d_{1,p-2,p+2}(\mathbf{m}^{(2)}) &= a_1 + \cdots + a_{p-3} + a_{p-1} + a_{p+1} + a_{p+2} - (p-2)n \\
&= a_{p-1} + a_{p+1} + a_{p+2} - a_{p-2} - D_p \qquad (> 0 \to \mathbf{m}^{(3,1)}).
\end{aligned}$$

In the following, we assume $\mathbf{m}^{(2)}_{3,p+2} < \mathbf{m}^{(2)}_{3,p+3}$, namely, $a_p + a_{p+3} > D_p$. Then $|I_{1,\dots,p+3}| = p-1$.

$$\begin{aligned}
d_{1,p-2,p+3}(\mathbf{m}^{(2)}) &= 2a_1 + \cdots + 2a_{p-3} + a_{p-2} + a_{p-1} + \cdots + a_{p+3} - 2(p-2)n \\
&\le a_1 + \cdots + a_{p+3} + (p-3)n - (2p-4)n \\
&= a_1 + \cdots + a_{p+3} - (p-1)n \le 0, \\
d_{1,p-1,p+3}(\mathbf{m}^{(2)}) &= a_{p+1} + a_{p+3} - D_p \qquad (> 0 \to \mathbf{m}^{(3)}), \\
d_{2,p-2,p+3}(\mathbf{m}^{(2)}) &= a_1 + \cdots + a_{p-3} + a_{p-1} + a_p + a_{p+3} - (p-2)n \\
&= a_{p-1} + a_p + a_{p+3} - a_{p-2} - D_p \qquad (> 0 \to \mathbf{m}^{(3,2)}).
\end{aligned}$$

First, suppose that $(D_p : 3)$ holds.

We define $\mathbf{m}^{(\ell)}$ inductively for $\ell = 3, \dots, N_{\mathbf{m}}$ and show that $\mathbf{m}^{(N_{\mathbf{m}})}$ is fundamental if $N_{mathbfm} \ge 3$.

We assume that

$$\begin{aligned}
\mathbf{m}^{(4k+3)} = \big[&[\underline{(2k+2)D_p + n - a_{[p-1,p+4k+1]} - a_{p+4k+3}}, \\
&(2k+2)D_p + n - a_{[p-1,p+4k+2]}], \\
&[n - a_1, \dots, n - a_{p-2}, (2k+2)D_p + n - a_{[p-1,p+4k+3]}, \\
&\underline{(2k+1)D_p + n - a_{[p-1,p+4k+1]}}], \\
&[0^p, D_p - a_{p-1}, \dots, D_p - a_{p+4k+1}, \underline{a_{p+4k+4}}, a_{p+4k+5}, \dots]\big],
\end{aligned}$$

which follows from $\mathbf{m}^{(3)} = \mathrm{mc}_{(1,p-1,p+3)}\,\mathbf{m}^{(2)}$ when $k = 0$. Note that $\mathbf{m}^{(4k+3)}_{1,1} \ge \mathbf{m}^{(4k+3)}_{1,2}$.

Suppose first that $\mathbf{m}^{(4k+3)}_{3,p+4k+3} \ge \mathbf{m}^{(4k+3)}_{3,p+4k+4}$.

$$\begin{aligned}
d_{1,p-2,p+4k+3}(\mathbf{m}^{(4k+3)}) &= a_{[p-1,p+4k+2]} - a_{p+4k+1} - a_{p-2} - (2k+1)D_p < 0 && \Leftarrow (4.9), \\
d_{1,p-1,p+4k+3}(\mathbf{m}^{(4k+3)}) &= -d_{1,p-1,p+4k+3}(\mathbf{m}^{(4k+2)}) = D_p - a_{p+4k+1} - a_{p+4k+3} < 0 && \Leftarrow (4.3), \\
d_{1,p,p+4k+3}(\mathbf{m}^{(4k+3)}) &= a_{p+4k+2} - a_{p+4k+1} \le 0 && \Leftarrow (4.4).
\end{aligned}$$

Here we have used the fact $a_{[p-1,p+4k+2]} = a_{[p-1,p+4k+3]} - a_{p+4k+3}$ and the first inequality follows from (4.9) in Lemma 4.9 with $(j_1, j_2) = (p+4k+1, p+4k+3)$. Hence $\mathbf{m}^{(4k+3)}$ is fundamental in this case.

Now suppose that $\mathbf{m}^{(4k+3)}_{3,p+4k+3} < \mathbf{m}^{(4k+3)}_{3,p+4k+4}$, that is,

$$a_{p+4k+1} + a_{p+4k+4} > D_p.$$

Then

$$\begin{aligned}
d_{1,p-2,p+4k+4}(\mathbf{m}^{(4k+3)}) &= a_{[p-1,p+4k+4]} - a_{p+4k+3} - a_{p-2} - (2k+2)D_p < 0 && \Leftarrow (4.8), \\
d_{1,p-1,p+4k+4}(\mathbf{m}^{(4k+3)}) &= a_{p+4k+4} - a_{p+4k+3} \le 0 && \Leftarrow (4.4), \\
d_{1,p,p+4k+4}(\mathbf{m}^{(4k+3)}) &= a_{p+4k+2} + a_{p+4k+4} - D_p \qquad (> 0 \to \mathbf{m}^{(4k+4)}).
\end{aligned}$$

Thus, $\mathbf{m}^{(4k+3)}$ is fundamental if $(D_p : 4k+4)$ does not hold.

Suppose now that $(D_p : 4k+4)$ holds. Then $\mathbf{m}^{(4k+4)} = \mathrm{mc}_{1,p,p+4k+4}\mathbf{m}^{(4k+3)}$.

Replacing k by $k+1$, we obtain $\mathbf{m}^{(4k)} = \mathrm{mc}_{(1,p,p+4k)}\,\mathbf{m}^{(4k-1)}$ for $k \ge 1$, under the assumption that $(D_p : 4k)$ holds, where

$$\begin{aligned}
\mathbf{m}^{(4k)} = \big[&[(2k+1)D_p + n - a_{[p-1,p+4k]}, \underline{2kD_p + n - a_{[p-1,p+4k-2]}}], \\
&[n - a_1, \dots, n - a_{p-2}, 2kD_p + n - a_{[p-1,p+4k-1]}, \underline{2kD_p + n - a_{[p-1,p+4k-2]} - a_{p+4k}}], \\
&[0^p, D_p - a_{p-1}, \dots, D_p - a_{p+4k-2}, \underline{a_{p+4k+1}}, a_{p+4k+2}, \dots]\big]
\end{aligned}$$

By the assumption $a_{p+4k} + a_{p+4k-2} > D_p$, we have the relations:

$$\begin{aligned}
d_{1,p-2,p+4k}(\mathbf{m}^{(4k)}) &= a_{[p-1,p+4k-3]} - a_{p-2} - (2k-1)D_p \le 0 && \Leftarrow (4.9),\\
d_{1,p,p+4k}(\mathbf{m}^{(4k)}) &= -d_{1,p,p+4k}(\mathbf{m}^{(4k-1)}) = D_p - a_{p+4k} - a_{p+4k-2} < 0 && \Leftarrow (4.3),\\
d_{2,p-2,p+4k}(\mathbf{m}^{(4k)}) &= a_{[p-1,p+4k]} - a_{p+4k-2} - a_{p-2} - 2kD < 0 && \Leftarrow (4.8),\\
d_{2,p,p+4k}(\mathbf{m}^{(4k)}) &= a_{p+4k-1} - a_{p+4k-2} \le 0 && \Leftarrow (4.4),\\
d_{1,p-2,p+4k+1}(\mathbf{m}^{(4k)}) &= a_{[p-1,p+4k+1]} - a_{p+4k} - a_{p+4k-1} - a_{p-2} - 2kD_p < 0 && \Leftarrow (4.8),\\
d_{1,p,p+4k+1}(\mathbf{m}^{(4k)}) &= a_{p+4k+1} - a_{p+4k} \le 0 && \Leftarrow (4.4),\\
d_{2,p-2,p+4k+1}(\mathbf{m}^{(4k)}) &= a_{[p-1,p+4k+1]} - a_{p-2} - (2k+2)D_p < 0 && \Leftarrow (4.8),\\
d_{2,p,p+4k+1}(\mathbf{m}^{(4k)}) &= a_{p+4k+1} + a_{p+4k-1} - D_p \qquad (> 0 \to \mathbf{m}^{(4k+1)}).
\end{aligned}$$

For $k \ge 1$, we have $\mathbf{m}^{(4k+1)} = \mathrm{mc}_{(2,p,p+4k+1)}\,\mathbf{m}^{(4k)}$, where

$$\begin{aligned}
\mathbf{m}^{(4k+1)} =\big[&[(2k+1)D_p + n - a_{[p-1,p+4k]}, \underline{(2k+1)D_p + n - a_{[p-1,p+4k-1]} - a_{p+4k+1}}],\\
&[n-a_1, \dots, n-a_{p-2}, \underline{2kD_p + n - a_{[p-1,p+4k-1]}}, (2k+1)D_p + n - a_{[p-1,p+4k+1]}],\\
&[0^p, D_p - a_{p-1}, \dots, D_p - a_{p+4k-1}, \underline{a_{p+4k+2}}, a_{p+4k+3}, \dots]\big]
\end{aligned}$$

$$\begin{aligned}
d_{2,p-2,p+4k+1}(\mathbf{m}^{(4k+1)}) &= a_{[p-1,p+4k]} - a_{p+4k-1} - a_{p-2} - 2kD_p < 0 && \Leftarrow (4.8),\\
d_{2,p-1,p+4k+1}(\mathbf{m}^{(4k+1)}) &= a_{p+4k} - a_{p+4k-1} \le 0 && \Leftarrow (4.4),\\
d_{2,p,p+4k+1}(\mathbf{m}^{(4k+1)}) &= -d_{2,p,p+4k+1}(\mathbf{m}^{(4k)}) = D_p - a_{p+4k+1} - a_{p+4k-1} < 0 && \Leftarrow (4.3),\\
d_{2,p-2,p+4k+2}(\mathbf{m}^{(4k+1)}) &= a_{[p-1,p+4k+2]} - a_{p+4k+1} - a_{p-2} - (2k+1)D_p < 0 && \Leftarrow (4.8),\\
d_{2,p-1,p+4k+2}(\mathbf{m}^{(4k+1)}) &= a_{p+4k+2} + a_{p+4k} - D_p \qquad (> 0 \to \mathbf{m}^{(4k+2)}),\\
d_{2,p,p+4k+2}(\mathbf{m}^{(4k+1)}) &= a_{p+4k+2} - a_{p+4k+1} \le 0 && \Leftarrow (4.4).
\end{aligned}$$

For $k \ge 1$, we have $\mathbf{m}^{(4k+2)} = \mathrm{mc}_{(2,p-1,p+4k+2)}\,\mathbf{m}^{(4k+1)}$, where

$$\begin{aligned}
\mathbf{m}^{(4k+2)} =\big[&[\underline{(2k+1)D_p + n - a_{[p-1,p+4k]}}, (2k+2)D_p + n - a_{[p-1,p+4k+2]}],\\
&[n-a_1, \dots, n-a_{p-2}, \underline{(2k+1)D_p + n - a_{[p-1,\dots,p+4k]} - a_{p+4k+2}}\\
&\quad (2k+1)D_p + n - a_{[p-1,p+4k+1]}],\\
&[0^p, D_p - a_{p-1}, \dots, D_p - a_{p+4k}, \underline{a_{p+4k+3}}, a_{p+4k+4}, \dots]\big].
\end{aligned}$$

$$\begin{aligned}
d_{1,p-2,p+4k+2}(\mathbf{m}^{(4k+2)}) &= a_{[p-1,p+4k+2]} - a_{p+4k} - a_{p-2} - (2k+1)D_p < 0 && \Leftarrow (4.8),\\
d_{1,p-1,p+4k+2}(\mathbf{m}^{(4k+2)}) &= a_{p+4k+1} - a_{p+4k} \le 0 && \Leftarrow (4.4),\\
d_{2,p-2,p+4k+2}(\mathbf{m}^{(4k+2)}) &= a_{[p-1,p+4k-1]} - a_{p-2} - 2kD_p < 0 && \Leftarrow (4.8),\\
d_{2,p-1,p+4k+2}(\mathbf{m}^{(4k+2)}) &= -d_{2,p-1,p+4k+2}(\mathbf{m}^{(4k+1)}) = D_p - a_{p+4k+2} - a_{p+4k} < 0 && \Leftarrow (4.3),\\
d_{1,p-2,p+4k+3}(\mathbf{m}^{(4k+2)}) &= a_{[p-1,p+4k+3]} - a_{p-2} - (2k+2)D_p < 0 && \Leftarrow (4.8),\\
d_{1,p-1,p+4k+3}(\mathbf{m}^{(4k+2)}) &= a_{p+4k+3} + a_{p+4k+2} - D_p \quad (> 0 \to \mathbf{m}^{(4k+3)}),\\
d_{2,p-1,p+4k+3}(\mathbf{m}^{(4k+2)}) &= a_{[p-1,p+4k+3]} - a_{p+4k+2} - a_{p+4k+1} - a_{p-2} - (2k+1)D_p < 0 && \Leftarrow (4.8),\\
d_{2,p-2,p+4k+3}(\mathbf{m}^{(4k+2)}) &= a_{p+4k+3} - a_{p+4k+2} \le 0 && \Leftarrow (4.4).
\end{aligned}$$

If $a_{p+4k+3}+a_{p+4k+2}-D_p\le 0$, $\mathbf{m}^{(4k+2)}$ is fundamental.
If $a_{p+4k+3}+a_{p+4k+2}-D_p>0$, we have $\mathbf{m}^{(4k+3)}=\mathrm{mc}_{(1,p-1,p+4k+3)}\,\mathbf{m}^{(4k+2)}$.
Thus, by induction, $\mathbf{m}^{(N_{\mathbf{m}})}$ is fundamental except in the cases (C:3.1) and (C:3.2).

Suppose that (C:3.1) holds. Then $\mathbf{m}^{(3,1)}=\mathrm{mc}_{(1,p-2,p+2)}\,\mathbf{m}^{(2)}$, where

$$\begin{aligned}\mathbf{m}^{(3,1)}=\big[&[\underline{2D_p+n+a_{p-2}-2a_{p-1}-a_p-a_{p+1}-a_{p+2}},\\ &\quad 2D_p+n-a_{p-1}-a_p-a_{p+1}-a_{p+2}],\\ &[n-a_1,\dots,n-a_{p-3},\underline{D_p+n-a_{p-1}-a_{p+1}-a_{p+2}},\\ &\quad D_p+n-a_{p-1}-a_p-a_{p+2},D_p+n-a_{p-1}-a_p-a_{p+1}]\\ &[0^p,D_p-a_{p-1},2D_p+a_{p-2}-a_{p-1}-a_p-a_{p+1}-a_{p+2},a_{p+3},a_{p+4},\dots]\big].\end{aligned}$$

Since $a_{[1,p-3]}+a_{p-1}+a_{p+1}+a_{p+2}>(p-2)n$, Lemma 4.9 shows that we may assume (4.7) with $i_p=p+2$. Since $I_{[1,p+2]}=I_{1,\dots,p-3,p-1,p,p+1}=\{1,\dots,p-1\}$, we have $a_k=m_{p-2,2}$ with a suitable $k\in\{p-1,p,p+1\}$. Moreover, Lemma 4.9 shows $I_j=\{p-1\}$ for $j\in\{p-1,p,p+1,p+2\}\setminus\{k\}$ and $I_{1,\dots,p-3,p-1,p+1,p+2}=I_{1,\dots,p-3,p-1,p,p+2}=\{1,\dots,p-1\}$. Then we must have $k=p-1$. Hence

$$a_j=m_{j,1}\quad(j=1,\dots,p-2),\qquad a_{p-1}=m_{p-2,2},\qquad a_{p+\nu}=m_{p-1,\nu+1}\quad(\nu=0,1,2).\tag{4.10}$$

In particular, $|I_{1,\dots,p-4,p-2,\dots,p+3}|=p-2$ when $p>3$.

Here, $E_8^{(1)}=33,222,111111$ and $(6-\epsilon)\epsilon,33,222,111111$ are examples (cf. Remark 4.6).

Note that

$$a_p+a_{p+3}<D_p.\tag{4.11}$$

Indeed, if $a_p+a_{p+3}\ge D_p$, then $|I_{[1,p+3]}|=p-1$, and hence

$$\begin{aligned}0&<(a_p+a_{p+3}-D_p)+(a_{p-1}+a_{p+1}+a_{p+2}-a_{p-2}-D_p)\\ &=a_{[1,p+3]}+a_{[1,p-3]}-2(p-2)n\\ &\le(p-1)n+(p-3)n-2(p-2)n=0,\end{aligned}$$

a contradiction.

We have the following relations:

$$\begin{aligned}&d_{1,p-2,p+1}(\mathbf{m}^{(3,1)})=a_p-a_{p-1}\le 0 && \Leftarrow (4.4),\\ &d_{1,p-2,p+2}(\mathbf{m}^{(3,1)})=D_p+a_{p-2}-a_{p-1}-a_p-a_{p+3}<0 && \Leftarrow (4.3),\\ &d_{1,p-2,p+3}(\mathbf{m}^{(3,1)})=a_p+a_{p+3}-D_p<0 && \Leftarrow (4.11)\end{aligned}$$

Moreover, if $p>3$, we have

$$\begin{aligned}&d_{1,p-3,p+1}(\mathbf{m}^{(3,1)})=a_{[1,p+2]}-a_{p-3}-a_{p-1}-(p-2)n<0 && \Leftarrow\ |I_{1,\dots,p-4,p-2,p,p+1,p+2}|=p-2,\\ &d_{1,p-3,p+2}(\mathbf{m}^{(3,1)})=a_{p-2}-a_{p-3}\le 0 && \Leftarrow (4.4)\\ &d_{1,p-3,p+3}(\mathbf{m}^{(3,1)})=a_{[p-1,p+3]}-a_{p-3}-2D_p\\ &\qquad=(a_{[p-1,p+2]}-a_p-a_{p-3}-D_p)+(a_p+a_{p+3}-D_p)\\ &\qquad<a_{[1,p+2]}-a_{p-3}-a_p-(p-2)n\le 0 && \Leftarrow\ |I_{1,\dots,p-4,p-2,p-1,p+1,p+2}|=p-2.\end{aligned}$$

Hence, $\mathbf{m}^{(3.1)}\in\mathcal{F}$.

Suppose that (C:3.2) holds. Then, $\mathbf{m}^{(3,2)} = \mathrm{mc}_{(2,p-2,p+3)}\,\mathbf{m}^{(2)}$, where

$$\begin{aligned}\mathbf{m}^{(3,2)} = \big[&[D_p + n - a_{p-1} - a_p, 3D_p + n + a_{p-2} - 2a_{p-1} - 2a_p - a_{p+1} - a_{p+2} - a_{p+3}],\\ &[n - a_1, \dots, n - a_{p-3}, \underline{D_p + n - a_{p-1} - a_p - a_{p+3}},\\ &\quad D_p + n - a_{p-1} - a_p - a_{p+2}, D_p + n - a_{p-1} - a_p - a_{p+1}],\\ &[0^p, D_p - a_{p-1}, D_p - a_p, D_p + a_{p-2} - a_{p-1} - a_p, a_{p+4}, a_{p+5}, \dots].\big]\end{aligned}$$

We may assume (4.7) with $i_p = p + 3$. There are two cases.

If $I_{[p-1,p]} = \{p-1\}$, the condition $|I_{1,\dots,p-3,p-1,p,j}| = p - 1$ for $j \in \{p+1, p+2, p+3\}$ implies that $I_{p+1,p+2,p+3} = \{p-2\}$ and hence

$$a_j = m_{j,1} \quad (j = 1, \dots, p-2), \quad a_{p-1} = m_{p-1,1}, \quad a_p = m_{p-1,2}, \quad a_{p+\nu} = m_{p-2,\nu+1} \quad (\nu = 1, 2, 3).$$

If there exists $k \in \{p-1, p\}$ such that $I_k \neq \{p-1\}$, then $I_{p+1,p+2,p+3} = \{p-1\}$ and the condition $|I_{1,\dots,p-3,k,p+1,p+2}| = p - 1$ implies $I_k = \{p-2\}$. Therefore,

$$a_j = m_{j,1} \quad (j = 1, \dots, p-2), \quad \{a_{p-1}, a_p\} = \{m_{p-2,2}, m_{p-1,1}\}, \quad a_{p+\nu} = m_{p-1,\nu+1} \quad (\nu = 1, 2, 3).$$

Here, the spectral types $9333, 882, 1^{18}$ and $99, 6444, 333333$ are corresponding examples.

We have the following relations:

$$\begin{aligned}
d_{1,p-2,p+3}(\mathbf{m}^{(3,2)}) &= a_{p+1} + a_{p+2} - D_p && \Leftarrow (4.4)\\
&< (a_{p+1} + a_{p+2} - D_p) + (a_{p-1} + a_p + a_{p+3} - a_{p-2} - D_p) && \Leftarrow (\mathrm{C{:}3.2})\\
&= a_{[1,p+3]} + a_{[1,p-3]} - 2(p-2)n\\
&\le (p-1)n + (p-3)n - 2(p-2)n \le 0,\\
d_{2,p-2,p+3}(\mathbf{m}^{(3,2)}) &= D_p + a_{p-2} - a_{p-1} - a_p - a_{p+3} < 0 && \Leftarrow (4.3),\\
d_{1,p-2,p+4}(\mathbf{m}^{(3,2)}) &= a_{[p-1,p+4]} - a_{p+3} - a_{p-2} - 2D_p\\
&\le a_{[p-1,p+3]} - a_{p-2} - 2D_p \le 0 && \Leftarrow (4.8),\\
d_{2,p-2,p+4}(\mathbf{m}^{(3,2)}) &= a_{p+4} - a_{p+3} \le 0 && \Leftarrow (4.4).
\end{aligned}$$

Moreover, if $p > 3$, we have

$$\begin{aligned}
d_{1,p-3,p+3}(\mathbf{m}^{(3,2)}) &= a_{[p-1,p+3]} - a_{p-3} - 2D_p \le 0 && \Leftarrow (4.8).\\
d_{2,p-3,p+3}(\mathbf{m}^{(3,2)}) &= a_{p-2} - a_{p-3} \le 0 && \Leftarrow (4.4),\\
d_{1,p-3,p+4}(\mathbf{m}^{(3,2)}) &= a_{[p-1,p+4]} + a_{p-1} + a_p - a_{p-3} - a_{p-2} - 3D_p\\
&\le a_{p+4} + a_{p-1} + a_p - a_{p-3} - D_p && \Leftarrow (4.8)\\
&\le a_{[1,p-4]} + a_{[p-2,p]} - (p-2)n \le 0 && \Leftarrow |I_{1,\dots,p-4,p-2,p-1,p}| = p-2,\\
d_{2,p-3,p+4}(\mathbf{m}^{(3,2)}) &= a_{p-1} + a_p + a_{p+4} - a_{p-3} - D_p\\
&\le a_{[1,p+1]} - a_{p-3} - (p-2)n \le 0 && \Leftarrow |I_{1,\dots,p-4,p-2,\dots,p+1}| = p-2.
\end{aligned}$$

Thus, $\mathbf{m}^{(3,2)} \in \mathcal{F}$, completing the proof of Theorem 4.1. □

We show the following proposition, which corresponds to Lemma 3.6 (4) and completes the proof of Theorem 3.1.

Proposition 4.12. *We have* $\operatorname{ord} \mathcal{RS}\mathbf{m} > \operatorname{ord} \mathbf{m}$ *for* $\mathbf{m} \in \mathcal{F} \setminus \mathcal{F}^{E_8}$.

Proof. Since $\operatorname{ord}\mathbf{m}^{(0)} = pn - (a_1 + \cdots + a_n)$, the inequality (4.6) gives $\operatorname{ord}\mathbf{m}^{(0)} > \frac{3}{2}n$.

Suppose that $N_{\mathbf{m}} \geq 1$. Then $\operatorname{ord}\mathbf{m}^{(0)} - \operatorname{ord}\mathbf{m}^{(1)} = a_1 + \cdots + a_{p-1} + a_{p+1} - (p-2)n < \frac{n}{2}$ and hence $\operatorname{ord}\mathbf{m}^{(1)} > n$.

Thus, we may assume that $N := N_{\mathbf{m}} \geq 2$. Then Lemma 4.9 gives $|I_{[1,p+N]}| = p-1$, and we may further assume $a_j = m_{j,1}$ for $j = 1, \ldots, p-2$ and

$$\{a_{p+N+1}, a_{p+N+2}, \cdots, a_{\tilde{n}}\} \supset \{m_{p1}, m_{p2}, \ldots, m_{pn_p}\}. \tag{4.12}$$

Note that $m_{p1} + \cdots + m_{pn_p} = n$ and that the third partition of $\mathbf{m}^{(N)}$ equals

$$[0^p, D_p - a_{p-1}, \ldots, D_p - a_{p+N-2}, a_{p+N+1}, a_{p+N+2}, \cdots, a_{\tilde{n}}].$$

Therefore, if

$$D_p - a_{p-1} = (n - m_{11}) + \cdots + (n - m_{p-2,1}) - a_{p-1}$$

is positive, then $\operatorname{ord}\mathbf{m}^{(N)} > n$

If $|I_{[1,p-1]}| = p-1$, then $m_{11} + \cdots + m_{p-2,1} + a_{p-1} < m_{11} + \cdots + m_{p1} \leq (p-2)n$, which implies $D_p - a_{p-1} > 0$ and hence $\operatorname{ord}\mathbf{m}^{(N)} > n$. Thus, we assume that $|I_{1,\ldots,p-1}| = p-2$. Then, there exists $j \leq p-2$ such that $a_{p-1} = m_{j2}$. Hence, if $D_p = a_{p-1}$, then $j = 1$ and

$$p = 3,\ (a_1, a_2) = (m_{11}, m_{12}) \ \text{ and } \ m_{11} + m_{12} = n. \tag{4.13}$$

If $D_p = a_{p-1} = a_p$, then $m_{11} + a_3 = n$, which is impossible since $m_{11} + m_{21} + m_{31} \leq n$. Thus $\operatorname{ord}\mathbf{m}^{(N)} > n$.

Suppose that $\mathcal{RS}\mathbf{m} = \mathbf{m}^{(3,2)}$. Since $a_{[1,p-3]} + a_{p-1} + a_{p+1} + a_{p+3} > (p-2)n$, we have (4.12) with $N = 3$, and and the same argument gives $\operatorname{ord}\mathbf{m}^{(3,2)} > n$.

Finally, suppose that $\mathcal{RS}\mathbf{m} = \mathbf{m}^{(3,1)}$ and $\operatorname{ord}\mathbf{m}^{(3,1)} \leq n$. Then $N = 2$ and the two sides of (4.12) are equal, and $\operatorname{ord}\mathbf{m}^{(3,1)} = n$. Moreover, $D_p = a_p$, and we have (4.13). Furthermore $p + N = n_1 + n_2$, which means $n_2 = 3$. Hence, $\mathbf{m}$ is of $(2, 3, *)$-type. □

Remark 4.13. To obtain the results in this paper, we developed several programs for the computer algebra system `Risa/Asir`. They compute examples and display the calculations in a suitable format. Some of these programs are available in `os_muldif.rr` [13]. For example, it contains the following functions.

`chkspt()` tests irreducibly realizability and computes $\operatorname{idx}\mathbf{m}$, $\mathcal{R}\mathbf{m}$, etc. ($\mathbf{m}$: spectral type)

`spgen()` computes $\mathcal{F}_\ell$ for $\ell \leq 0$ and, more generally,
computes all $\mathbf{m} \in \mathcal{P}$ such that $\mathbf{m} \underset{\text{mc}}{\sim} \mathbf{n}$ and $\operatorname{ord}\mathbf{m} \leq n$, ($\mathbf{n} \in \mathcal{P}$, $n \geq 0$)

`refinements()` computes all refinements of $\mathbf{m}$ ($\mathbf{m} \in \mathcal{P}$).

`spfundclass()` computes $\mathcal{F}_\ell/! \sim$ ($\ell \leq 0$).

`spfundauto()` computes the automorphism group of the Fuchsian systems with a given spectral type $\mathbf{m}$.

`spmcv()` calculates $\operatorname{mc}_\sigma(\mathbf{m})$ ($\mathbf{m} \in \mathcal{P}$, $\sigma \in \mathbb{Z}^p_{\geq 0}$).

`spharnad()` calculates $\operatorname{Hd}_\sigma \mathbf{m}$ ($\mathbf{m}$ is a spectral type with refinement structure, $\sigma \in \mathbb{Z}^p_{\geq 0}$).

`spfundroot()` calculates $\mathcal{U}\mathbf{m}$ ($\mathbf{m} \in \mathcal{P}$).

`spmcr()` calculates $\mathcal{R}\mathbf{m}$ with showing steps of the Katz reductions $\mathbf{m} \to \mathcal{R}\mathbf{m}$ ($\mathbf{m} \in \mathcal{P}$),
where the multiplicities $m_{j,\nu}$ may contain infinitesimal positive numbers (cf. Remark 4.6).

`spDEfromE8()` computes all $\mathbf{m} \in \mathcal{F}^T$ such that $\mathbf{m} \sim \mathbf{n}$ for $T = D_4, E_6, E_7$ ($\mathbf{n} \in \mathcal{F}^{E_8}$).

`linprog()` performs linear programming that may involve infinitesimal positive numbers.

`spharnadcv()` computes all $\mathcal{R}\,\mathrm{Hd}_\sigma\,\mathbf{n}$ such that $\mathbf{n} \underset{\mathrm{mc}}{\sim} \mathbf{m}$ and $\operatorname{ord}\mathbf{n} \le \operatorname{ord}\mathbf{m} + \ell$ $(\mathbf{m} \in \mathcal{F}$ and $\ell \in \mathbb{Z}_{\ge 0})$.

`sp2tab()` generates a TEX table for a given list, such as $\mathcal{F}_\ell/\sim$.

`spfunddiag()` generates a TEX figure using `TikZ` that displays the relations $\sim$ for a given set of spectral types.

Here the arguments of the functions are indicated in "( )", for example, $(\mathbf{m} \in \mathcal{P})$.

The examples in §2.7 were obtained by using `spfundclass()`, `sp2harnadcv()`, `sp2tab()` and `spfunddiag()`. The lists $\mathcal{F}_\ell/\sim$ with $\ell \ge -100$ obtained by `spfundclass()` are available (cf. [18]).

Remark 4.14. We assume that the differential equations have no ramified irregular singularities. It is expected that the results of this paper can be extended to the ramified case by studying of such singularities, including their confluences, unfoldings and degenerations of HTL canonical forms (cf. [11]). An extension to linear difference equations is also interesting.

4.2 D_4-fundamental spectral types

In this subsection, we examine $\mathcal{F}^{D_4}_{[\mathbf{n}]}$ for $\mathbf{n} \in \mathcal{F}^{E_8}$. Here, we define as follows.

Definition 4.15. For $\mathbf{n} \in \mathcal{P}$ and $T = E_6$, E_7, E_8 or D_4, we set

$$\mathcal{F}^T_{[\mathbf{n}]} := \mathcal{F}_{[\mathbf{n}]} \cap \mathcal{F}^T.$$

For an ordered D_4-fundamental spectral type

$$\mathbf{m} = m_{11}m_{12},\, m_{21}m_{22},\, m_{31}m_{32},\, m_{41}m_{42}\cdots m_{4,n_4}, \tag{4.14}$$

we calculate $\mathcal{U}\mathbf{m}$ by Theorem 4.1. We may assume $m_{11} \ge m_{21} \ge m_{31}$. Then

$$\begin{aligned}
&m_{11} \ge m_{12} > 0,\ m_{21} \ge m_{22} > 0,\ m_{31} \ge m_{32},\\
&m_{41} \ge m_{42} \ge \cdots \ge m_{4,n_4} > 0,\ m_{11} \ge m_{21} \ge m_{31},\\
&n := \operatorname{ord}\mathbf{m} = m_{11} + m_{12} = m_{21} + m_{22} = m_{31} + m_{32} = m_{41} + \cdots + m_{4,n_4},\\
&d_{1,1,1,1}(\mathbf{m}) = m_{11} + m_{21} + m_{31} + m_{41} - 2n = m_{31} + m_{41} - m_{12} - m_{22} \le 0.
\end{aligned}$$

Since $m_{41} \le m_{12} + m_{22} - m_{31} \le m_{12}$, we have

$$m_{11} \ge m_{21} \ge m_{31} \ge m_{32} \ge m_{22} \ge m_{12} \ge m_{41} \ge \cdots \ge m_{4n_4} > 0 \tag{4.15}$$

and $\mathbf{m}$ is expressed in normal form. Hence, under the notation (4.1),

$$\begin{aligned}
&a_1 = m_{11},\ a_2 = m_{21},\ a_3 = m_{31},\ a_4 = m_{32},\ a_5 = m_{22},\ a_6 = m_{12},\ a_{6+k} = m_{4,k} \quad (1 \le k \le n_4),\\
&D_4 = 2n - a_1 - a_2 = m_{12} + m_{22} = a_5 + a_6,\ a_4 + a_6 \ge D_4 \ge a_5 + a_5
\end{aligned}$$

and therefore we may put $N_{\mathbf{m}} = 2$. Then, Theorem 4.1 with the relations

$$\begin{aligned}
&(a_2 + D_4) - (a_6 + a_5 + a_3) = (a_2 + a_5 + a_6) - (a_3 + a_5 + a_6) = a_2 - a_3 \ge 0,\\
&(a_2 + D_4) - (a_7 + a_4 + a_3) = (a_2 + a_5 + a_6) - (a_3 + a_4 + a_7) = a_6 - a_7 \ge 0
\end{aligned}$$

proves

$$\begin{aligned}\mathcal{RS}\mathbf{m} &= \mathbf{m}^{(2)}\\ &= \big[[D_4+n-a_3-a_4, 2D_4+n-a_3-a_4-a_5-a_6],\\ &\quad ([n-a_1, n-a_2, D_4+n-a_3-a_4-a_6, D_4+n-a_3-a_4-a_5],\\ &\quad [D_4-a_3, D_4-a_4, a_7, a_8, \ldots]\big]\\ &= \big[[m_{12}+m_{22}, m_{12}+m_{22}], [m_{12}, m_{22}, m_{22}, m_{12}],\\ &\quad [m_{12}+m_{22}-m_{32}, m_{12}+m_{22}-m_{31}, m_{41}, \ldots, m_{4,n_4}]\big].\end{aligned}$$

Here we note that

$$m_{12}+m_{22} \ge m_{22} \ge m_{12} \ge m_{12}+m_{22}-m_{32} \ge m_{12}+m_{22}-m_{31} \ge m_{41} \ge \cdots \ge m_{4,n_4}.$$

Set

$$\begin{aligned}&a_1 = a_2 = m_{12}+m_{22},\ a_3 = a_4 = m_{22},\ a_5 = a_6 = m_{12},\\ &a_7 = m_{12}+m_{22}-m_{32},\ a_8 = m_{12}+m_{22}-m_{31},\ a_{8+k} = m_{4k} \quad (1 \le k \le n_4),\\ &n = a_1+a_2 = 2(m_{12}+m_{22}),\ D_3 = n-a_1 = m_{12}+m_{22},\\ &\mathbf{m} = \big[[a_1, a_2], [a_3, a_4, a_5, a_6], [a_7, a_8, \cdots, a_{8+n_4}]\big].\end{aligned}$$

Since $a_3+a_5 = D_3$, Theorem 4.1 proves

$$\begin{aligned}\mathcal{RS}\mathbf{m} &= \mathbf{m}^{(1)}\\ &= \big[[D_3+n-a_2-a_3, D_3+n-a_2-a_4], [n-a_1, n-a_2, D_3+n-a_2-a_3-a_4]\\ &\quad [D_3-a_2, a_5, a_6, \ldots, a_{8+n_4}]\big]\\ &= \big[[2m_{12}+m_{22}, 2m_{12}+m_{22}], [m_{12}+m_{22}, m_{12}+m_{22}, 2m_{12}],\\ &\quad [0, m_{12}, m_{12}, m_{12}+m_{22}-m_{32}, m_{12}+m_{22}-m_{31}, m_{41}, \ldots, m_{4,n_5}]\big].\end{aligned}$$

Here we note that

$$\begin{aligned}&2m_{12}+m_{22} \ge m_{12}+m_{22} \ge 2m_{12} \ge m_{12} \ge m_{12}+m_{22}-m_{32} \ge m_{12}+m_{22}-m_{31}\\ &\ge m_{41} \ge \cdots \ge m_{4,n_4}.\end{aligned}$$

Theorem 4.2. (1) *Let* $\mathbf{m} = m_{11}m_{12}, m_{21}m_{22}, m_{31}m_{32}, m_{41}\cdots m_{4,n_4}$ *be an ordered fundamental spectral type satisfying* $m_{11} \ge m_{21} \ge m_{31}$. *Then*

$$\begin{aligned}\mathcal{RS}\mathbf{m} = \big[&[m_{12}+m_{22}, m_{12}+m_{22}], [m_{12}, m_{12}, m_{22}, m_{22}],\\ &[m_{12}+m_{22}-m_{32}, m_{12}+m_{22}-m_{31}, m_{41}, \ldots, m_{4,n_4}]\big]\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}\mathbf{m} = \big[&[2m_{12}+m_{22}, 2m_{12}+m_{22}], [m_{12}+m_{22}, m_{12}+m_{22}, 2m_{12}]\\ &[m_{12}, m_{12}, m_{12}+m_{22}-m_{32}, m_{12}+m_{22}-m_{31}, m_{41}, \ldots, m_{4,n_4}]\big].\end{aligned} \tag{4.16}$$

All the expressions above are ordered.
(2) *For any* $\mathbf{n} \in \mathcal{F}^{E_8}$, *we have*

$$|\mathcal{F}^{D_4}_{[\mathbf{n}]}| \le 1.$$

Moreover, $|\mathcal{F}^{D_4}_{[\mathbf{n}]}| = 1$ *if and only if*

$$\bar{m}_{11} = \bar{m}_{12} = \bar{m}_{21}+\bar{m}_{31},\ \bar{m}_{21} = \bar{m}_{22} \ \text{ and } \ \bar{m}_{31} = \bar{m}_{32} \tag{4.17}$$

under the ordered expression

$$\mathbf{n} = \bar{m}_{11}\bar{m}_{12},\ \bar{m}_{21}\bar{m}_{22}\bar{m}_{23},\ \bar{m}_{31}\bar{m}_{32}\bar{m}_{33}\bar{m}_{34}\bar{m}_{35}\cdots\bar{m}_{3,n_3},$$

namely, $\mathbf{n}$ *has the ordered expression*

$$\mathbf{n} = \big[[\bar{m}_{21}+\bar{m}_{31}, \bar{m}_{21}+\bar{m}_{31}], [\bar{m}_{21}, \bar{m}_{21}, 2\bar{m}_{31}], [\bar{m}_{31}, \bar{m}_{31}, \bar{m}_{33}, \bar{m}_{34}, \bar{m}_{35}, \dots, \bar{m}_{3,n_3}]\big]. \tag{4.18}$$

Then the element $\mathbf{m} \in \mathcal{F}^{D_4}_{[\mathbf{n}]}$ *whose expression is given by* (4.14) *is*

$$\begin{aligned}&\big[[\operatorname{ord}\mathbf{m} - \bar{m}_{31}, \bar{m}_{31}], [\operatorname{ord}\mathbf{m} - \bar{m}_{21} + \bar{m}_{31}, \bar{m}_{21} - \bar{m}_{31}],\\ &\quad [\operatorname{ord}\mathbf{m} - \bar{m}_{21} + \bar{m}_{33}, \bar{m}_{21} - \bar{n}_{33}], [\bar{m}_{35}, \bar{m}_{36}, \dots, \bar{m}_{3,n_3}]\big],\end{aligned}$$

where

$$\operatorname{ord}\mathbf{m} = 2\bar{m}_{21} - \bar{m}_{33} - \bar{m}_{34}.$$

Proof. The first part of the theorem has already been proved.

To prove the second part, suppose that $\mathbf{n} = \mathcal{U}\mathbf{m}$ for some $\mathbf{m}$ as in (1). Note that (4.17) follows from the ordered expression (4.16). Then, $\bar{m}_{23} = (\bar{m}_{11}+\bar{m}_{12}) - (\bar{m}_{21}+\bar{m}_{22}) = 2\bar{m}_{31}$ and hence we obtain (4.18). Comparing (4.16) and (4.18), we obtain

$$\begin{aligned}
m_{12} &= \bar{m}_{31}, & m_{22} &= \bar{m}_{21} - \bar{m}_{31},\\
m_{32} &= m_{12} + m_{22} - \bar{m}_{33} = \bar{m}_{21} - \bar{m}_{33}, & m_{31} &= m_{12} + m_{22} - \bar{m}_{34} = \bar{m}_{21} - \bar{m}_{34},\\
m_{4,\nu} &= \bar{m}_{3,\nu+4} \quad (1 \le \nu \le n_4 - 4), & \operatorname{ord}\mathbf{m} &= m_{31} + m_{32} = 2\bar{m}_{21} - \bar{m}_{33} - \bar{m}_{34},\\
m_{11} &= \operatorname{ord}\mathbf{m} - m_{12}, & m_{21} &= \operatorname{ord}\mathbf{m} - m_{22}.
\end{aligned}$$

Hence the map $\mathcal{U} : \mathcal{F}^{D_4} \to \mathcal{F}^{E_8}$ is injective.

Moreover, since the expressing (4.18) of $\mathbf{n}$ is ordered and fundamental, we have

$$\begin{aligned}
m_{31} - m_{32} &= \bar{m}_{33} - \bar{m}_{34} \ge 0,\\
m_{32} - m_{22} &= \bar{m}_{21} - \bar{m}_{33} \ge 0,\\
m_{22} - m_{12} &= \bar{m}_{21} - 2\bar{m}_{31} \ge 0,\\
m_{12} - m_{41} &= \bar{m}_{31} - \bar{m}_{35} \ge 0,\\
m_{11} + m_{21} + m_{31} + m_{41} - 2\operatorname{ord}\mathbf{m} &= m_{31} + m_{41} - (m_{12} + m_{22}) = \bar{m}_{35} - \bar{m}_{34} \le 0,
\end{aligned}$$

where the $m_{j,\nu}$ are defined from $\mathbf{n}$ by the above relations. This means that $\{m_{j,\nu}\}$ is an ordered D_4-fundamental spectral type in normal form. Thus, $m_{j,\nu}$ is an ordered D_4-fundamental spectral type in □

4.3 E_7-fundamental spectral types

Let

$$\mathbf{m} = \big[[m_{11}, m_{12}], [m_{21}, m_{22}, m_{23}, m_{24}], [m_{31}, m_{32}, \dots, m_{3n_3}]\big]$$

be an ordered fundamental E_7-tuple, and put $n = \operatorname{ord}\mathbf{m}$. Then

$$\begin{aligned}
&m_{11} \ge m_{12} > 0,\ m_{21} \ge m_{22} \ge m_{23} \ge m_{24} > 0,\ m_{31} \ge \cdots \ge m_{3n_3} > 0\\
&n = m_{11} + m_{12} = m_{21} + m_{22} + m_{23} + m_{24} = m_{31} + \cdots + m_{3n_3},\\
&d_{1,1,1}(\mathbf{m}) = m_{11} + m_{21} + m_{31} - n = m_{21} + m_{31} - m_{12} \le 0.
\end{aligned} \tag{4.19}$$

It follows that

$$m_{31} \le m_{12} - m_{21} \le \tfrac{n}{2} - \tfrac{1}{2}(m_{21} + m_{22}) = \tfrac{1}{2}(m_{23} + m_{24}) \le m_{23},$$
$$m_{11} \ge m_{12} \ge m_{21} \ge m_{22} \ge m_{23} \ge m_{31} \ge \cdots \ge m_{3n_3},\ m_{23} \ge m_{24}.$$

Under the notation (4.1),

$$\begin{aligned} &a_1 = m_{11},\ a_2 = m_{12},\ a_3 = m_{21},\ a_4 = m_{22},\ a_5 = m_{23},\ a_6 = m_{24} \ \text{ or } \ m_{31}, \\ &a_3 + a_5 \ge D_3 = n - m_{11} = m_{12} \ge a_5 + a_7. \end{aligned} \tag{4.20}$$

Here the last inequality follows from the fact that Lemma 4.9: if $a_5 + a_7 > D_3$, then $|I_{[1,7]}| = 2$. Similarly, if $a_1 + a_3 + a_6 > n$, then $a_6 = m_{24}$. Thus, Theorem 4.1 shows that $\mathcal{U}\mathbf{m}$ is one of $\mathbf{m}^{(2)}$, $\mathbf{m}^{(3)}$, $\mathbf{m}^{(3,1)}$ and $\mathbf{m}^{(3,2)}$. More precisely, we have the following cases.

If $m_{11} - m_{21} - m_{23} = a_1 + a_4 + m_{24} - n > 0$, then $a_6 = m_{24}$ and

$$\begin{aligned} \mathcal{U}\mathbf{m} = \mathbf{m}^{(3)} = \big[&[2D_3 + n - a_{[2,4]} - a_6, D_3 + n - a_{[2,5]}], \\ &[n - a_1, 2D_3 + n - a_{[2,6]}, D_3 + n - a_{[2,4]}], [D_3 - a_2, D_3 - a_3, D_3 - a_4, a_7, a_8, \ldots]\big] \end{aligned}$$

If $m_{22} + m_{23} - m_{11} = a_2 + a_4 + a_5 - n > 0$,

$$\begin{aligned} \mathcal{U}\mathbf{m} = \mathbf{m}^{(3,1)} = \big[&[2D_3 + n + a_1 - a_2 - a_{[2,5]}, 2D_3 + n - a_{[2,5]}], \\ &[D_3 + n - a_2 - a_4 - a_5, D_3 + n - a_2 - a_3 - a_5, D_3 + n - a_2 - a_3 - a_4], \\ &[D_3 - a_2, 2D_3 + a_1 - a_2 - a_3 - a_4 - a_5, a_6, a_7, a_8, \ldots]\big]. \end{aligned}$$

If $m_{21} + m_{24} - m_{11} = a_2 + a_3 + m_{24} - n > 0$, we have $m_{24} = a_6$ and

$$\begin{aligned} \mathcal{U}\mathbf{m} = \mathbf{m}^{(3,2)} = \big[&[D_3 + n - a_2 - a_3, 3D_3 + n + a_1 - 2a_2 - 2a_3 - a_4 - a_5 - a_6], \\ &[D_3 + n - a_2 - a_3 - a_6, D_3 + n - a_2 - a_3 - a_5, D_3 + n - a_2 - a_3 - a_4], \\ &[D_3 - a_2, D_3 - a_3, D_3 + a_1 - a_2 - a_3, a_6, a_7, a_8, \ldots]\big]. \end{aligned}$$

In the remaining case,

$$\begin{aligned} \mathcal{U}\mathbf{m} = \mathbf{m}^{(2)} = \big[&[D_3 + n - a_2 - a_3, 2D_3 + n - a_{[2,5]}], \\ &[n - a_1, D_3 + n - a_2 - a_3 - a_5, D_3 + n - a_{[2,4]}], [D_3 - a_2, D_3 - a_3, a_6, a_7, a_8, \ldots]\big]. \end{aligned}$$

Applying (4.20) to the above cases, we obtain the following theorem.

Theorem 4.3. *$\mathcal{U}\,\mathbf{m}$ equals*

$$\begin{aligned} \mathbf{m}^{(3)} =& \big[[m_{12} + m_{23}, m_{12} + m_{24}], [m_{12}, m_{12}, m_{23} + m_{24}], \\ &[m_{12} - m_{22}, m_{12} - m_{21}, m_{31}, \ldots, m_{3n_3}]\big] \quad \textit{if } \ m_{11} > m_{21} + m_{23}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \mathbf{m}^{(3,1)} =& \big[[m_{11} + m_{24}, m_{12} + m_{24}], [m_{21} + m_{24}, m_{22} + m_{24}, m_{23} + m_{24}], \\ &[m_{24}, m_{24}, m_{31}, \ldots, m_{3n_3}] \qquad\qquad\qquad \textit{if } \ m_{11} < m_{22} + m_{23}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} \mathbf{m}^{(3,2)} =& \big[[m_{22} + m_{23} + m_{24}, m_{22} + m_{23} + m_{24}], [m_{22} + m_{23}, m_{22} + m_{24}, m_{23} + m_{24}], \\ &[m_{11} - m_{21}, m_{12} - m_{21}, m_{31}, \ldots, m_{3n_3}] \quad \textit{if } \ m_{11} < m_{21} + m_{24}, \end{aligned} \tag{4.23}$$

$$\begin{aligned} \mathbf{m}^{(2)} =& \big[[m_{22} + m_{23} + m_{24}, m_{12} + m_{24}], [m_{12}, m_{22} + m_{24}, m_{23} + m_{24}], \\ &[m_{24}, m_{12} - m_{21}, m_{31}, \ldots, m_{3n_3}] \\ &\textit{if } \ m_{11} \ge m_{22} + m_{23},\ m_{11} \le m_{21} + m_{23} \ \textit{ and } \ m_{11} \ge m_{21} + m_{24}. \end{aligned} \tag{4.24}$$

Although the tuples (4.23) *and* (4.24) *are ordered,* $m_{24} < m_{31}$ *may occur in* (4.22)*, and* $m_{12} < m_{23} + m_{24}$ *may happen in* (4.21)*. The tuple* (4.24) *satisfies* $d_{1,1,1}(\mathcal{U}\,\mathbf{m}) = 0$.

Remark 4.16. Using Theorem 4.3, we can determine $\mathcal{F}^{E_7}_{[\mathbf{n}]}$ for for any $\mathbf{n} \in \mathcal{F}^{E_8}$ as follows.

Since every $\mathbf{m} \in \mathcal{F}^{E_7}_{[\mathbf{n}]}$ is obtained from one of the four cases in the theorem, $\mathbf{n}$ satisfies at least one of the following four conditions:

$$\bar{m}_{21} = \bar{m}_{22} \quad \text{or} \quad \bar{m}_{22} = \bar{m}_{23}, \tag{4.25}$$

$$\text{there exists } \nu \text{ satisfying } \bar{m}_{3,\nu} = \bar{m}_{3,\nu+1}, \tag{4.26}$$

$$\bar{m}_{11} = \bar{m}_{12}, \tag{4.27}$$

$$\bar{m}_{12} = \bar{m}_{21} + \bar{m}_{31}. \tag{4.28}$$

Here, $\mathbf{n}$ is written in the ordered form

$$\mathbf{n} = \bar{m}_{11}\bar{m}_{12},\ \bar{m}_{21}\bar{m}_{22}\bar{m}_{23},\ \bar{m}_{31}\bar{m}_{32}\bar{m}_{33}\cdots\bar{m}_{3,\bar{n}_3}. \tag{4.29}$$

For example, if (4.28) holds, $\mathbf{m}$ may be given by (4.24), which implies

$$\begin{aligned} &m_{21} = \bar{m}_{21} - \bar{m}_{32},\ m_{22} = \bar{m}_{22} - \bar{m}_{31},\ m_{23} = \bar{m}_{23} - \bar{m}_{31},\ m_{24} = \bar{m}_{31}, \\ &\operatorname{ord}\mathbf{m} = \bar{m}_{11} + \bar{m}_{21} - \bar{m}_{32},\ m_{12} = \bar{m}_{21},\ m_{11} = \bar{m}_{11} - \bar{m}_{32}. \end{aligned}$$

Checking inequalities (4.19) and those in (4.24), we determine whether $\mathbf{m}$ is indeed given by (4.24).

In the same way, we can determine the elements $\mathbf{m} \in \mathcal{F}^{E_7}_{[\mathbf{n}]}$ corresponding to each case in Theorem 4.3. Here we note that there may be several $\mathbf{m}$ corresponding to the case (4.22) with ν satisfying (4.26), whereas there is at most one $\mathbf{m}$ corresponding to each of the other cases.

Example 4.17. (1) Put

$$\mathbf{n} = \texttt{ee,a99,443333221111} \in \mathcal{F}^{E_8}.$$

Then $\operatorname{idx}\mathbf{n} = -50$, $\#\mathcal{F}^{E_7}_{[\mathbf{n}]} = 5$ and $\mathcal{F}^{E_7}_{[\mathbf{n}]}$ is obtained as is explained in Remark 4.16:

Case (4.21) : `b9,5555,3333221111`

Case (4.22) with $m_{24} = 1,\ 2,\ 3$:

`dd,9881,4433332211` `cc,8772,4433331111` `bb,7663,4433221111`

Case (4.24) : `aa,6554,3333221111`

Here `a` $= 10$, `b` $= 11$, `c` $= 12$, `d` $= 13$, `e` $= 14, \ldots$

(2) For any $\mathbf{n} \in \mathcal{F}^{E_8}$ satisfying $\operatorname{idx}\mathbf{n} \geq -100$, we have calculated $\mathcal{F}_{[\mathbf{n}]}$ (cf. [18]) and found $\bar{\mathbf{n}} \in \mathcal{F}^{E_7}_{[\mathbf{n}]}$ such that $\operatorname{ord}\bar{\mathbf{n}} \geq \operatorname{ord}\mathbf{m}$ for any $\mathbf{m} \in \mathcal{F}_{[\mathbf{n}]} \setminus \{\mathbf{n}\}$. However, in particular,

$$\mathbf{n} = \big[[50, 49], [34, 33, 32], [14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4]\big] \tag{4.30}$$

with $\operatorname{idx}\mathbf{n} = -630$ does not satisfy any of the conditions in Remark 4.16. Therefore,

$$\mathcal{F}^{E_7}_{[\mathbf{n}]} = \emptyset.$$

4.4 E_6-fundamental spectral types

In this subsection, we calculate $\mathcal{U}\mathbf{m}$ for an E_6-fundamental spectral type $\mathbf{m}$. The result is as follows.

Theorem 4.4. *Let*

$$\mathbf{m} = \big[[m_{11}, m_{12}, m_{13}], [m_{21}, m_{22}, m_{23}], [m_{31}, \ldots, m_{3n_3}]\big]$$

be an ordered E_6-fundamental spectral type. Then $\mathcal{U}\,\mathbf{m}$ equals

$$\begin{aligned} &\big[[m_{11} + 2m_{13}, m_{12} + 2m_{13}], [m_{13} + m_{21}, m_{13} + m_{22}, m_{13} + m_{23}] \\ &\quad [m_{13}, m_{13}, m_{13}, m_{31}, \ldots]\big] \\ &\qquad \textit{if}\ \ m_{21} < m_{12}, \end{aligned} \tag{4.31}$$

$$\begin{aligned}&\big[[m_{13}+m_{22}+m_{23}, m_{12}+2m_{13}], [m_{12}+m_{13}, m_{13}+m_{22}, m_{13}+m_{23}],\\ &\quad [m_{13}, m_{13}, m_{12}+m_{13}-m_{21}, m_{31}, \ldots]\big]\\ &\qquad \textit{if}\ \ m_{22}\le m_{12}\le m_{21},\end{aligned}\tag{4.32}$$

$$\begin{aligned}&\big[[m_{12}+m_{13}+m_{23}, m_{12}+m_{13}+m_{23}], [m_{12}+m_{13}, m_{12}+m_{23}, m_{13}+m_{23}],\\ &\quad [m_{23}, m_{12}+m_{13}-m_{22}, m_{12}+m_{13}-m_{21}, m_{31}, \ldots]\big]\\ &\qquad \textit{if}\ \ m_{12}<m_{22}\ \ \textit{and}\ \ m_{13}\ge m_{23},\end{aligned}\tag{4.33}$$

$$\begin{aligned}&\big[[m_{12}+m_{13}+m_{23}, m_{12}+2m_{13}], [m_{12}+m_{13}, m_{12}+m_{13}, m_{13}+m_{23}],\\ &\quad [m_{13}, m_{12}+m_{13}-m_{22}, m_{12}+m_{13}-m_{21}, m_{31}, \ldots]\big]\\ &\qquad \textit{if}\ \ m_{23}\le m_{12}<m_{22}\ \ \textit{and}\ \ m_{13}<m_{23},\end{aligned}\tag{4.34}$$

$$\begin{aligned}&\big[[2m_{12}+m_{13}, m_{12}+2m_{13}],\ [m_{12}+m_{13}, m_{12}+m_{13}, m_{12}+m_{13}]\\ &\quad [n-m_{11}-m_{23}, n-m_{11}-m_{22}, n-m_{11}-m_{21}, m_{31}, \ldots]\big]\\ &\qquad \textit{if}\ \ m_{12}<m_{23}\ \ \textit{and}\ \ m_{13}<m_{23}.\end{aligned}\tag{4.35}$$

In the case (4.31)*, it may happen that* $m_{13}<m_{31}$*. In the other cases, the above tuples in* $\mathcal{F}^{E_8}$ *are ordered.*

Proof. Note that

$$\begin{aligned}&m_{11}\ge m_{12}\ge m_{13}>0,\ m_{21}\ge m_{22}\ge m_{23}>0,\ m_{31}\ge\cdots\ge m_{3n_3}>0,\\ &n=m_{11}+m_{12}+m_{13}=m_{21}+m_{22}+m_{23}=m_{31}+\cdots+m_{3n_3},\\ &m_{11}\ge m_{21}\ge m_{31},\\ &m_{12}+m_{13}\ge m_{21}+m_{31}.\end{aligned}$$

We calculate $\mathcal{RT}m$ as the proof of Theorem 4.1. Hence, consider the following transformations.

$$\begin{aligned}&\mathcal{T}\mathbf{m}\simeq\big[[2n,n],[n,n,n],[m_{11},m_{21},m_{22},m_{12},m_{23},m_{13},m_{31},\ldots]\big]\\ &\xrightarrow[1,1,1]{m_{11}}\xrightarrow[1,2,2]{m_{21}}\xrightarrow[1,3,3]{m_{22}}\\ &\big[[2n-m_{11}-m_{21}-m_{22},\underline{n}],[n-m_{11},n-m_{21},\underline{n-m_{22}}],[0,0,0,\underline{m_{12}},m_{23},m_{13},m_{31},\ldots]\big]\\ &\xrightarrow[2,3,4]{m_{21}-m_{13}}\quad((n-m_{22})+m_{21}-(2n-m_{11}-m_{21}-m_{22})=m_{21}-m_{13},\quad m_{21}\ge\tfrac{n}{3}\ge m_{13})\\ &\big[[m_{12}+m_{13}+m_{23},m_{13}+m_{22}+m_{23}],[m_{12}+m_{13},m_{22}+m_{23},m_{13}+m_{23}],\\ &\quad [0,0,0,n-m_{11}-m_{21},m_{23},m_{13},m_{31},\ldots]\big].\end{aligned}$$

Here we note the relations

$$\begin{aligned}&(m_{12}+m_{13}+m_{23})-(m_{13}+m_{22}+m_{23})=m_{11}-m_{22},\\ &(m_{22}+m_{23})-(m_{12}+m_{13})=m_{11}-m_{12}\ge 0,\\ &(m_{22}+m_{23})-(m_{13}+m_{23})=m_{22}-m_{13}.\end{aligned}$$

Case $m_{12}\ge m_{22}$ $(\Rightarrow m_{13}\le m_{23})$:

$$\begin{aligned}&\big[[\underline{m_{12}+m_{13}+m_{23}},m_{13}+m_{22}+m_{23}],[m_{12}+m_{13},\underline{m_{22}+m_{23}},m_{13}+m_{23}],\\ &\quad [0,0,0,n-m_{11}-m_{21},\underline{m_{23}},m_{13},m_{31},\ldots]\big]\\ &\xrightarrow[1,2,5]{m_{23}-m_{13}}\\ &\big[[m_{12}+2m_{13},\underline{m_{13}+m_{22}+m_{23}}],[\underline{m_{12}+m_{13}},m_{13}+m_{22},m_{13}+m_{23}],\\ &\quad [0,0,0,\underline{m_{12}+m_{13}-m_{21}},m_{13},m_{13},m_{31},\ldots]\big].\end{aligned}$$

Here, we remark

$$\begin{aligned}
&(m_{13}+m_{22}+m_{23})-(m_{12}+2m_{13})=m_{11}-m_{21}\geq 0,\\
&(m_{12}+m_{13})-(m_{13}+m_{22})=(m_{12}+m_{13}-m_{22})-m_{13}=m_{12}-m_{22}\geq 0,\\
&m_{13}-(m_{12}+m_{13}-m_{21})=m_{21}-m_{12},\\
&m_{12}+m_{13}-m_{21}=n-m_{11}-m_{21}\geq m_{31},\\
&(m_{12}+2m_{13})-(m_{12}+m_{13})-m_{13}=0.
\end{aligned}$$

Hence, if $m_{21}\geq m_{12}$, we have the theorem.

We assume $m_{12}>m_{21}$ and apply $\mathrm{mc}_{(2,1,4)}$ to the last tuple. Then

$$\begin{aligned}
\xrightarrow[2,1,4]{m_{12}-m_{21}}\ &\big[[m_{12}+2m_{13},m_{11}+2m_{13}],[m_{13}+m_{21},m_{13}+m_{22},m_{13}+m_{23}]\\
&\qquad [0,0,0,m_{13},m_{13},m_{13},m_{31},\ldots]\big],
\end{aligned}$$

which is E_8-fundamental since

$$\begin{aligned}
(m_{12}+2m_{13})-(m_{13}+m_{21})-m_{13}&=m_{12}-m_{21}>0,\\
(m_{12}+2m_{13})-(m_{13}+m_{21})-m_{31}&=m_{12}+m_{13}-m_{31}\\
&=n-m_{11}-m_{21}-m_{31}\geq 0.
\end{aligned}$$

When

$$\mathbf{m}=972,6^3,3^6\quad\text{or}\quad \mathbf{m}=972,6^3,1^{18},$$

$m_{13}<m_{31}$ or $m_{13}>m_{31}$, respectively.

Case $m_{12}<m_{22}$ and $m_{13}\geq m_{23}$:

$$\begin{aligned}
&\big[[m_{12}+m_{13}+m_{23},\underline{m_{13}+m_{22}+m_{23}}],[m_{12}+m_{13},\underline{m_{22}+m_{23}},m_{13}+m_{23}],\\
&\quad [0,0,0,n-m_{11}-m_{21},m_{23},\underline{m_{13}},m_{31},\ldots]\big]\\
&\xrightarrow[2,2,6]{m_{22}-m_{12}}\\
&\big[[m_{12}+m_{13}+m_{23},m_{12}+m_{13}+m_{23}],[\underline{m_{12}+m_{13}},m_{12}+m_{23},m_{13}+m_{23}],\\
&\quad [0,0,0,m_{12}+m_{13}-m_{21},\underline{m_{23}},m_{12}+m_{13}-m_{22},m_{31},\ldots]\big].
\end{aligned}$$

Here

$$\begin{aligned}
&m_{23}-(m_{12}+m_{13}-m_{22})=m_{22}+m_{23}-m_{12}-m_{13}=m_{11}-m_{21}\geq 0,\\
&m_{12}+m_{13}-m_{22}\geq m_{12}+m_{13}-m_{21}=n-m_{11}-m_{21}\geq m_{31},\\
&(m_{12}+m_{13}+m_{23})-(m_{12}+m_{13})-m_{23}=0.
\end{aligned}$$

Hence we have the ordered E_8-fundamental tuple (4.33).

Case $m_{12}<m_{22}$ and $m_{13}<m_{23}$:

$$\begin{aligned}
&\big[[m_{12}+m_{13}+m_{23},\underline{m_{13}+m_{22}+m_{23}}],[m_{12}+m_{13},\underline{m_{22}+m_{23}},m_{13}+m_{23}\big],\\
&\quad [0,0,0,n-m_{11}-m_{21},\underline{m_{23}},m_{13},m_{31},\ldots]]\\
&\xrightarrow[2,2,5]{m_{11}-m_{21}}\\
&\mathbf{n}=\big[[\underline{m_{12}+m_{13}+m_{23}},m_{12}+2m_{13}],[m_{12}+m_{13},m_{12}+m_{13},m_{13}+m_{23}],\\
&\qquad [0,0,0,m_{12}+m_{13}-m_{21},m_{12}+m_{13}-m_{22},\underline{m_{13}},m_{31},\ldots]\big].
\end{aligned}$$

Here

$$\begin{aligned}
&(m_{12}+m_{13}+m_{23})-(m_{12}+2m_{13})=m_{12}-m_{13}\geq 0,\\
&(m_{12}+m_{13})-(m_{13}+m_{23})=m_{12}-m_{23},\\
&m_{13}-(m_{12}+m_{13}-m_{22})=m_{22}-m_{12}>0,\\
&m_{12}+m_{13}-m_{22}\geq m_{12}+m_{13}-m_{21}=n-m_{11}-m_{21}\geq m_{31},\\
&(m_{12}+m_{13}+m_{23})-(m_{12}+m_{13})-m_{13}=m_{23}-m_{13}>0.
\end{aligned}$$

Then, if $m_{12}\geq m_{23}$, $\mathbf{n}$ is E_8-fundamental. We assume $m_{12}<m_{23}$. Then

$$\begin{aligned}
\mathbf{n}\xrightarrow[1,3,6]{m_{23}-m_{12}}\big[&[\underline{2m_{12}+m_{13}},m_{12}+2m_{13}],\ [\underline{m_{12}+m_{13}},m_{12}+m_{13},m_{12}+m_{13}],\\
&[0,0,0,n-m_{11}-m_{21},n-m_{11}-m_{22},\underline{n-m_{11}-m_{23}},m_{31},\ldots]\big].
\end{aligned}$$

Since

$$(m_{12}+2m_{13})-(m_{12}+m_{13})-(n-m_{11}-m_{23})=m_{23}-m_{12}>0,$$

this spectral type equals $\mathcal{U}\,\mathbf{m}$. □

Remark 4.18. For $\mathbf{n}\in\mathcal{F}^{E_8}$ with $\operatorname{idx}\geq -100$, we have $\mathcal{F}^{E_6}_{[\mathbf{n}]}\neq\emptyset$ (cf. Remark 4.13). Note that $\mathcal{F}^{T}_{[\mathbf{n}]}=\emptyset$ for $\mathbf{n}$ given by (4.30) and $T=D_4$, E_6 or E_7.

References

[1] W. Crawley-Boevey. On matrices in prescribed conjugacy classes with no common invariant subspaces and sum zero. *Duke Math. J.*, 118:339–352, 2003.

[2] M. Dettweiler and S. Reiter. An algorithm of Katz and its applications to the inverse Galois problems. *J. Symbolic Comput.*, 30:761–798, 2000.

[3] K. Fuji and T. Suzuki. Higher order Painlevé system of type A, Drinfeld-Sokolov hierarchies and Fuchsian systems. *RIMS Kokyouroku Bessatsu*, B30 1:181–208, 2012.

[4] J. Harnad. Dual isomonodromic deformations and moment maps to loop algebras. *Commun. Math. Phys.*, 166:337–365, 1994.

[5] K. Hiroe. Linear differential equations on the Riemann sphere and representations of quivers. *Duke Math. J.*, 166:855–935, 2017.

[6] K. Hiroe. Deformation of moduli spaces of meromorphic-connections on $\mathbb{P}^1$ via unfolding of irregular singularities. *ArXiv.2407.20486*, 2025.

[7] K. Hiroe, H. Kawakami, A. Nakamura, and H. Sakai. *4-dimensional Painlevé-type equations.* Number 37 in MSJ Memoirs. Mathematical Society of Japan, 2018.

[8] K. Hiroe and T. Oshima. A classification of roots of symmetric Kac-Moody root systems and its application. In *Symmetries, Integrable Systems and Representations*, volume 40 of *Springer Proceedings in Mathematics and Statistics*, pages 195–241. Springer, 2012.

[9] V. G. Kac. *Infinite dimensional Lie algebras*, Third Edition. Cambridge University Press, 1990.

[10] N. M. Katz. *Rigid local systems.* Number 139 in Annals of Mathematics Studies. Princeton University Press, 1996.

[11] H. Kawakami. 4-dimensional Painlevé-type difference equations. *Experimental Mathematics*, 34:563–577, 2025.

[12] V. P. Kostov. The Deligne-Simpson problem - a survey. *J. Algebra*, 281:83–108, 2004.

[13] T. Oshima. os_muldif.rr, a library of computer algebra `Risa/Asir`, 2008–2026, `http://www.ms.u-tokyo.ac.jp/~oshima/`.

[14] T. Oshima. *Fractional calculus of Weyl algebra and Fuchsian differential equations.* Number 28 in MSJ Memoirs. Mathematical Society of Japan, 2012.

[15] T. Oshima. Confluence and versal unfolding of Pfaffian systems. *Josai Mathematical Monographs*, 12:117–151, 2020.

[16] T. Oshima. Versal unfolding of irregular singularities of a linear differential equation on the Riemann sphere. *Publ. RIMS Kyoto Univ.*, 57:893–920, 2021.

[17] T. Oshima. Algorithm of classifying roots of star-shaped Kac-Moody root systems. *ArXiv.2504.18542*, 2025.

[18] T. Oshima. spect10.pdf, `http://www.ms.u-tokyo.ac.jp/~oshima/`paper/spect10.pdf. 2026.

[19] Y. Sasano. Higher order Painlevé equations of type $D_\ell^{(1)}$. *RIMS Koukyuroku*, 1473:143–163, 2005.

[20] T. Suzuki. Six-dimensional Painlevé systems and their particular solutions in terms of rigid systems. *J. Math. Phys.*, 55(10), 2014.

[21] T. Tsuda. UC hierarchy and monodromy preserving deformation, *J. reine angew.*, 690:1–34, 2014.