FOURIER ANALYSIS ON SEMISIMPLE SYMMETRIC SPACES

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1. Introduction

A homogeneous space $X = G/H$ of a connected Lie group G is called a symmetric homogeneous space if there exists an involution σ of G such that H lies between the fixed point group G^σ and its identity component G^σ_o.

Example 0. For a connected Lie group G', put $G = G' \times G'$, $\sigma\big((g_1, g_2)\big) = (g_2, g_1)$ and $H = G^\sigma$. Then the homogeneous space $X = G/H$ is naturally isomorphic to G' by the map $(g_1, g_2) \mapsto g_1 g_2^{-1}$. Then the action of G on X corresponds to the left and right translations on G' by elements of G'. Hence any connected Lie group is an example of symmetric homogeneous space.

If G' is the abelian group $\mathbb{R}^n$ in Example 0, then the ring $\mathbb{D}(\mathbb{R}^n)$ of invariant differential operators on $\mathbb{R}^n$ equals the ring of differential operators with constant coefficients and $L^2(\mathbb{R}^n)$ is naturally unitary representation space of $\mathbb{R}^n$. The irreducible decomposition of the representation is given by Fourier transformations and it is also regards as a spectral resolution of $\mathbb{D}(\mathbb{R}^n)$ or expansions of functions in $L^2(\mathbb{R}^n)$ by joint eigenfunctions of $\mathbb{D}(\mathbb{R}^n)$.

Considering the above, we give a method of Fourier analysis on X when G is semisimple. Hereafter we assume G is semisimple and first cite more examples.

Example 1. G/K: a Riemannian symmetric space of non-compact type.

Example 2. A Riemannian symmetric space of compact type.

Example 3. G'_c/K'_c: a complex semisimple symmetric space, where G'_c is a complex semisimple Lie group and K'_c is a complexification of a maximal compact subgroup K' of a real form G' of G'_c (for example, $SL(n,\mathbb{C})/SO(n,\mathbb{C})$).

Example 4. G/K_ε: this is defined in [7] (cf. Definition 8). The complexification of the Lie algebra of K_ε coincides with that of K (for example, $SL(n,\mathbb{R})/SO(p,n-p)$, $Sp(n,\mathbb{R})/U(p,n-p)$, $Sp(n,\mathbb{R})/GL(n,\mathbb{R})$.)

Berger [1] gives a list of irreducible pairs $(\mathfrak{g},\mathfrak{h})$ of Lie algebras corresponding to (G,H) under outer automorphism. (Several easy ones are missed in the list.) If $\mathfrak{g}$ are of exceptional type, there are 131 inequivalent irreducible pairs $(\mathfrak{g},\mathfrak{h})$ and among them 49 ones belong to Example 4 and 36 ones do not belong to any one of Example 0 $\sim$ Example 4.

The group G acts by left translations on the space $L^2(X)$ of square integrable functions on X with respect to a G-invariant measure. Then $L^2(X)$ is a unitary representation space of G. In Example 0 an irreducible decomposition of $L^2(X)$ is obtained by Harish-Chandra. One of our purpose is to obtain that of $L^2(X)$ in the

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case of general X. Although we have not yet a general formula about it, we want to mention a method to do it.

Remark 5. Let $\widehat{G}_r$ denote the support of Plancherel measure of $L^2(G)$. Then that of $L^2(G/H)$ is not necessary contained in $\widehat{G}_r$. In general and if $n > 2$, there appears complementary series representations of $SO_o(n+1,1)$ in $L^2(SO_o(n+1,1)/SO_o(n,1))$.

2. Notation

Let G be a connected real form of a connected complex semisimple Lie group G_c and $\mathfrak{g}$ a Lie algebra of G. Let σ be any involutive automorphism of G and H a subgroup of G lies between G^σ and G^σ_o. Then the homogeneous space $X = G/H$ is called a semisimple symmetric space. We fix a Cartan involution θ of G commuting with σ and also denote by σ and θ the corresponding $\mathbb{C}$-linear involutions of Lie algebra $\mathfrak{g}_c$ of G_c. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (resp. $\mathfrak{h} + \mathfrak{q}$) be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces for θ (resp. σ). Then we have the direct sum decomposition

$$\mathfrak{g} = \mathfrak{k}\cap\mathfrak{h} + \mathfrak{k}\cap\mathfrak{q} + \mathfrak{p}\cap\mathfrak{h} + \mathfrak{p}\cap\mathfrak{q}.$$

Let $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$, $\mathfrak{a}_\mathfrak{p}$ a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$, and $\tilde{\mathfrak{j}}$ a Cartan subalgebra of $\mathfrak{g}$ containing both $\mathfrak{a}_\mathfrak{p}$ and a maximal abelian subspace of $\mathfrak{m}\cap\mathfrak{q}$, where $\mathfrak{m}$ denotes the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$. Furthermore we put $\mathfrak{j} = \tilde{\mathfrak{j}}\cap\mathfrak{q}$, $\mathfrak{t} = \mathfrak{j}\cap\mathfrak{k}$, $\ell' = \dim\mathfrak{j}$ and $\ell = \dim\mathfrak{a}$. We call ℓ' the rank of G/H and ℓ the split rank of G/H. Thus we have the inclusion relations:

$$\begin{array}{ccccccccc} & & & & \mathfrak{q} & & & & \\ & & & & \cup & & & & \\ & & & \supset & \mathfrak{j} & \supset & & & \\ \mathfrak{g} & \supset & \tilde{\mathfrak{j}} & & & & \mathfrak{a} & \subset & \mathfrak{p}\cap\mathfrak{q} \\ & & & \supset & \mathfrak{a}_\mathfrak{p} & \supset & & & \\ & & & & \cap & & & & \\ & & & & \mathfrak{p} & & & & \end{array}$$

For a linear subspace $\mathfrak{b}$ of $\mathfrak{g}$, $\mathfrak{b}_c$ denotes the complexification of $\mathfrak{b}$. If $\mathfrak{b}$ is a subalgebra, $U(\mathfrak{b})$ denotes the universal enveloping algebra of $\mathfrak{b}_c$. Let Ad (resp. ad) denote the adjoint representation of G_c (resp. $\mathfrak{g}_c$) on $\mathfrak{g}_c$ or $U(\mathfrak{g})$. For a θ-invariant linear subspace $\tilde{\mathfrak{a}}$ of $\tilde{\mathfrak{h}}$, $\tilde{\mathfrak{a}}^*$ denotes the dual space of $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}^*_c$ the complexification of $\tilde{\mathfrak{a}}^*_c$. Then we put $\mathfrak{g}_c(\tilde{\mathfrak{a}};\lambda) = \{X \in \mathfrak{g}_c;\, \mathrm{ad}(Y)X = \lambda(X) \text{ for all } Y \in \tilde{\mathfrak{a}}\}$ and $\mathfrak{g}(\tilde{\mathfrak{a}};\lambda) = \mathfrak{g}_c(\tilde{\mathfrak{a}};\lambda)\cap\mathfrak{g}$ for any λ in $\tilde{\mathfrak{a}}^*_c$ and moreover $\Sigma(\tilde{\mathfrak{a}}) = \{\lambda \in \tilde{\mathfrak{a}}^*_c - \{0\};\, \mathfrak{g}_c(\tilde{\mathfrak{a}};\lambda) \neq \{0\}\}$. By the Killing form $\langle\ ,\ \rangle$ of the complex Lie algebra $\mathfrak{g}_c$, we identify $\tilde{\mathfrak{a}}^*_c$ and $\tilde{\mathfrak{a}}_c$, and therefore $\tilde{\mathfrak{a}}^*_c$ is identified with a subspace of $\tilde{\mathfrak{j}}^*_c$. Let K, $A_\mathfrak{p}$ and A denote the analytic subgroups of $\mathfrak{k}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}$, respectively, and let M (resp. M^*) denote the centralizer (resp. normalizer) of $\mathfrak{a}_\mathfrak{p}$ in K. Then the quotient group M^*/M, which will be denoted by $W(\mathfrak{a}_\mathfrak{p})$, is the restricted Weyl group. Under the above notation we have easily

Lemma 6. i) $\mathfrak{j}$ *is a maximal abelian subspace of* $\mathfrak{q}$.
ii) $\Sigma(\mathfrak{j})$ *and* $\Sigma(\mathfrak{a})$ *satisfy the axiom of root systems. Let* $W(\mathfrak{j})$ *and* $W(\mathfrak{a})$ *denote the corresponding Weyl groups.*
iii) *Put*

$$W(\mathfrak{j})_\theta = \{w \in W(\mathfrak{j});\, w|_\mathfrak{a} = \mathrm{id}\}, \qquad W^\theta(\mathfrak{j}) = \{w \in W(\mathfrak{j}) :\, w(\mathfrak{a}) = \mathfrak{a}\},$$
$$W(\mathfrak{a}_\mathfrak{p})_\sigma = \{w \in W(\mathfrak{a}_\mathfrak{p});\, w|_\mathfrak{a} = \mathrm{id}\}, \quad W^\sigma(\mathfrak{a}_\mathfrak{p}) = \{w \in W(\mathfrak{a}_\mathfrak{p});\, w(\mathfrak{a}) = \mathfrak{a}\},$$

and $W(\mathfrak{a}_\mathfrak{p};H) = (M^*\cap H)/(M\cap H)$. *Then* $W(\mathfrak{a}_\mathfrak{p})_\sigma \subset W(\mathfrak{a}_\mathfrak{p};H) \subset W^\sigma(\mathfrak{a}_\mathfrak{p})$ *and the quotient group* $W^\theta(\mathfrak{j})/W(\mathfrak{j})_\theta$ *and* $W^\sigma(\mathfrak{a}_\mathfrak{p})/W(\mathfrak{a}_\mathfrak{p})_\sigma$ *are naturally isomorphic to* $W(\mathfrak{a})$.

iv) *We can define compatible systems of positive roots* $\Sigma(\tilde{\mathfrak{j}})^+$, $\Sigma(\mathfrak{j})^+$, $\Sigma(\mathfrak{a}_\mathfrak{p})^+$ *and* $\Sigma(\mathfrak{a})^+$.

We put $\rho = \frac{1}{2}\Sigma_{\alpha\in\Sigma(\tilde{\mathfrak{j}})^+}\alpha$ and $\mathfrak{n} = \Sigma_{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+}\mathfrak{g}(\mathfrak{a}_\mathfrak{p};\alpha)$ and denote by N the analytic subgroup of G with Lie algebra $\mathfrak{n}$. Then $P = MA_\mathfrak{p}N$ is a minimal parabolic subgroup. We define another parabolic subgroup $P_\sigma = \bigcup_{w\in W(\mathfrak{a}_\mathfrak{p})_\sigma} PwP$ and its Langlands decomposition $P_\sigma = M_\sigma A_\sigma N_\sigma$ so that $M_\sigma A_\sigma$ is the centralizer of $\mathfrak{a}$ in G. Let $\mathfrak{m}_\sigma = \mathfrak{m}(\sigma) + \mathfrak{g}(\sigma)$ be the direct sum decomposition into ideals of the Lie algebra $\mathfrak{m}_\sigma$ of M_σ so that the corresponding analytic subgroup $M(\sigma)_o$ (resp. $G(\sigma)$) of G is compact (resp. semisimple of non-compact type). We put $M(\sigma) = M(\sigma)_o(K\cap\exp\sqrt{-1}\mathfrak{a}_\mathfrak{p})$.

Lemma 7. *$M(\sigma)\subset M$, $G(\sigma)\subset H$ and $M_\sigma = M(\sigma)G(\sigma)$.*

Let $w_1,\dots,w_r$ be representatives of the factor set $W(\mathfrak{a}_\mathfrak{p};H)\backslash W^\sigma(\mathfrak{a}_\mathfrak{p})$, where $r = [W^\sigma(\mathfrak{a}_\mathfrak{p}) : W(\mathfrak{a}_\mathfrak{p};H)]$. We choose representatives $\bar{w}_i$ of w_i in M^* satisfying $\mathrm{Ad}(\bar{w}_i)\tilde{\mathfrak{j}} = \tilde{\mathfrak{j}}$, $\mathrm{Ad}(\bar{w}_i)\mathfrak{j} = \mathfrak{j}$ and $\bar{w}_i(\Sigma(\mathfrak{j})^+_\theta) = \Sigma(\mathfrak{j})^+_\theta$. Here we put $\Sigma(\mathfrak{j})^+_\theta = \{\alpha\in\Sigma(\mathfrak{j})^+;\ \alpha|_\mathfrak{a} = 0\}$.

Finally we give a definition of K_ε (cf. Example 4).

Definition 8. For any homomorphism of ε of $\Sigma(\mathfrak{a}_\mathfrak{p})\cup\{0\}$ to $\{\pm1\}$ (i.e., $\varepsilon(\alpha+\beta) = \varepsilon(\alpha)+\varepsilon(\beta)$ if α, $\beta\in\Sigma(\mathfrak{a}_\mathfrak{p})\cap\{0\}$) we define an involution θ_ε of $\mathfrak{g}$ so that

$$\theta_\varepsilon(X) = \varepsilon(\alpha)\theta(X) \quad \text{for } X\in\mathfrak{g}(\mathfrak{a}_\mathfrak{p};\alpha).$$

Putting $\mathfrak{k}_\varepsilon = \{X\in\mathfrak{g};\ \theta_\varepsilon(X) = X\}$, we define an analytic subgroup $(K_\varepsilon)_o$ of G with the Lie algebra $\mathfrak{k}_\varepsilon$. Then $K_\varepsilon = (K_\varepsilon)_oM$.

3. A realization of X

We want to construct a compact real analytic manifold $\mathbb{X}$ where our analysis is laid. For example, if X is $SL(2,\mathbb{R})/SO(2)$, X is naturally identified with an upper half plane and $\mathbb{X}$ is $\mathbb{P}^1_\mathbb{C}$. In general $\mathbb{X}$ has the following properties.

i) The group G acts analytically on $\mathbb{X}$ and then $\mathbb{X}$ is decomposed into finite number of G-orbits. An open orbit is isomorphic to G/H and hence we identify X with the orbit. The set of closed G-orbits contained in the closure of X in $\mathbb{X}$ is isomorphic to $\{G/Q_i;\ i=1,\dots,r\}$, where $Q_i = (M_\sigma\cap H_i)A_\sigma H_\sigma$ and $H_i = \bar{w}_i^{-1}H\bar{w}_i$.

ii) The orbital decomposition is of normal crossing type (i.e., there exists a local coordinate system $(t,x) = (t_1,\dots,t_m,x_1,\dots,x_n)$ around any point in $\mathbb{X}$ such that $G(t,x) = G(t',x')$ if and only if $\operatorname{sgn} t = \operatorname{sgn} t'$ ($\in\{-1,0,1\}^m$)).

iii) Any element of the ring $\mathbb{D}(X)$ of invariant differential operators on X is analytically extended to a differential operators on $\mathbb{X}$.

A method of the construction of $\mathbb{X}$ is as follows (cf. [5]). Since $G = KAH$, we think to compactify A. Paying attention to a Weyl chamber $\mathfrak{a}_+$ of $\mathfrak{a}$, we have the identification

$$\begin{array}{ccc} G\times A & \xrightarrow{\sim} & G\times(0,\infty)^\ell \hookrightarrow G\times\mathbb{R}^\ell \\ (g,a) & \mapsto & (g,t) = \big(g,(a^{-\alpha_1},\dots,a^{-\alpha_\ell})\big) \end{array}$$

where $\{\alpha_1,\dots,\alpha_\ell\}$ is the fundamental system of $\Sigma(\mathfrak{a})^+$. We define the equivalence relation $\sim$ in $G\times A$ such that $X\simeq(G\times A)/\sim$. Using the above identification, this equivalence relation is naturally extended to that in $G\times\mathbb{R}^\ell$, which is also denoted by $\sim$. Putting $\mathbb{X}_o = (G\times\mathbb{R}^\ell)/\sim$, we can construct $\mathbb{X}$ by patching finite copies of $\mathbb{X}_o$. (In Example 1, $\mathbb{X}_o = \mathbb{X}$ because $G = K\exp\mathfrak{a}_+K$ with $\mathfrak{a}_+ = \{X\in\mathfrak{a};\ \alpha(Z)\geq 0$ for any $\alpha\in\Sigma(\mathfrak{a})^+\}$.)

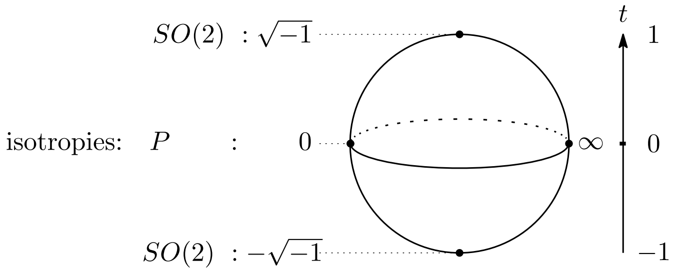
In the above construction, if $t_j = a^{-2\alpha_j}$ $(j = 1, \dots, \ell)$ in place of $t_j = a^{-\alpha_j}$, we have a difference $\mathbb{X}$ (cf. [7, §2]):

i) $X = SL(2,\mathbb{R})/SO(2)$ and $t = a^{-\alpha}$.

$\mathbb{X} = \mathbb{P}^1_{\mathbb{C}} \ni z \mapsto \dfrac{az+b}{cz+d}$,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$

$\mathbb{X} = \mathbb{X}_o$

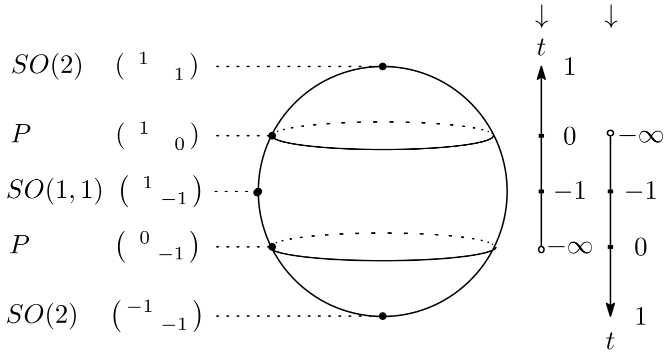


ii) $X = SL(2,\mathbb{R})/SO(2)$ and $t = a^{-2\alpha}$.

$\mathbb{X} = \Big\{ \begin{pmatrix} u & w \\ w & v \end{pmatrix};\ (u,v,w) \in (\mathbb{R}^3 - \{0\})/\mathbb{R}^{\times}_{+} \Big\} \simeq S^2,$ $\qquad \mathbb{X} = \mathbb{X}_o \cup \mathbb{X}_0$

$U \mapsto gU{}^tg \quad (g \in G).$

isotropies:



4. Boundary values of joint eigenfunctions of $\mathbb{D}(X)$

For a D in $U(\mathfrak{g})$ we define a D_j in $U(\mathfrak{j})$ so that $D - D_j$ belongs to $U(\mathfrak{g})\mathfrak{h} + \sum_{\alpha \in \Sigma(\mathfrak{j})^+} \mathfrak{g}_c(\mathfrak{j}; -\alpha)U(\mathfrak{g})$, and we put $\tilde{\iota}(D) = e^{\rho} \circ D_j \circ e^{-\rho}$. Then the identification $D(X) \simeq U(\mathfrak{g})^H / U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$ (where $U(\mathfrak{g})^H = \{D \in U(\mathfrak{g});\ \mathrm{Ad}(g)D = D$ for any g in $H\}$) and the map $\tilde{\iota}$ induces the following algebra isomorphism.

Lemma 9. $\qquad \iota : \mathbb{D}(X) \xrightarrow{\sim} I(\mathfrak{j}),$
where $I(\mathfrak{j})$ denotes the ring of all $W(\mathfrak{j})$-invariants in $U(\mathfrak{j})$.

We fix a $\nu \in \mathfrak{j}_c^*$, naturally identify the element of $U(\mathfrak{j})$ with a polynomial function on j_c^* and define a system of differential equations on X:

$$\mathcal{M}_\nu :\ Du = \big(\iota(D)(-\nu)\big)u \quad \text{for any } D \text{ in } \mathbb{D}(X).$$

Let $\mathcal{B}(X; \mathcal{M}_\nu)$ denote the space of hyperfunction solutions of $\mathcal{M}_\nu$. Since the system has regular singularities along the boundaries of X in $\mathbb{X}$, we can define boundary values of closed G-orbits G/Q_i define G-equivariant maps

$$\beta^i_\nu :\ \mathcal{B}(X; \mathcal{M}_\nu) \ \to\ \mathcal{B}(G/Q_i; L_\nu),$$

where

$$\mathcal{B}(G/Q_i; L_\nu) = \Big\{ f \in \mathcal{B}(G);\ f(g) = \chi_\nu^{M(\sigma)_o}(e) \int_{M(\sigma)_o} \chi_\nu^{M(\sigma)_o}(m) f(gm)dm$$
$$\text{and } f(gm'an) = f(g)a^{-\nu-\rho} \text{ for amy } m' \in M_\sigma \cap H_i,\ a \in A_\sigma \text{ and } n \in N_\sigma \Big\}.$$

Here we use the following notation: $\mathcal{B}(G)$ denotes the space of hyperfunctions on G. $\chi_\nu^{M(\sigma)_o}$ is the character of an irreducible unitary representation of $M(\sigma)_o$ with the highest weight $(\nu+\rho)|_{\mathfrak{t}}$ but $\chi_\nu^{M(\sigma)_o}=0$ if such representation does not exist. (Hence $\mathcal{B}(G/Q_i;L_\nu)=\{0\}$ if $\langle\nu,\alpha\rangle/\langle\alpha,\alpha\rangle\notin\{1,2,3,\ldots\}$ for a suitable $\alpha\in\Sigma(\mathfrak{j})_\theta^+$.) Furthermore in this note any measure on any compact group is the Haar measure so normalized that the total measure equals one.

Since $M(\sigma)$ normalizes $G(\sigma)N_\sigma$ and centralizes A_σ, the G-module $\mathcal{B}(G/Q_i;L_\nu)$ is decomposed into finite number of G-invariant subspaces according to the representation with respect to the right translations by $M(\sigma)$. We will explain it in the next section.

5. Principal series for X

Let $\tilde{J}$ be the Cartan subgroup of G associated with $\tilde{\mathfrak{j}}$ and $\tilde{\Pi}$ the set of homogeneous of $\tilde{J}$ to the multiplicative group $\mathbb{C}^\times$. For $i=1,\ldots,r$ we put

$$\Pi_i=\{\tau\in\tilde{\Pi};\ \tau(\tilde{J}\cap H_i)=\{1\}\text{ and }\langle d\tau,\alpha\rangle\geq 0\text{ for any }\alpha\in\Sigma(\mathfrak{j})_\theta^+\}$$

and $\Pi=\bigcup_{i=1}^r\Pi_i$, where $d\tau$ $(\in\tilde{\mathfrak{j}}_c^*)$ denotes the differential of τ. Let Π_i' be the set of equivalent classes of finite dimensional irreducible representations of P_σ with non-zero $(P_\sigma\cap H_i)N_\sigma$-fixed vectors. For $\delta\in\Pi_i'$, let (π_δ,V_δ) and $(d\pi_\delta,V_\delta)$ denote the corresponding representations of P_σ and its Lie algebra, respectively. Let V_σ be a highest weight vector of the representation $(d\pi_\delta|_{\mathfrak{m}_\sigma+\mathfrak{a}_\sigma},V_\delta)$ with respect to $\tilde{\mathfrak{j}}\cap(\mathfrak{m}_\sigma\cap\mathfrak{a}_\sigma)$ and the ordering defined in Lemma 6. Then if

$$\pi_\delta(g)v_\delta=\tau(g)v_\delta\text{ for any }g\text{ in }\tilde{J},$$

we identify a $\delta\in\Pi_i'$ and a $\tau\in\Pi_i$. This defines a bijection between Π_i' and Π_i, so we identify Π_i' and Π_i.

Let τ be an element of Π and (π_τ,V_τ) a corresponding representation of P_σ. The space V_τ has a Hermitian inner product $(\ ,\ )$ by which $\pi_\tau|_{M(\sigma)}$ is unitary. Let V_τ^* be the dual space of V_τ, $\langle\ ,\ \rangle$ the canonical bilinear map of $V_\tau\times V_{\tau^*}$ to $\mathbb{C}$ and (π_τ^*,V_τ^*) the contragredient representation. Then (π_τ^*,V_τ^*) is isomorphic to $(\pi_{\tau^*},V_{\tau^*})$ with a suitable $\tau^*\in\Pi$. We choose a unit vector u_τ in V_τ fixed by the identity component of $M_\sigma\cap H$ and define u_τ^* in V_τ^* by

$$(u,u_\tau)=\langle u,u_\tau^*\rangle\quad\text{for any }u\text{ in }V_\tau.$$

Definition 10. For a τ in Π, the G-module

$$\Bigl\{\sum f_j(g)\otimes v_j\in\mathcal{B}(G)\otimes V_\tau;\ \sum f_j(gman)\otimes v_j=a^{-\rho}f_j(g)\otimes\pi_\tau^{-1}(man)v_j$$
$$\text{for any }m\in M_\sigma,\ a\in A_\sigma\text{ and }n\in N_\sigma\Bigr\}$$

is called the space of (hyperfunction valued sections of non-unitary) principal series for X and denoted by $\mathcal{B}(G/P_\sigma;V_\tau)$.

We remark that the space

$$L^2(G/P_\sigma;V_\tau)=\Bigl\{\sum f_j\otimes v_j\in\mathcal{B}(G/P_\sigma;V_\tau);\ f_j|_K\in L^2(K)\Bigr\}$$

is naturally a unitary representation space of G if $d\tau|_{\mathfrak{a}}\in\sqrt{-1}\mathfrak{a}^*$.

We have $r=1$ in Example 0, 1, 2 or 3. In particular, in Example 1, $Q_1=P$, $\Pi=\{1\}$ and

$$\mathcal{B}(G/P_\sigma;V_\tau)=\{f\in\mathcal{B}(G);\ f(gman)=a^{-d\tau-\rho}f(g)$$
$$\text{for any }m\in M,\ a\in A_{\mathfrak{p}}\text{ and }n\in N\}.$$

In Example 0, $Q_1=\{(ma_1n_1,ma_2\theta'(n_2));\ m\in M',\ a_j\in A_{\mathfrak{p}}'\text{ and }n_j\in N'\}$ and $\Pi\simeq\widehat{M'}$. Then $\mathcal{B}(G/P_\sigma;V_\tau)$ is the space of hyperfunction-valued sections of

$U_\delta^\lambda \otimes U_{w^*\delta^{*\prime}}^{-w^*\lambda}$, where U_δ^λ is the usual non-unitary principal series parametrized by $\delta \in \widehat{M}'$ and $\lambda \in (\mathfrak{a}'_\mathfrak{p})^*_c$, δ^* is the contragredient representation and w^* is the element of $W(\mathfrak{a}'_\mathfrak{p})$ satisfying $w^*\Sigma(\mathfrak{a}'_\mathfrak{p})^+ = -\Sigma(\mathfrak{a}'_\mathfrak{p})^+$. In Example 4, the space of principal series equals that in Example 1. On the other hand, for example, if $X = SL(n,\mathbb{R})/SO(p,n-p)$, $r = n!/\bigl(p!(n-p)!\bigr)$.

Lemma 11. *Using the above notation, we have the G-isomorphism for each i:*

$$\bigoplus_{\substack{\tau\in\Pi_i\\ d'\tau=\nu}} \mathcal{B}(G/P_\sigma; V_\tau) \;\simeq\; \mathcal{B}(G/Q_i; L_\nu)$$
$$(F_\tau) \;\leftrightarrow\; f = \sum \langle F_\tau, u_\tau^* \rangle$$

In the above, $d'\tau = d\tau + \rho|_\mathfrak{t}$ and $F_\tau = p_\tau f$, where

$$p_\tau f = \chi_\tau(e)\int_{M(\sigma)} \chi_\tau(m) f(gm)dm \otimes u_\tau$$

with the character χ_τ of the representation π_τ.

Considering $\mathcal{M}_{w\nu} = \mathcal{M}_\nu$ for $w \in W(\mathfrak{j})$, we have $u = 0$ by Holmgren's theorem if a function u in $\mathcal{B}(X;\mathcal{M}_\nu)$ satisfies $\beta^1_{w\nu}(u) = 0$ for any w in $W(\mathfrak{j})$. Thus we have

Proposition 12. *Any irreducible representation of G realized in a subspace of $\mathcal{B}(X;\mathcal{M}_\nu)$ is isomorphic to a subrepresentation of a principal series representation for X.* (*Considering unitary representations and their matrix coefficients, we can prove* (cf. Example 0) *that any irreducible unitary representation of G is isomorphic to a subrepresentation of a usual non-unitary principal series of G.*)

6. Poisson kernels

In §4 we define an intertwining operator which maps the space of joint eigenfunction on X to the space of principal series for X. Since G/P_σ is compact, its inverse map is considered to be an integral transformation by certain kernel function, which should be a H-invariant section of principal series because of the G-equivariance.

Lemma 13. (cf. [4]) i) *$\bigcup_{i=1}^r H\bar{w}_i P_\sigma$ is a union of the open subsets $H\bar{w}_i P_\sigma$ and the union is dense in G.*
ii) $$H_i \cap M(\sigma)A_\sigma N_\sigma = M(\sigma)\cap H_i.$$

For $i = 1,\dots,r$, $\tau \in \Pi_i$ and $g \in G$ we put

$$h^i(\tau; g) = \begin{cases} a^{-\rho}\displaystyle\int_{M(\sigma)\cap H_i} \chi_\tau(m'ma)dm' & \text{if } g \in H\bar{w}_i maN_\sigma \\ \quad\text{with } m \in M(\sigma) \text{ and } a \in A_{\sigma'} & \\ 0 & \text{if } g \notin H\bar{w}_i P_\sigma \end{cases}$$

and call this a Poisson kernel.

Lemma 14. i) *If* $\operatorname{Re}\langle d\tau - \rho, \alpha\rangle > 0$ *for any $\alpha \in \Sigma(\mathfrak{a})^+$, $h^i(\tau; g)$ is a continuous function of $g \in G$. It is meromorphically extended for all $\tau \in \Pi_i$ as a distribution on G and defines an H-invariant section of $\mathcal{B}(G/Q_i; L_{d'\tau^*})$. By* Lemma 11 *this function corresponds to the element $h^i(\tau; g) \otimes u_\tau^*$ in $\mathcal{B}(G/P_\sigma; V_\tau^*)$. For simplicity we will omit $\otimes u_\tau^*$.*
ii) *The function $G \ni g \mapsto h^i(\tau; g^{-1})$ belongs to $\mathcal{B}(X;\mathcal{M}_{d'\tau})$.*

Definition 15. Partial Poisson transformations $\mathcal{P}_\tau^i$ and Poisson transformations $\mathcal{P}_\nu$ are G-equivaliant maps defined by

$$\begin{aligned} \mathcal{P}_\tau^i :\ \mathcal{B}(G/P_\sigma; V_\tau) \;&\to\; \mathcal{B}(X;\mathcal{M}_{d'\tau}) \\ f \;&\mapsto\; (\mathcal{P}_\tau^i f)(g) = \int_K \langle f(k), h^i(\tau; g^{-1}k)\rangle dk \end{aligned}$$

and

$$\begin{aligned}\mathcal{P}_\nu : \bigoplus_{i=1}^{r} \bigoplus_{\substack{\tau\in\Pi_i\\ d'\tau=\nu}} \mathcal{B}(G/P_\sigma; V_\tau) &\to \mathcal{B}(X;\mathcal{M}_\nu)\\ (f_i^\tau) &\mapsto \sum \mathcal{P}_\tau^i f_i^\tau .\end{aligned}$$

7. Integral representations of eigenfunctions

By a similar argument as in [7] we can prove

Theorem 16. i) *For a* ν *in* $\mathfrak{j}_c^*$, $\mathcal{B}(X;\mathcal{M}_\nu)\neq\{0\}$ *if and only if there exists a* w *in* $W(\mathfrak{j})$ *and a* τ *in* Π *such that* $w\nu = d'\tau$.
ii) *For a generic* ν *in* $\mathfrak{j}_c^*$ *which satisfies the condition* $\mathcal{B}(X;\mathcal{M}_\nu)\neq\{0\}$ (cf. [6] *for the precise assumption*) *there exists a* w *in* $W(\mathfrak{j})$ *such that* $\mathcal{P}_{w\nu}$ *is an onto isomorphism. Moreover we have*

$$\mathcal{P}_{w\nu}\Big(\bigoplus_{i=1}^{r} \bigoplus_{\substack{\tau\in\Pi_i\\ d'\tau=w\nu}} \mathcal{D}'(G/P_\sigma; V_\tau)\Big) = \mathcal{C}'_*(X;\mathcal{M}_\nu).$$

Here $\mathcal{D}'(G/P_\sigma;V_\tau)$ *denotes the space of distribution sections of the principal series,* $\mathcal{C}'_*(X;\mathcal{M}_\nu)=\mathcal{C}'_*(X)\cap\mathcal{B}(X;\mathcal{M}_\nu)$ *and* $\mathcal{C}'_*(X)$ *is the dual space of the Fréchet space*

$$\mathcal{C}_*(X)=\{f\in\mathcal{C}^\infty(X);\ \sup_{(k,Y)\in K\times\mathfrak{a}} |(Df)(k\exp Y)e^{j\langle Y,Y\rangle^{\frac12}}|<\infty \quad \text{for any } j\in\mathbb{Z} \text{ and } D\in U(\mathfrak{g})\}.$$

8. c-function

The map of taking the boundary values and the Poisson transformation are mutually inverse mappings up to constant multiple. Then we have

Definition 17. For $\tau\in\Pi$ we put $I(\tau)=\{i\in\{1,\dots,r\};\ \tau\in\Pi_i\}$ and

$$c(\tau)=\big(p_\tau\circ\beta_\tau^i\circ\mathcal{P}_\tau^j\big)_{i,j\in I(\tau)}.$$

We call $c(\tau)$ the c-function for X, which is a meromorphic function of $\tau\in\Pi$.

For the explicit calculation of the c-function the following i) $\sim$ iv) are important.

i) $c(\tau)$ is given by integral of a product of certain power of polynomial functions over $\theta(N_\sigma)$.
ii) By the technique due to Harish-Chandra, Gindikin-Karpelevič, Helgason and Shiffmann (cf. [7, §4]) we can prove that $c(\tau)$ is a product of c-function for semisimple symmetric spaces of split rank 1.

By i) and ii) we can reduce the calculation to the integrals $\int(1+x^2)^\lambda dx$ in Example 1, $\int|1\pm x^2|_\pm^\lambda dx$ in Example 4 (cf. [7, §4]) and $\int(1+z^2)^\lambda(1+\bar z^2)^{\lambda+n}dzd\bar z$ in Example 3. For the general cases we prepare the following:

We put

$$\begin{aligned}\mathfrak{g}^d &= \mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\mathfrak{p}\cap\mathfrak{q},\\ \mathfrak{h}^d &= \mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}),\\ \mathfrak{k}^d &= \mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}),\\ \mathfrak{g}(\sigma)^d &= [\mathfrak{m}(\sigma),\mathfrak{m}(\sigma)]\cap\mathfrak{h}+\sqrt{-1}([\mathfrak{m}(\sigma),\mathfrak{m}(\sigma)]\cap\mathfrak{q})\end{aligned}$$

and we denoted by G^σ, H^d, K^d and $G(\sigma)^d$ the analytic subgroups of G_c with the Lie algebras $\mathfrak{g}^\sigma$, $\mathfrak{h}^d$, $\mathfrak{k}^d$ and $\mathfrak{g}(\sigma)^d$, respectively. Then G^d/K^d and $G(\sigma)^d/G(\sigma)^d\cap H$ are Riemannian symmetric spaces of non-compact type. Denoting by $c_{G^d}^R$ (resp.

$c^R_{G(\sigma)^d}$) the c-function for G^d/K^d (resp. $G(\sigma)^d/(G(\sigma)^d\cap K)$ with the variables $\mathfrak{j}_c^*$ (resp. $\mathfrak{t}_c^*$), we put

$$c^G_{G^d}(\tau)=c^R_{G(\sigma)^d}(d'\tau|_{\mathfrak{t}})\chi_\tau(e)c^R_{G^d}(d'\tau)^{-1}c(\tau).$$

We remark that the c-functions for Riemannian symmetric spaces of non-compact type are well-known.

iii) $c^G_{G^d}(\tau)$ does not depend on the discrete parameter $d\tau|_{\mathfrak{t}}$.

iv) If the split rank of X equals one, K-invariant eigenfunctions in $\mathcal{B}(X;\mathcal{M}_{d'\tau})$ are expressed by Gauss' hypergeometric functions.

The facts i) and iii) assure that we have only to consider the cases when $\dim\mathfrak{a}=1$ and $d\tau|_{\mathfrak{t}}=0$. Then using ii) and iv), we have $c(\tau)$ by the connection formula for Gauss' hypergeometric functions. Thue $c^G_{G^d}$ is expressed in terms of trigonometric functions.

In Example 1, $G^d\simeq G$ and $c^G_{G^d}=c^R_{G(\sigma)^d}=\chi_\tau=1$.

In Example 2, $c^G_{G^d}=1$, $c^R_{G(\sigma)^d}=c^R_{G^d}$ and $c(\tau)=\chi_\tau(e)$.

In Example 3, $G^d\simeq G'\times G'$, $c^R_{G(\sigma)^d}=\chi_\tau=1$, $\Sigma(\mathfrak{a})\simeq\Sigma(\mathfrak{a}'_{\mathfrak{p}})$,

$$c^G_{G^d}=\prod_{\substack{\alpha\in\Sigma(\mathfrak{a})^+\\ \frac{\alpha}{2}\notin\Sigma(\mathfrak{a})^+}} I\Big(-\frac{\langle d'\tau,\tilde\alpha\rangle}{\langle\alpha,\alpha\rangle};m_\alpha,m_{2\alpha}\Big),$$

$$m'_\alpha=\dim\mathfrak{g}'(\mathfrak{a}'_{\mathfrak{p}};\alpha)$$

and

$$I(\lambda;m,n)=\frac{\Gamma(\frac{m+n}{2})\Gamma(\frac{m}{2}+n+1)\Gamma(\frac{n}{2}+1)\sin\frac{\pi}{2}\lambda}{\Gamma(m+n)\Gamma(n+1)2\cos\frac{\pi}{4}(\lambda+m)\sin\frac{\pi}{4}(\lambda+m+2n)},$$

where each $\tilde\alpha$ is a root in $\Sigma(\mathfrak{j})$ satisfying $\tilde\alpha|_{\mathfrak{a}}=\alpha$.

In Example 4, $c^G_{G^d}$ equals $E_\varepsilon A^\varepsilon_{w^*}$ which is defined and calculated in [7, §4].

In Example 0, $G^d\simeq G'_C$, $\Sigma(\mathfrak{a})\simeq\Sigma(\mathfrak{a}'_{\mathfrak{p}})$ and if $\tau|_{K\cap\sqrt{-1}\mathfrak{a}_{\mathfrak{p}}}=1$,

$$c^G_{G^d}=\prod_{\substack{\alpha\in\Sigma(\mathfrak{a})^+\\ \frac{\alpha}{2}\notin\Sigma(\mathfrak{a})^+}} I\Big(\frac{\langle d'\tau,\tilde\alpha\rangle}{\langle\alpha,\alpha\rangle};m_\alpha,m_{2\alpha}\Big)^{-1},$$

where we use the same notation as in Example 3.

9. Fourier-Laplace transform and their inverse

Let $\mathcal{C}^\infty_o(X)$ denote the space of $\mathcal{C}^\infty$-functions on X with compact support. Then the Fourier-Laplace transform $\mathcal{F}\psi$ or $\hat\psi$ of the function ψ in $\mathcal{C}^\infty_0(X)$ is given by

Definition 18. $$\hat\psi(i,\tau;g)=\int_X\psi(xH)h^i(\tau^*;x^{-1}g)d(xH)$$
for $i=1,\dots,r$, $\tau\in\Pi_i$ and $g\in G$.

Then $\hat\psi(i,\tau;\cdot)$ belongs to $\mathcal{B}(G/P_\sigma;V_\tau)$. Here the invariant measure $d(xH)$ is defined so that for $\phi\in\mathcal{C}^\infty_o(X)$

$$\int_X\phi(xH)d(xH)=\frac{r}{\#W(\mathfrak{a})}\int_{K\times\mathfrak{a}}\phi(k\exp Y\,H)\prod_{\alpha\in\Sigma(\mathfrak{a})^+}D(\alpha;\exp Y)dkDY,$$

$$D(\alpha;a)=|a^\alpha+a^{-\alpha}|^{m^-_\alpha}|a^\alpha-a^{-\alpha}|^{m^+_\alpha},$$

$$m^-_\alpha=\dim\mathfrak{g}(\mathfrak{a};\alpha)\cap\sqrt{-1}\mathfrak{g}^d,\quad m^+_\alpha=\dim\mathfrak{g}(\mathfrak{a};\alpha)\cap\mathfrak{g}^d$$

and the Euclid measure on $\mathfrak{a}$ is determined so that the Fourier inversion formula between functions on $\mathfrak{a}$ and $\mathfrak{a}^*$ holds without a multiplicative constant.

We put $\Pi_i^t = \{\tau|_{\tilde{J}\cap K};\, \tau \in \Pi_i\}$ and $\Pi^t = \bigcup_{i=1}^r \Pi_i^t$. By the bijection $\tau \mapsto (\tau|_{\tilde{J}\cap K}, d\tau|_{\mathfrak{a}})$ of Π_i onto $\Pi_i^t \times \mathfrak{a}_c^*$ we identify Π_i and $\Pi_i^t \times \mathfrak{a}_c^*$. When $\delta = \tau|_{\tilde{J}\cap K}$ and $\lambda = d\tau|_{\mathfrak{a}}$, we write $\mathcal{P}^i_{\delta,\lambda}$, $I(\delta)$, $c(\delta,\lambda)$, $c(\delta,\lambda)$, $\hat{\psi}(i,\delta,\lambda;g)$, etc. instead of $\mathcal{P}^i_\tau$, $I(\tau)$, $c(\tau)$, $\hat{\psi}(i,j;g)$, etc., respectively. Then the inversion formula is expected to be something like

$$\psi \sim I_A(\psi) \equiv \frac{1}{\#W(\mathfrak{a})} \sum_{\delta\in\Pi^t} \mathrm{Tr} \int_A \Big(\mathcal{P}^i_{\delta,\sqrt{-1}\lambda} \hat{\psi}(j,\delta,\sqrt{-1}\lambda;g)\Big)_{i,j\in I(\delta)}$$
$$\times\, c(\delta,\sqrt{-1}\lambda)^{-1} c(\delta,-\sqrt{-1}\lambda)^{-1} d\lambda,$$

where Tr denotes the trace of $\#I(\delta)\times\#I(\delta)$-matrix.

Rosenberg [8] gives a proof of the Plancherel theorem for G/K. Here we will review the proof: Since the origin in G/K is K-invariant, we can assume ψ is K-invariant. For $R>0$, let B'_R denote the R-ball about 0 in $\mathfrak{a}$ and put $B_R = K\cdot\exp B'_R\cdot K$. Then a Paley-Wiener theorem says that $I_{\mathfrak{a}^*}(\psi)$ has support in B_R if ψ has support in B_R (which is proved as follows: Suppose ψ has support in B_R. Then the entire function $\hat{\psi}$ of λ has some estimate for its growth order when λ tends to infinity. Rewriting the integrand of $I_{\mathfrak{a}^*}(\psi)$ by the use of spherical functions and changing the path $\mathfrak{a}^*$ of the integral in $\mathfrak{a}_c^*$, we can prove $I_{\mathfrak{a}^*}(\psi)(x)=0$ by the estimate if $x\notin B_R$). Since the map $\psi\mapsto I_{\mathfrak{a}^*}(\psi)$ does not increase the support, $(I_{\mathfrak{a}^*}(\psi))(e) = (D\psi)(e)$ with a suitable differential operator D. Using an estimate for norms and the G-equivariance of the map and moreover considering ψ with support contained in a small neighborhood of the boundary, we can conclude D should be 1.

We want to apply this method to general cases. For the function ψ in $\mathcal{C}_o^\infty(X)$, $\hat{\psi}$ is meromorphic for λ. To have a Paley-Wiener theorem we change the path $\mathfrak{a}^*$ of the integral $I_{\mathfrak{a}^*}(\psi)$. Since the integrand is meromorphic, there appears poles for λ and thus by calculating the residues, $I_{\mathfrak{a}^*}(\psi)$ should be replaced by

$$I(\psi) = I_{\mathfrak{a}^*}(\psi) + \sum \mathrm{Res}.$$

If ψ is K-finite with respect to the left translations, $I(\psi)$ is well-defined because the number of poles are finite. To calculate the residues we use the facts that any K-finite eigenfunction of $\mathbb{D}(G/H)$ is uniquely corresponds to H^d-finite eigenfunction of $\mathbb{D}(G^d/K^d)$ (cf. [2]) and that the latter is known to be expressed by Poisson integral of its boundary value because G^d/K^d is a Riemannian symmetric space of non-compact type. Thus we prove a Paley-Wiener theorem by putting $B_R = K\cdot\exp B'_R\cdot H$. Moreover we prove that the map $\psi\mapsto I(\psi)$ of the space of K-finite functions in $\mathcal{C}_o^\infty(X)$ commutes with the left action of $\mathfrak{g}$. If we prove that $I(\psi)$ is well-defined for any $\psi\in C_o^\infty(X)$, (which is reduced to a problem on an analysis on a compact Lie group,) we can proceed in a similar way as in [8] by the following lemma.

Lemma 19. *The G_o^σ-orbits containing in the K-orbit of the origin eH in X consist of finite points.*

We have not yet succeeded in obtaining a general inversion formula but I believe the above procedure is possible for general cases.

Here we give inversion formulas for simplest cases.

In Example 1, $$\psi = \int_{(\mathfrak{a}_\mathfrak{p}^*)_+} \mathcal{P}_{i\lambda}\hat{\psi}(i\lambda)|c(i\lambda)|^{-2}d\lambda$$

and

$$\|\psi\|^2_{L^2(X)} = \int_{(\mathfrak{a}_\mathfrak{p}^*)_+} \|\hat{\psi}(i\lambda)|_K\|^2_{L^2(K)} \frac{d\lambda}{|c(i\lambda)|^2}.$$

In Example 3, $\displaystyle \psi = \sum_{\delta\in\widehat{\exp i\mathfrak{a}'_{\mathfrak{p}}}} \int_{(\mathfrak{a}^*_{\mathfrak{p}})_+} \mathcal{P}_{\delta,i\lambda}\hat{\psi}(\delta,i\lambda)|c(\delta,i\lambda)|^{-2}d\lambda$

and

$$\|\psi\|^2_{L^2(X)} = \sum_{\delta\in\widehat{\exp i\mathfrak{a}'_{\mathfrak{p}}}} \int_{(\mathfrak{a}^*_{\mathfrak{p}})_+} \|\hat{\psi}(\delta,i\lambda)\|^2_{L^2(G'_c/P_\sigma;V_{\delta,i\lambda})}\frac{d\lambda}{|c(\delta,i\lambda)|^2}.$$

When G' is complex semisimple, this coincides with the Plancherel theorem for $L^2(G')$.

If the rank of G/K_ε equals one in Example 4, we have

$$\psi = \sum_{n=1}^{r}\int_o^\infty \mathcal{P}^n_{i\lambda}\hat{\psi}(n,i\lambda)\frac{d\lambda}{|c(i\lambda)|^2} + 2\pi\sum_{j=1}^{\infty}\mathcal{P}_{(j)}A_j^{-1}\hat{\psi}_j$$

and

$$\|\psi\|^2_{L^2(G/K_\varepsilon)} = \sum_{n=1}^{r}\int_0^\infty \|\hat{\psi}(n,i\lambda)\|^2\frac{d\lambda}{|c(i\lambda)|^2} + 2\pi\sum_{i=1}^{\infty}(\hat{\psi}_j|_K, A_j^{-1}\hat{\psi}_j|_K)_{L^2(K)}$$

with

$$\begin{aligned}
\hat{\psi}_j &= \int_X P_{(j)}(x^{-1}g)\psi(xH)d(xH),\\
\mathcal{P}_{(j)}\phi &= \int_K \langle P_{(j)}(g^{-1}k),\phi(k)\rangle dk,\\
P_{(j)} &= \big(\operatorname*{Res}_{\lambda=j} h^n(\lambda;\mathfrak{g})c(\lambda)^{-1}\big)_{n=1,\ldots,r},\\
A_j &= \Big(A(\lambda,w^*)\Gamma\big(-\frac{\lambda}{2}\big)^{-1}\Big)_{\lambda=j}c(j)^{-1}\operatorname*{Res}_{\lambda=j}\Gamma\big(-\frac{\lambda}{2}\big)\Big(A^\varepsilon_{w^*}(\lambda)-(-1)^{\frac{\lambda}{2}}\Big).
\end{aligned}$$

Here by the simple root α in $\Sigma(\mathfrak{a})^+$ we identify λ with $-2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$. See [7] for other notation.

An extended result and the precise argument will appear in another paper.

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