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Riemann-Liouville transform and linear differential equations on the Riemann sphere

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Abstract. We study the Riemann-Liouville transformation of solutions to linear differential equations on the Riemann sphere. The transformation corresponds to the middle convolution of the equations. Under the transformation, we examine the asymptotic behavior of the solutions at the singular points of the equations. When the singular points are regular, we studied it in [**O1**] and solved a connection problem for the general rigid Fuchsian equations. In this paper we mainly study the case when some singular points are irregular.

1. Introduction

Classical special functions such as Gauss hypergeometric functions, Bessel functions and Whittaker functions are solutions to linear differential equations on the Riemann sphere and the analysis of the equations gives fundamental properties of the special functions. Our purpose is to analyze solutions to linear differential equations on the Riemann sphere as in the case of the classical special functions.

N. Katz [**Kz**] defined and studied the middle convolution and the addition of local systems. They are invertible transformations keeping the index of rigidity of the system. The index gives the number of accessory parameters which cannot be determined by the local structure of the system. The corresponding transformations of the solution are given by Riemann-Liouville integral and multiplication by a function. The equation is called rigid if there is no accessory parameter.

Any rigid Fuchsian equation is reduced to and constructed from the trivial equation $u' = 0$ by successive applications of additions and middle convolutions, which is proved by [**Kz**] for Fuchsian systems of the first order and by [**O1**] for scalar higher order linear ordinary differential equations together with the definition of the middle convolution of the equations. In this case the addition corresponds to the multiplication by the function $(x-c)^\lambda$ and a study of the Riemann-Liouville integral in [**O1**] gives a global information of the solution such as a connection problem describing the relation between analytic continuations of local solutions.

We will generalize some results in [**O1**] when an equation has unramified irregular singularities. In particular, we study the Riemann-Liouville integral of the

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solutions and clarify the change of the asymptotic behavior of the solutions under the integral transform at the singular points. Note that the rigid single equation without ramified irregular singularities is reduced to or constructed from the trivial equation by successive applications of additions and middle convolutions (cf. [**O6**]). These additions correspond to the multiplications by any functions ϕ such that $\frac{\phi'}{\phi}$ are rational functions.

Here we give some standard notation used in this paper. We mainly consider functions of one variable x and then $\mathbb{C}[x]$, $\mathbb{C}(x)$, $\mathbb{C}[\![x]\!]$ and $W[x]$ are the space of polynomials, the space of rational functions, the space of formal power series and the ring of linear differential operators with polynomial coefficients, respectively. Then $W[x]$ is generated by x and $\frac{d}{dx}$ with the relation $\frac{d}{dx}x = x\frac{d}{dx} + 1$. Moreover we put $W(x) := \mathbb{C}(x) \otimes W[x]$.

Let $\mathrm{C}^m(a,b)$ denote the space of functions f on the interval $(a,b) \subset \mathbb{R}$ such that $f^{(j)}$ exist and define continuous functions on (a,b) for $j = 0,\dots,m$. The space $\mathrm{C}^m(a,b]$ is similarly defined. We also use the notation

$$\begin{aligned}
&\partial = \tfrac{d}{dx}, \quad \vartheta = x\partial,\\
&\operatorname{Re} z = a,\ \operatorname{Im} z = b \quad (z = a+bi,\ a,\, b \in \mathbb{R}),\\
&\mathbb{Z}_{>0} = \{k \in \mathbb{Z} \mid k > 0\},\\
&(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1) \quad (\lambda \in \mathbb{C},\, m \in \mathbb{Z}_{>0}) \text{ and } (\lambda)_0 = 1,\\
&{}_mF_k(\alpha_1,\dots,\alpha_m;\beta_1,\dots,\beta_k;x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_m)_n}{(\beta_1)_n\dots(\beta_k)_n(1)_n} x^n \in \mathbb{C}[\![x]\!],\\
&F(\alpha,\beta;\gamma;x) = {}_2F_1(\alpha,\beta;\gamma;x).
\end{aligned}$$

Under this notation the addition $\mathrm{Ad}(\phi)$ of the equation $Pu = 0$ corresponds to the map $u \mapsto \phi u$ and then $x \mapsto x$ and $\partial \mapsto \partial - \frac{\phi'}{\phi}$ in $W(x)$. And the middle convolution mc_μ with $\mu \in \mathbb{C}$ corresponds to the map $u \mapsto \partial^{-\mu}u$ and then $\vartheta \mapsto \vartheta - \mu$ and $\partial \mapsto \partial$ in $W[x]$. We can define $\mathrm{Ad}(\partial^{-\mu})$ in the ring $\mathbb{C}[\partial^{-1}] \otimes W[x]$, which is isomorphic to the ring $W(x)$ by the correspondence $(x,\partial) \mapsto (-\partial, x)$.

In §2 Riemann-Liouville transform of a function $u(x)$ is defined by a regularization of the Riemann-Liouville integral

$$(I_a^\mu u)(x) = \frac{1}{\Gamma(\mu)} \int_a^x u(t)(x-t)^{\mu-1}dt$$

and we examine its fundamental properties. In §3 we prove Theorem 3.1 and Theorem 3.7 which describe the asymptotic of Riemann-Liouville transform $I_a^\mu u$ of a function $u(x)$ in terms of the asymptotic of $u(x)$. Namely, when a and b are singularities of $u(x)$, we examine the asymptotic of $(I_a^\mu u)(x)$ when $x \to a$ or $x \to b$. We show that the equation $Pu = 0$ implies $\bigl(\mathrm{mc}_\mu(P)\bigr)(I_a^\mu u) = 0$ in §4. In §5 we review the linear differential equations on the Riemann sphere. Important concepts are characteristic exponents, a generalized Riemann scheme, index of rigidity and middle convolution of the equation. In §6 we give some examples to understand the results in this paper.

Using versal additions introduced in [**O1**, §2.3], we define and construct in [**O6**] a versal unfolding of any rigid differential equation on $P^1_{\mathbb{C}}$ whose irregular singularities are unramified. Hence the equation is naturally realized as a confluent limit of rigid Fuchsian equations and we can study the original equation by the

Fuchsian equations and therefore in this paper we also consider regular singular points in parallel. Moreover in [**O5**] we regard the singular points as new variables and then the versal differential equation can be extended to a versal KZ equation with several variables, which will be useful to study the confluence of singularities.

2. Definition of Riemann-Liouville transform

In this section we define Riemann-Liouville transform by a regularization of Riemann-Liouville integral and show fundamental properties of the transform. We will apply the transform to solutions to linear differential equations with irregular singularities. Other regularizations are given in [**O1**, p.5] and [**Ha**, §2] mainly for the purpose to study Fuchsian differential equations.

Definition 2.1. For $\mu \in \mathbb{C}$ and $a \in \mathbb{C}$ the Riemann-Liouville transform (Euler transformation) I_a^μ of a holomorphic function $u(x)$ is defined by the complex integral

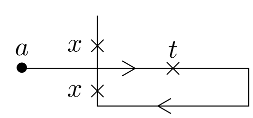
$$(I^\mu_{a+e^{i\theta_0}0}u)(x) := \frac{1}{\Gamma(\mu)}\int_a^x u(t)(x-t)^{\mu-1}dt = \frac{1}{\Gamma(\mu)}\int_{a+e^{i\theta_0}0}^x u(t)(x-t)^{\mu-1}dt \tag{2.1}$$

along a piecewise C^1-path

$$\begin{aligned}
&L\,:\,[\alpha,\beta]\ni t\mapsto L(t)\in\mathbb{C},\ L(\alpha)=a,\ L(\beta)=x,\ \theta_0=\operatorname{Arg}L'(\alpha),\\
&\alpha=t_0<t_1<\cdots<t_m=\beta \text{ and } L|_{[t_{j-1},t_j]}\in\mathrm{C}^1[t_{j-1},t_j]\quad(j=1,\dots,m),\\
&L(s)\neq L(t)\quad\text{ for }\alpha\le s<t\le\beta.
\end{aligned}\tag{2.2}$$

Sometimes the above $I^\mu_{a+e^{i\theta_0}0}$ will be denoted by I_a^μ and the resulting function $I_a^\mu u$ will be denoted by $I_a^\mu(u)$. We assume that $u(x)$ is holomorphic along the path $L(t)$ for $t\in(\alpha,\beta]$ and may have a singularity at the starting point $a = L(\alpha)$ of the path. When $u(x)$ is bounded along L and $\operatorname{Re}\mu > 0$, $(I_a^\mu u)(x)$ is naturally defined and holomorphic in a neighborhood of $L(t)$ for $t \in (\alpha,\beta]$.

Here $(I_a^\mu u)(L(t))$ is defined by the path $L|_{[\alpha,t]}$ and the condition $L(s)\neq L(t)$ for $\alpha\le s<t\le\beta$ is essential which assures that $(I_a^\mu u)(x)$ is holomorphic along the path L.



We also assume that L can be replaced by a piecewise linear path, namely,

$$L(t)=\tfrac{t_j-t}{t_j-t_{j-1}}L(t_{j-1})+\tfrac{t-t_{j-1}}{t_j-t_{j-1}}L(t_j)\text{ for }t\in[t_{j-1},t_j]\text{ and }j=1,\dots,m. \tag{2.3}$$

We will study the case that $u(x)$ is a solution to a differential equation $Pu = 0$ with $P \in W[x]$ and a is a singular point of the equation. In this case (2.3) can be always assumed.

When a, $x\in\mathbb{R}$ and $x < a$, we may consider

$$(\check I_a^\mu u)(x)=\frac{1}{\Gamma(\mu)}\int_x^a u(t)(t-x)^{\mu-1}dt,$$

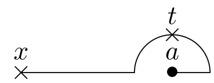
which has a clear meaning.

If we use the path L satisfying $\operatorname{Im}L(t)\ge 0$ in the case a, $x\in\mathbb{R}$, we have

$$\check I_a^\mu u = e^{-\mu\pi i}I_a^\mu u, \tag{2.4}$$

which follows from

$$\int_a^x u(t)(x-t)^{\mu-1}dt = \int_a^x u(t)\big(e^{\pi i}(t-x)\big)^{\mu-1}dt$$
$$= e^{\mu\pi i}\int_x^a u(t)(t-x)^{\mu-1}dt.$$



For $K>0$ we have

$$(I^\mu_{a+e^{i\theta_0}0}u)(Kx) = K^\mu I^\mu_{\frac{a}{K}+e^{i\theta_0}0}\big(u(Kx)\big). \tag{2.5}$$

EXAMPLE 2.2. First we consider a typical example :

$$u(x) = x^\lambda(c-x)^{\lambda'}e^{\frac{C'}{c-x}-\frac{C}{x}} \quad (c>0,\ \lambda,\,\lambda',\,C,\,C'\in\mathbb{C}), \tag{2.6}$$

$$v(x) := (I_0^\mu u)(x) = \frac{1}{\Gamma(\mu)}\int_0^x t^\lambda(c-t)^{\lambda'}e^{\frac{C'}{1-t}-\frac{C}{t}}(x-t)^{\mu-1}dt. \tag{2.7}$$

If $\operatorname{Re}\mu>0$ and one of the conditions

$$\operatorname{Re}C>0 \tag{2.8}$$

or

$$C=0 \text{ and } \operatorname{Re}\lambda>-1 \tag{2.9}$$

is valid, then $(I_0^\mu u)(x)$ is naturally defined for $x\in[0,c)$ and gives a holomorphic function on $\mathbb{C}\setminus\Big((-\infty,0)\cup[c,\infty)\Big)$. By the argument in this section we see that the condition $\operatorname{Re}\mu>0$ can be removed and the condition for λ in (2.9) is replaced by $\lambda\notin\mathbb{Z}_{<0}$. In fact $u(x)$ holomorphically depends on $(\lambda,\mu)\in(\mathbb{C}\setminus\mathbb{Z}_{<0})\times\mathbb{C}$.

If the path L satisfies $\operatorname{Re}\frac{C}{L'(\alpha)}>0$, we can define $v(x)$. In particular if $\operatorname{Re}C<0$, we may take $-L'(\alpha)>0$ and $v(x)$ is naturally a holomorphic function on $\mathbb{C}\setminus\big((-\infty,0]\cup[c,\infty)\big)$.

Since $\frac{u'(x)}{u(x)}=\frac{a_0(x)}{a_1(x)}$ with $a_0(x)$, $a_1(x)\in\mathbb{C}[x]$, $u(x)$ satisfies $Pu=0$ with $P=a_1(x)\partial-a_0(x)$. Here

$$\frac{u'(x)}{u(x)} = \frac{\lambda}{x}+\frac{C}{x^2}-\frac{\lambda'}{c-x}-\frac{C'}{(c-x)^2}. \tag{2.10}$$

Then $v=I_0^\mu u$ satisfies a differential equation $Qv=0$ with $Q=\mathrm{mc}_\mu(P)$, where $\mathrm{mc}_\mu(P)$ is given by (4.2) and (4.1). If $C=C'=0$, the differential equation $Qu=0$ is reduced to Gauss hypergeometric equation and

$$u(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}F(-\lambda',\lambda+1;\mu+1;x), \tag{2.11}$$

which is obtained by an easy calculation (cf. [**O1**, (1.47)] and (3.4)).

If $\lambda'=C'=0$, it is reduced to Kummer's equation by a simple transformation. If $CC'\neq0$, it is a confluence of Jordan-Pochhammer's equation of order 4.

We will examine the asymptotics of $(I_0^\mu u)(x)$ for $x\to+0$ and $x\to c-0$ and $x\to\infty$. In fact we show general theorems, which give these asymptotics.

Now we return to study $I_a^\mu u$ in the general case. First we assume that $u(x)$ is bounded on the path L and $\operatorname{Re}\mu>0$ and $\operatorname{Re}\mu'>0$. We will show

$$I_a^{\mu'}\circ I_a^\mu u = I_a^{\mu+\mu'}u. \tag{2.12}$$

Since $(I_a^{\mu'} \circ I_a^{\mu} u)(x)$ and $(I_a^{\mu+\mu'} u)(x)$ holomorphically depend on the variable x in the path L, it is sufficient to prove (2.12) for $x = L(t)$ for $t \in [t_0, t_1]$. Hence we may assume that $L(t) = \frac{t_1 - t}{t_1 - t_0} L(t_0) + \frac{t - t_0}{t_1 - t_0} L(t_1)$. By a change of the variable we may moreover assume that $a = 0$ and $x \in \mathbb{R}_{>0}$. Then

$$\begin{aligned}(I_0^{\mu'} \circ I_0^{\mu} u)(x) &= \frac{1}{\Gamma(\mu')} \int_0^x (x-t)^{\mu'-1} \Big(\frac{1}{\Gamma(\mu)} \int_0^t u(s)(t-s)^{\mu-1} ds \Big) dt \\ &= \frac{1}{\Gamma(\mu)\Gamma(\mu')} \int_0^x u(s) \int_s^x (x-t)^{\mu'-1}(t-s)^{\mu-1} dt\, ds,\end{aligned}$$

$$\begin{aligned}\int_s^x (x-t)^{\mu-1}(t-s)^{\mu'-1} dt &= \int_0^{x-s} (x-s-t)^{\mu'-1} t^{\mu-1} dt \\ &= (x-s)^{\mu+\mu'-1} \int_0^1 (1-t_1)^{\mu'-1} t_1^{\mu-1} dt_1 \quad (t = (x-s)t_1) \\ &= (x-s)^{\mu+\mu'-1} \frac{\Gamma(\mu)\Gamma(\mu')}{\Gamma(\mu+\mu')}\end{aligned}$$

and therefore we have (2.1).

By the same reason as above we may assume $a = 0$ and $x \in (0, c)$ with $c > 0$ and $L = [0, x]$ to prove several relations satisfied by $I_a^{\mu} u$.

Put $C_0^m[0,c) := \{u \in C^m[0,c) \mid (\partial^j u)(0) = 0 \quad (0 \le j \le m)\}$ for $m \in \mathbb{Z}_{\ge 0}$. If $u \in C_0^m[0,c)$ and $\operatorname{Re} \mu > 0$,

$$\int_a^x u(t)(x-t)^{\mu-1} dt = \int_a^x u(x-t) t^{\mu-1} dt$$

and $I_0^{\mu}(u) \in C_0^m[0,c)$. Here we note that $(I_0^{\mu} u)(x)$ holomorphically depends on μ.

Suppose $u \in C_0^1[0,c)$. Since

$$\begin{aligned}\tfrac{d}{dt}(u(t)(x-t)^{\mu}) &= u'(t)(x-t)^{\mu} - \mu u(t)(x-t)^{\mu-1} \\ &= u'(t)(x-t)^{\mu} - \tfrac{d}{dx}(u(t)(x-t)^{\mu}) \\ &= xu'(t)(x-t)^{\mu-1} - tu'(t)(x-t)^{\mu-1} - \mu u(t)(x-t)^{\mu-1}\end{aligned}$$

we have

$$I_0^{\mu}(u) = I_0^{\mu+1}(u') = \partial I_0^{\mu+1}(u)$$

for $\operatorname{Re} \mu > 0$ and

$$I_0^{\mu}(\vartheta u) = (\vartheta - \mu) I_0^{\mu}(u) \tag{2.13}$$

for $\operatorname{Re} \mu > 1$.

Hence for $u \in C_0^m[0,c)$ with $m \in \mathbb{Z}_{\ge 0}$ we have

$$I_0^{\mu}(u) = \partial^m I_0^{\mu+m}(u) = I_0^{\mu+m}(\partial^m u) \tag{2.14}$$

and define $I_0^{\mu}(u)$ for $\operatorname{Re} \mu > -m$ by this equality. In particular

$$I_0^0(u) = I_0^1(\partial u) = u$$

for $u \in C_0^1[0,c)$. If $0 < \operatorname{Re} \mu < m$ and $u \in C_0^m[0,c)$,

$$\begin{aligned}I_0^{-\mu} I_0^{\mu}(u) &= I_0^{m-\mu} \partial^m I_0^{\mu}(u) = \partial^m I_0^{m-\mu} I_0^{\mu}(u) = \partial^m I_0^m(u) = u, \\ I_0^{\mu}\ I_0^{-\mu}(u) &= I_0^{\mu} \circ I_0^{m-\mu}(\partial^m u) = I_0^m(\partial^m u) = u,\end{aligned}$$

namely, if $|\mathrm{Re}\,\mu| < m$ and $u \in C_0^m[0,c)$, we have

$$I_0^{-\mu} \circ I_0^{\mu}(u) = I_0^0(u) = u. \tag{2.15}$$

Let $k \in \mathbb{Z}_{\ge 0}$. If $\mathrm{Re}\,\lambda > m$ and $\phi(x) \in C^\infty[0,c)$, then $\phi(x)x^\lambda \log^k x \in C_0^m[0,x)$ and we may apply I_0^λ to u. In particular

$$I_0^\mu(x^\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu} \tag{2.16}$$

if $\mathrm{Re}\,\lambda > -1$ and $\mathrm{Re}\,\mu > 0$. Applying $\frac{d^k}{d\lambda^k}$ to the both side of the equation, we have

$$I_0^\mu(x^\lambda \log^k x) = \sum_{j=0}^k \binom{k}{j} \left(\frac{d^{k-j}}{d\lambda^{k-j}} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} \right) x^{\lambda+\mu} \log^j x. \tag{2.17}$$

For $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ The right hand side of the above is holomorphic for $(\lambda,\mu) \in (\mathbb{C}\setminus\mathbb{Z}_{<0}) \times \mathbb{C}$ and we define $I_0^\mu(x^\lambda \log^k x)$ by this equality. Owing to this definition, we have (2.13) and (2.14) for $u(x) = x^\lambda \log^k x$ if $\lambda \notin \mathbb{Z}_{<0}$. Moreover we have (2.12) and (2.15), namely,

$$I_0^{\mu'} \circ I_0^\mu(u) = I_0^{\mu+\mu'}(u) \quad \text{and} \quad I_0^0(u) = u \tag{2.18}$$

for $x^\lambda \log^k x$ if $\lambda \notin \mathbb{Z}_{<0}$ and $\lambda + \mu \notin \mathbb{Z}_{<0}$.

We examine $I_0^\mu(u)$ for the solution u to a differential equation with a singularity at the origin. If the origin is a regular singular point of the equation, the solutions

$$u(x) = \sum_{j=0}^k \phi_j(x) x^\lambda \log^j x \tag{2.19}$$

with $\lambda \in \mathbb{C}$ and suitable holomorphic functions $\phi_j(x)$ at the origin span a space of the local solutions. We may assume that c is also a singular point and $x \in (0,c)$ are regular point. Then $u(x) \in \mathrm{C}_0^m[0,c)$ if $\mathrm{Re}\,\lambda > m$. For any given $m \in \mathbb{Z}_{>0}$, $u(x)$ in (2.19) is a finite sum of the functions $C_{i,j}x^{\lambda+i}\log^j x$ with $i \in \mathbb{Z}_{\ge 0}$ and a function in $\mathrm{C}_0^m[0,c)$ and therefore if $\mathrm{Re}\,\lambda \notin \mathbb{Z}$, $I_0^\mu(u)$ is naturally defined. As a consequence, $I_0^\mu(x^\tau u(x))$ is defined when $\mathrm{Re}\,\tau$ and $\mathrm{Re}\,\mu$ are sufficiently large and $I_0^\mu(x^\tau u(x))$ extends holomorphically for $(\tau,\mu) \in \{\tau \in \mathbb{C} \mid \tau + \lambda \notin \mathbb{Z}_{<0}\} \times \mathbb{C}$ and satisfies (2.13) and (2.14). Moreover if $\lambda + \mu \notin \mathbb{Z}_{<0}$, we have (2.18). Thus we have the following proposition.

Proposition 2.3. *Suppose $u(x)$ is of the form* (2.19) *with convergent series $\phi_j(x)$ and $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, Then $I_0^\mu(u)$ is naturally defined for any $\mu \in \mathbb{C}$ and μ is a holomorphic parameter of $I_0^\mu(u)$ and we have* (2.13) *and* (2.14). *Moreover we have* (2.18) *if $\lambda + \mu \notin \mathbb{Z}_{<0}$.*

If the origin is an unramified irregular singular point of the equation, the local solutions $u(x)$ with the asymptotic expansion

$$\begin{gathered} u(x) \sim \sum_{j=0}^k \phi_j(x) x^\lambda \log^j x \cdot \exp(-\tfrac{C_1}{x^{m_1}} - \tfrac{C_2}{x^{m_2}} - \cdots - \tfrac{C_K}{x^{m_K}}) \text{ for } x \to e^{i\theta_0}0, \\ \phi_j \in \mathbb{C}[\![x]\!],\ C_j \in \mathbb{C} \text{ and } m_j \in \mathbb{Z} \text{ with } m_1 > m_2 > \cdots > m_K > 0 \end{gathered} \tag{2.20}$$

span the space of the local solutions. Here the condition $x \to e^{i\theta_0}0$ means that

$$V_{\theta_0,\epsilon,r} := \{se^{i\theta} \in \mathbb{C} \mid 0 < s < r,\ |\theta - \theta_0| \le \epsilon\} \ni x \to 0 \tag{2.21}$$

for suitable $\epsilon > 0$ and $r > 0$. Namely if $\phi_j(x) = a_{j,0} + a_{j,1}x + a_{j,2} + \cdots$, then

$$\lim_{r\to+0} \sup_{x\in V_{\theta_0,\epsilon,r}} |E_N(x)x^{-N}| = 0 \quad \text{for } N \in \mathbb{Z}_{\geq 0}, \text{ where}$$

$$E_N(x) = u(x)x^{-\lambda}\exp\bigl(\tfrac{C_1}{x^{m_1}} + \tfrac{C_2}{x^{m_2}} + \cdots + \tfrac{C_K}{x^{m_K}}\bigr) - \sum_{\nu=0}^{N}\sum_{j=0}^{k} a_{j,\nu}x^\nu \log^j x.$$

We define $\exp(-\frac{C_1}{x^{m_1}} - \frac{C_2}{x^{m_2}} - \cdots - \frac{C_K}{x^{m_K}}) = 1$ when $K = 0$. We note that $\partial^m u$ satisfies a similar asymptotic which is obtained by applying ∂^m to the right hand side of (2.20).

Moreover we note that if the origin is a ramified singular point, it changes into an unramified singular point under a new coordinate $y = x^{\frac{1}{q}}$ with a suitable $q \in \mathbb{Z}_{>0}$.

Theorem 2.4. *Put $\theta_0 = \operatorname{Arg} L'(\alpha)$ for the path (2.2) of the integration (2.1) to define $I_0^\mu u$. Suppose $u(x)$ has the asymptotic (2.20) with the condition*

$$K = 0 \text{ and } \lambda \in \mathbb{C} \setminus \mathbb{Z}_{<0} \tag{2.22}$$

or

$$\operatorname{Re} C_1 e^{-im_1\theta_0} > 0. \tag{2.23}$$

Then $I_0^\mu(u)$ is defined for any $\mu \in \mathbb{C}$ and μ is a holomorphic parameter of $I_0^\mu(u)$ and we have (2.13) and (2.14).

If the condition (2.22) together with $\lambda+\mu \notin \mathbb{Z}_{<0}$ is valid or the condition (2.23) is valid, (2.18) holds for any $\mu' \in \mathbb{C}$.

Proof. We may assume that $x = L(t)$ with $t \in [t_0, t_1]$ and $L(s) = \frac{s-\alpha}{t_1-\alpha}L(t_1)$ for $s \in [\alpha, t_1]$. Then under the new coordinate $y = e^{-i\theta_0}x$ we may assume $L|_{[\alpha,t_0]}$ equals $[0, c]$ with $c > 0$. Then the theorem is clear by the argument we have done. □

Remark 2.5. i) Suppose $K = 0$ in Theorem 2.4. Put $\tilde u = x^{\tilde\lambda-\lambda}u$. It follows from (2.17) that $\frac{1}{\Gamma(\tilde\lambda+1)}I_0^\mu \tilde u$ is well defined and holomorphic for $(\tilde\lambda, \mu) \in \mathbb{C}^2$. In this interpretation $\frac{1}{\Gamma(\lambda+1)}I_0^\mu u$ is well-defined even if $\lambda \in \mathbb{Z}_{<0}$.

ii) Since

$$I_a^{-n}(u) = \partial^n I_0^0(u) = \partial^n u,$$

we sometimes denote I_a^μ by $\partial^{-\mu}$.

Definition 2.6. We define a Riemann-Liouville integral $\tilde I^\mu_{e^{i\theta_0}\infty}$ of $u(x)$ by the complex integral

$$(\tilde I^\mu_{e^{i\theta_0}\infty}u)(x) := \frac{1}{\Gamma(\mu)}\int_x^\infty u(t)(t-x)^{\mu-1}dt = \frac{1}{\Gamma(\mu)}\int_x^{e^{i\theta_0}\infty} u(t)(t-x)^{\mu-1}dt$$

along a path

$$\begin{aligned}
&L : [\alpha, \infty) \ni t \mapsto L(t) \in \mathbb{C},\ L(\alpha) = x,\ \lim_{t\to\infty} |L(t)| = \infty,\ \lim_{t\to\infty} \operatorname{Arg} L(t) = \theta_0,\\
&\quad \alpha = t_0 < t_1 < t_2 < \cdots < t_m,\ L|_{[t_{j-1},t_j]} \in \mathrm{C}^1[t_{j-1}, t_j] \quad (j = 1, \ldots, m),\\
&\quad L|_{[t_m,\infty)} \in \mathrm{C}^1[t_m, \infty), \quad L(s) \neq L(t) \quad (\alpha \leq s < t).
\end{aligned}$$

Here $u(x)$ is holomorphic along L. For simplicity, we may denote $I^\mu_{e^{i\theta_0}\infty}$ by I_a^∞.

Suppose $u(x)$ has an asymptotic

$$u(x) \sim \sum_{j=0}^{k} \phi_j(\tfrac{1}{x}) x^{-\lambda} \log^j x \cdot \exp(-C_1 x^{m_1} - \cdots - C_k x^{m_K}) \quad \text{for } x \to e^{i\theta_0}\infty. \tag{2.24}$$

Here $m_j \in \mathbb{Z}$, $\phi_j \in \mathbb{C}[\![x]\!]$ and $m_1 > m_2 > \cdots > m_K > 0$ and the condition $x \to e^{i\theta_0}\infty$ is defined by $\frac{1}{x} \to e^{-i\theta_0}0$. Since

$$\begin{aligned}\int_x^\infty u(t)(t-x)^{\mu-1}dt &= \int_0^{\frac{1}{x}} u(\tfrac{1}{s})(\tfrac{1}{s} - \tfrac{1}{\frac{1}{x}})^{\mu-1}\tfrac{ds}{s^2}\\ &= x^{\mu-1}\int_0^{\frac{1}{x}} s^{-\mu-1}u(\tfrac{1}{s})(\tfrac{1}{x}-s)^{\mu-1}ds,\end{aligned}$$

we have

$$(\tilde{I}^\mu_{e^{i\theta_0}\infty}u)(x) = x^{\mu-1}\cdot\Bigl(I^\mu_{e^{-i\theta_0}0}x^{-\mu-1}u(\tfrac{1}{x})\Bigr)(\tfrac{1}{x}). \tag{2.25}$$

In particular

$$\tilde{I}^\mu_\infty(x^{-\lambda}) = x^{\mu-1}\bigl((I^\mu_0 x^{\lambda-\mu-1})\bigm|_{x\mapsto\frac{1}{x}}\bigr) = x^{\mu-1}\frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}x^{1-\lambda} = \frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}x^{-\lambda+\mu}.$$

Hence if

$$K = 0 \text{ and } \lambda-\mu \notin \mathbb{Z}_{\le 0} \tag{2.26}$$

or

$$\operatorname{Re} C_1 e^{im\theta_0} > 0 \tag{2.27}$$

is valid, $I^\mu_{e^{i\theta_0}\infty}u$ is defined and satisfies (2.13) and (2.14). Moreover we have (2.18) if the condition $\lambda+\mu+\mu' \notin \mathbb{Z}_{\le 0}$ and (2.26) are valid or the condition (2.27) is valid. In the same way we have

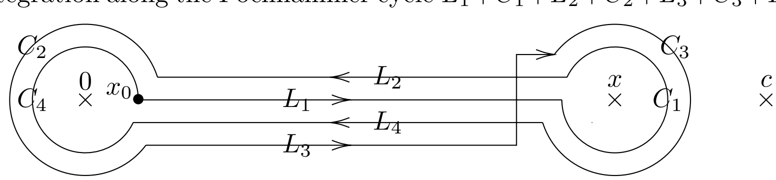
$$(I^\mu_c u)(x) = x^{\mu-1}\cdot\Bigl(\tilde{I}^\mu_{\frac{1}{c}}x^{-\mu-1}u(\tfrac{1}{x})\Bigr)(\tfrac{1}{x}) \tag{2.28}$$

for $x > 1$, which is useful to examine the asymptotic of $(I^\mu_c u)(x)$ for $x \to \infty$.

Remark 2.7. (Cf. [**O1**, p.5]) Let $Pu = 0$ be a differential equation with $P \in W[x]$ and $u_1, \ldots, u_n$ be linearly independent solutions to the equation in a neighborhood of a point $x_0 = \epsilon e^{i \operatorname{Arg} x}$. Here ϵ with $0 < \epsilon \ll 1$ and n is the order of P and $x \neq 0$ and we suppose that tx is not the singular point of the equation for $t \in (0,1)$. Then we define

$$(\hat{I}^\mu_0 u)(x) = \frac{1}{\Gamma(\mu)(1-e^{2\pi i\mu})}\int^{+x,+0,-x,-0} u(t)(t-x)^{\mu-1}dt$$

by the integration along the Pochhammer cycle $L_1+C_1+L_2+C_2+L_3+C_3+L_4+C_4$



We can define for any $\mu \in \mathbb{C}$ by the analytic continuation to $\mu \in \mathbb{Z}_{<0}$.

The analytic continuation along the path $[0, 2\pi] \ni t \to x_0 e^{i\theta t}$ induces transformations $u(x) \mapsto u^1(x) := u(e^{2\pi i}x)$ and $\mathbf{u} \mapsto \mathbf{u}A$. Here $A \in GL(n, \mathbb{C})$ is called the local monodromy matrix. This integration is used by [**DR**] to define a transformation of solutions to Fuchsian systems.

If $(I_0^\mu \mathbf{u})(x) = \int_0^x \mathbf{u}(t)(t-x)^{\mu-1}dt$ is naturally defined, then

$$\hat{I}_0^\mu u = I_0^\mu u - I_0^\mu u^1 \text{ and } \hat{I}_0^\mu \mathbf{u} = (I_0^\mu \mathbf{u})(1-A).$$

For example,

$$\hat{I}_0^\mu x^\lambda = \frac{(1-e^{2\pi i\lambda})\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu}.$$

Suppose $u(x)$ is a solution to the equation $Pu = 0$ with $P \in W[x]$. We may assume that the path has no intersection with the disc $D_\epsilon := \{x \in \mathbb{C} \mid |x| < \epsilon\}$. Then the above integration is defined even if there are several singular points of the equation in D_ϵ, which will be useful to examine the confluence of singular points. The monodromy around a circle enclosing several singular points is discussed in [**O4**].

3. Asymtotic of Riemann-Liouville integral

In this section we examine the asymptotic of holomorphic functions at singular points under the Riemann-Liouville transform so that the result can be applied to solutions to linear differential equations.

Theorem 3.1. *Assume a holomorphic function $u(x)$ has an asymptotic*

$$u(x) \sim \sum_{j=0}^k \phi_j(x) x^\lambda \log^j x \cdot \exp(-\tfrac{C_1}{x^{m_1}} - \tfrac{C_2}{x^{m_2}} - \cdots - \tfrac{C_K}{x^{m_K}}) \quad \text{for } x \to e^{i\theta_0}0 \tag{3.1}$$

with $\phi_j \in \mathbb{C}[\![x]\!]$, $\theta_0 \in \mathbb{R}$, k, $K \in \mathbb{Z}_{\geq 0}$, λ, $C_j \in \mathbb{C}$, $m_j \in \mathbb{Z}$ and $m_1 > m_2 > \cdots > m_K > 0$. Here $x \to e^{i\theta_0}0$ means $e^{-i\theta_0}x \to +0$.

To define $v(x) = (I_0^\mu u)(x)$ by (2.1) *we moreover assume the condition* (2.22)

$$K = 0 \text{ and } \lambda \notin \mathbb{Z}_{<0}$$

or the condition (2.23)

$$\operatorname{Re} C_1 e^{-m_1\theta_0} > 0 \qquad (\theta_0 := \operatorname{Arg} L'(\alpha)).$$

i) Then $v(x)$ has the asymptotic

$$v(x) \sim \sum_{j=0}^k \psi_j(x) x^{\lambda_0} \log^j x \cdot \exp\Big(-\tfrac{C_1}{x^{m_1}} - \tfrac{C_2}{x^{m_2}} - \cdots - \tfrac{C_K}{x^{m_K}}\Big) \quad \text{for } x \to e^{i\theta_0}0. \tag{3.2}$$

Here $\psi_j \in \mathbb{C}[\![x]\!]$ and $\lambda_0 \in \mathbb{C}$. Fix $k_0 \in \mathbb{Z}$ satisfying $\phi_j(0) = 0$ for $k_0 < j \leq k$ and $-1 \leq k_0 \leq k$. Then $\psi_j(0) = 0$ for $k_0 < j \leq k$.

(Case I) Assume (2.22). *Then*

$$\lambda_0 = \lambda + \mu, \quad \psi_{k_0}(0) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} \phi_{k_0}(0). \tag{3.3}$$

Moreover if $k = 0$ and $\phi_0(x) = a_0 + a_1 x + a_2 x + \cdots$, then

$$\psi_0 = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} \Big(a_0 + \frac{(\lambda+1)_1 a_1}{(\lambda+\mu+1)_1} x + \frac{(\lambda+1)_2 a_2}{(\lambda+\mu+1)_2} x^2 + \cdots\Big). \tag{3.4}$$

If ϕ_j are convergent power series and $u(x)$ equals the right hand side of (3.1), *then ψ_j are convergent power series and $v(x)$ equals the right hand side of* (3.2).

(Case II). Assume (2.23). *Then*

$$\lambda_0 = \lambda + (m_1+1)\mu \quad \text{and} \quad \psi_{k_0}(0) = (m_1C_1)^{-m_1\mu}\phi_{k_0}(0). \tag{3.5}$$

ii) Let $c \in \mathbb{C}$ and L be a piecewise C^1-path which satisfies

$$L : [\alpha, \beta] \ni t \to \mathbb{C} \text{ with } L(\alpha) = 0 \text{ and } L(\beta) = c.$$

For $x = L(t)$ with $\alpha \le t < \beta$ we define $I_0^\mu(u)(x)$ by the path $L|_{[\alpha,t]}$. We examine $u(x)$ and $(I_0^\mu u)(x)$ when $x = L(t)$ is close to c, namely, t is close to β. We define $\operatorname{Arg}\bigl(L(s_2) - L(s_1)\bigr)$ with $\alpha \le s_1 < s_2 \le \beta$, $\theta_0 = \operatorname{Arg} L'(\alpha)$, $\theta_1 = \operatorname{Arg} L'(\beta)$ and $(c-x)^\lambda$ so that they are compatible.

We note that $L(s_1) \ne L(s_2)$ but $L(\alpha)$ may equal $L(\beta)$. Assume

$$u(x) = \bigl(a' + o(1)\bigr)(c-x)^{\lambda'} \exp\Bigl(\tfrac{C_1'}{(c-x)^{m_1'}} + \cdots + \tfrac{C_{K'}}{(c-x)^{m'_{K'}}}\Bigr) \quad \text{for } x \to c - e^{i\theta_1}0.$$

Here a', $C_j' \in \mathbb{C}$, $m_j \in \mathbb{Z}$ and $m_0' > m_1' > \cdots > m'_{K'} > 0$.

(Case III) Assume $K' = 0$ and $\operatorname{Re}(\lambda' + \mu) < 0$. If $L(\alpha) = L(\beta)$ and $K = 0$, we moreover assume $\operatorname{Re}\lambda > \operatorname{Re}\lambda'$. Then

$$v(x) = \frac{\Gamma(-\lambda' - \mu)}{\Gamma(-\lambda')}\bigl(a' + o(1)\bigr)(c-x)^{\lambda'+\mu} \quad \text{for } x \to c - e^{i\theta_1}0.$$

(Case IV) Assume $\operatorname{Re} C_1' e^{-im_1'\theta_1} > 0$. Then

$$v(x) = (m_1'C_1')^{-m_1'\mu}\bigl(a' + o(1)\bigr)(c-x)^{\lambda + (m_0'+1)\mu} \exp\Bigl(\tfrac{C_1'}{(c-x)^{m_1'}} + \cdots + \tfrac{C_{K'}}{(c-x)^{m'_{K'}}}\Bigr).$$

For $s_0 \in (\alpha, \beta)$ we put $x_0 = L(s_0)$ and define $v_0(x) = (I_{x_0}^\mu u)(x)$ by the path $L|_{[x_0,t]}$ with $x = L(t)$. Then the asymptotic of $v_0(x)$ is same as that of $v(x)$ given in (Case III) and (Case IV).

(Case V) Assume $K' = 0$ and $\operatorname{Re}(\lambda' + \mu) > 0$ or $\operatorname{Re} C_1 e^{-im_1'\theta_1} < 0$. If $L(\alpha) = L(\beta)$ and $K = 0$, we moreover assume $\operatorname{Re}(\lambda + \mu) > 0$. Then

$$v(x) = \frac{1}{\Gamma(\mu)} \int_{e^{i\theta_0}0}^{c - e^{i\theta_1}0} u(t)(c-t)^{\mu-1}dt + o(1) \quad \text{for } x \to c - e^{i\theta_1}0.$$

Remark 3.2. Under the assumption in Theorem 3.1 (Case III) , we have $(\check I_c u)(x) = \frac{\Gamma(\lambda'+1)}{\Gamma(\lambda'+\mu+1)}(a' + o(c-x))(c-x)^{\lambda'+\mu}$ and therefore

$$I_0^\mu(u) = \frac{\sin(\lambda' + \mu)}{\sin\lambda'} \check I_c^\mu(u) + o\bigl((c-x)^{\lambda'+\mu}\bigr) \quad \text{for } x \to c - e^{i\theta_1}0.$$

Note that the connection coefficient between the Riemann-Liouville transforms of a function is written by trigonometric functions. This is commonly true because we can construct local solutions by successive applications of the transforms. See the examples (6.9) and (6.18).

Proof of Theorem 3.1. Since $I_0^\mu(u) = I_0^{\mu+m}(\partial^m u)$ with $m \in \mathbb{Z}_{\ge 0}$, we may assume $\operatorname{Re}\mu > 0$ to prove the theorem.

First suppose $K = 0$ and $\lambda \notin \mathbb{Z}$. Then the statement ii) follows from the value $T^\mu_{a,b}(u)(1)$ in [**O1**, Lemma 12.2] and the statement i) follows from (2.16) and the fact that $I_0^\mu(u) \in \mathrm{C}_0^m(0,1)$ if $u \in \mathrm{C}_0^m(0,1)$.

The statement i) in the theorem follows from [**O6**, Lemma 5.1]. We will show the statement (Case IV) in the theorem. The proof of other statements are easier.

Put $C_1' = C$, $m_1' = m$, $m_j = m_j'$ and $K' = K$ for simplicity. Suppose $\operatorname{Re} C > 0$. Let $\phi \in \mathrm{C}[0,1]$. Then

$$
\begin{aligned}
I(x) &:= \int_0^x (1-t)^\lambda e^{\frac{C}{(1-t)^m}+\frac{C_2}{(1-t)^{m_2}}+\cdots+\frac{C_K}{(1-t)^{m_K}}} (x-t)^{\mu-1}\phi(t)dt \\
&= x^\mu \int_0^1 (1-xs)^\lambda e^{\frac{C}{(1-xs)^m}+\frac{C_2}{(1-xs)^{m_2}}+\cdots} (1-s)^{\mu-1}\phi(xs)ds \quad (t=xs) \\
&= x^\mu \int_0^1 (1-x+xs_1)^\lambda e^{\frac{C}{(1-x+xs_1)^m}+\frac{C_2}{(1-x+xs_1)^{m_2}}+\cdots} s_1^{\mu-1}\phi(x-xs_1)ds_1 \\
&\qquad (s_1 = 1-s) \\
&= x^\mu(1-x)^\lambda \int_0^1 (1+\frac{xs_1}{1-x})^\lambda e^{\frac{C}{(1-x)^m(1+\frac{xs_1}{1-x})^m}+\frac{C_2}{(1-x)^{m_2}(1+\frac{xs_1}{1-x})^{m_2}}+\cdots} \\
&\qquad \cdot s_1^{\mu-1}\phi(x-xs_1)ds_1 \\
&= x^\mu(1-x)^{\lambda+\mu} \int_0^{\frac{1}{1-x}} (1+xs_2)^\lambda e^{\frac{C}{(1-x)^m(1+xs_2)^m}+\frac{C_2}{(1-x)^{m_2}(1+xs_2)^{m_2}}+\cdots} \\
&\quad \cdot s_2^{\mu-1}\phi\big(x-x(1-x)s_2\big)ds_2 \quad (s_1 = (1-x)s_2) \\
&= x^\mu(1-x)^{\lambda+\mu} e^{\frac{C}{(1-x)^m}+\frac{C_2}{(1-x)^{m_2}}+\cdots} \int_0^{\frac{1}{1-x}} (1+xs_2)^\lambda s_2^{\mu-1} \\
&\quad \cdot e^{\frac{C}{(1-x)^m}(\frac{1}{(1+xs_2)^m}-1)+\frac{C_2}{(1-x)^{m_2}}(\frac{1}{(1+xs_2)^{m_2}}-1)+\cdots}\phi\big(x-x(1-x)s_2\big)ds_2 \\
&\qquad (s_2 = (1-x)^m s_3) \\
&= x^\mu(1-x)^{\lambda+(m+1)\mu} e^{\frac{C}{(1-x)^m}+\frac{C_2}{(1-x)^{m_2}}+\cdots} \\
&\quad \cdot \int_0^{\frac{1}{(1-x)^{m+1}}} \big(1+x(1-x)^m s_3\big)^\lambda s_3^{\mu-1} e^{-f(x,(1-x)^m s_3)}\phi\big(x-x(1-x)^{m+1}s_3\big)ds_3
\end{aligned}
$$

with

$$
f(x,t) := \frac{C}{(1-x)^m}\left(1-\frac{1}{(1+xt)^m}\right)+\frac{C_2}{(1-x)^{m_2}}\left(1-\frac{1}{(1+xt)^{m_2}}\right)+\cdots.
$$

Suppose $1 \le s_2 \le \frac{1}{1-x}$, $\frac{1}{2} < x < 1$ and $m > 0$. Then $xs_2 > \frac{1}{2}$ and

$$
1-(\tfrac{2}{3})^m < 1-\frac{1}{(1+xs_2)^m} < 1.
$$

Hence putting $M = \max_{0\le t\le 1}|\phi(t)|$ and $A = \frac{1}{1-x}$, we have

$$
\begin{aligned}
&\lim_{x\to 1-0}\Big|\int_1^{\frac{1}{1-x}} (1+xs_2)^\lambda e^{\frac{C}{(1-x)^m}(\frac{1}{(1+xs_2)^m}-1)+\frac{C_2}{(1-x)^{m_2}}(\frac{1}{(1+xs_2)^{m_2}}-1)+\cdots} \\
&\quad \cdot\left(\frac{s_2}{(1-x)^m}\right)^{\mu-1} v\big(x-x(1-x)s_2\big)\frac{ds_2}{(1-x)^m}\Big| \\
&\le \lim_{A\to+\infty}(A-1)(1+A)^{|\lambda|}A^{(m+1)|\mu|+m}Me^{-\operatorname{Re}C(1-\frac{2^m}{3^m})A^m+|C_1|A^{m-1}+\cdots} = 0.
\end{aligned}
$$

Suppose $0 \le s_2 = (1-x)^m s_3 \le 1$, $\frac12 < x < 1$, $0 < m_2 < m$ and $t > 0$. Then

$$\begin{aligned}\frac{(1+xs_2)^m - 1}{(1-x)^m(1+xs_2)^m} &\ge \frac{mxs_2}{2^{m+1}(1-x)^m} = \frac{mxs_3}{2^{m+1}} \ge \frac{ms_3}{2^{m+2}},\\ \frac{\frac{1}{(1-x)^{m_2}}\bigl(1-\frac{1}{(1+t)^{m_2}}\bigr)}{\frac{1}{(1-x)^{m}}\bigl(1-\frac{1}{(1+t)^{m}}\bigr)} &= (1-x)^{m-m_2}\frac{(1+t)^m-(1+t)^{m-m_2}}{(1+t)^m-1}\\ &< (1-x)^{m-m_2} \xrightarrow{x\to 1-0} 0.\end{aligned}$$

Hence there exists $\delta > 0$ such that

$$|e^{-f(x,(1-x)^m s_3)}| \le e^{-\operatorname{Re} C \frac{ms_3}{2^{m+3}}} \quad \text{for } 0 \le s_3 \le \tfrac{1}{(1-x)^m} \text{ and } 1-\delta < x < 1.$$

Since

$$\begin{aligned}\lim_{x\to 1-0}\Bigl(&\frac{C}{(1-x)^m}\Bigl(1-\frac{1}{(1+x(1-x)^m s_3)^m}\Bigr)\\ &+\frac{C_2}{(1-x)^{m_2}}\Bigl(1-\frac{1}{(1+x(1-x)^m s_3)^{m_2}}\Bigr)+\cdots\Bigr) = mCs_3,\end{aligned}$$

Lebesgue's dominated convergence theorem proves

$$\begin{aligned}&\lim_{x\to 1-0}\int_0^{\frac{1}{(1-x)^{m+1}}}\bigl(1+x(1-x)^m s_3\bigr)^\lambda s_3^{\mu-1}e^{-f(x,(1-x)^m s_3)}\phi\bigl(x-x(1-x)^{m+1}s_3\bigr)ds_3\\ &=\lim_{x\to 1-0}\int_0^{\frac{1}{(1-x)^{m}}}\bigl(1+x(1-x)^m s_3\bigr)^\lambda s_3^{\mu-1}e^{-f(x,(1-x)^m s_3)}\phi\bigl(x-x(1-x)^{m+1}s_3\bigr)ds_3\\ &=\int_0^\infty s_3^{\mu-1}e^{-mCs_3}\phi(1)ds_3 = (mC)^{-\mu}\int_0^\infty s^{\mu-1}e^{-s}\phi(1)ds = (mC)^{-\mu}\Gamma(\mu)\phi(1),\end{aligned}$$

which implies

$$\frac{I(x)}{\Gamma(\mu)} = (mC)^{-\mu}\bigl(\phi(1)+o(1)\bigr)(1-x)^{\lambda+(m+1)\mu}e^{\frac{C}{(1-x)^m}+\frac{C_2}{(1-x)^{m_2}}+\cdots} \quad \text{for } x \to 1-0.$$

In the general case, we may assume that the path L is of the form (2.2) and (2.3). We may moreover assume that $x = L(t)$ with $t \in [t_{m-1}, t_m)$. Since we may assume $|(I_a^\mu u)(x) - (I^\mu_{L(t_{m-1}u)}(x)|$ is bounded for $x \to c - e^{i\theta_1}0$, we have only to examine the asymptotic $(I^\mu_{L(t_{m-1})}u)(x)$ when $x \to c - e^{i\theta_1}0$. Hence we have the statement (Case IV) from the estimate we have just proved . $\square$

Remark 3.3. i) The condition $\operatorname{Re}(\lambda'+\mu) < 0$ or $\operatorname{Re} C_1 e^{-im_0'\theta_1} > 0$ is essential for the claim ii) in the theorem, which assures that $\lim_{x\to c-e^{i\theta_1}0}|u(x)| = \infty$.

ii) If $\operatorname{Re}(\lambda'+\mu) > 0$ and $\lambda \notin \mathbb{Z}_{<0}$,

$$\bigl(I_0^\mu x^\lambda(1-x)^{\lambda'}\bigr)(1) = \frac{1}{\Gamma(\mu)}\int_0^1 x^\lambda(1-x)^{\lambda'+\mu-1}dx = \frac{\Gamma(\lambda+1)\Gamma(\lambda'+\mu)}{\Gamma(\mu)\Gamma(\lambda+\lambda'+\mu+1)}.$$

Example 3.4. Let $u(x)$ and $v(x)$ be the functions given in Example 2.2.

When $C = 0$, the series expansion of the function (3.4) in the theorem is obtained by that of the holomorphic function $(1-x)^{\lambda'}\exp(-\frac{C'}{1-x})$ at the origin. Hence if $C = C' = 0$, the series expansion

$$(1-x)^{\lambda'} = \sum_{n=0}^\infty \frac{(-\lambda')_n}{n!}x^n$$

shows

$$\begin{aligned}v(x) = I_0^\mu\bigl(x^\lambda(1-x)^{\lambda'}\bigr) &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}\sum_{n=0}^\infty \frac{(\lambda+1)_n(-\lambda')_n}{(\lambda+\mu+1)_n n!}x^n\\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}F(\lambda+1,-\lambda';\lambda+\mu+1;x).\end{aligned}$$

If $\operatorname{Re} C > 0$,

$$\lim_{x\to+0} v(x)x^{-\lambda-2\mu}e^{\frac{C}{x}} = C^{-\mu}.$$

If $C' = 0$ and $\operatorname{Re}(\lambda'+\mu) < 0$,

$$\lim_{x\to 1-0} v(x)(1-x)^{-\lambda'-\mu} = \frac{\Gamma(-\lambda'-\mu)}{\Gamma(-\lambda')}.$$

If $\operatorname{Re} C' > 0$,

$$\lim_{x\to 1-0} v(x)(1-x)^{-\lambda'-2\mu}e^{-\frac{C'}{1-x}} = (C')^{-\mu}.$$

Example 3.5. Here we give another example of the claim i) in the theorem. Assume $\operatorname{Re} C > 0$ and $\lambda' = C' = 0$ in Example 2.2. Then $v(x)$ is

$$\begin{aligned}(I_0^\mu x^\lambda e^{-\frac{C}{x}})(x) &= \frac{1}{\Gamma(\mu)}\int_0^x t^{\lambda+1}e^{-\frac{C}{t}}(x-t)^{\mu-1}\tfrac{dt}{t}\\
&= \frac{1}{\Gamma(\mu)}\int_1^\infty x^{\lambda+1}s_1^{-\lambda-1}e^{-C\frac{s_1}{x}}x^{\mu-1}(\tfrac{s_1-1}{s_1})^{\mu-1}\tfrac{ds_1}{s_1}\qquad (t=\tfrac{x}{s_1})\\
&= \frac{1}{\Gamma(\mu)}x^{\lambda+\mu}\int_1^\infty s_1^{-\lambda-\mu-1}(s_1-1)^{\mu-1}e^{-\frac{Cs_1}{x}}ds_1\\
&= \frac{1}{\Gamma(\mu)}x^{\lambda+\mu}e^{-\frac{c}{x}}\int_0^\infty s_1^\mu(s_1+1)^{-\lambda-\mu-1}e^{-\frac{c}{x}s_1}\tfrac{ds_1}{s_1}\\
&= \frac{1}{\Gamma(\mu)}x^{\lambda+2\mu}e^{-\frac{C}{x}}\int_0^\infty s_2^\mu(1+xs_2)^{-\lambda-\mu-1}e^{-Cs_2}\tfrac{ds_2}{s_2}\qquad (s_1 = xs_2)\\
&= \frac{1}{\Gamma(\mu)}x^{\lambda+2\mu}e^{-\frac{C}{x}}\int_0^\infty\sum_{n=0}^\infty(-x)^n s_2^{\mu+n}\frac{(\lambda+\mu+1)_n}{n!}e^{-Cs_2}\tfrac{ds_2}{s_2}\\
&= \frac{C^{-\mu}}{\Gamma(\mu)}x^{\lambda+2\mu}e^{-\frac{c}{x}}\int_0^\infty\sum_{n=0}^\infty(-\tfrac{x}{C})^n\frac{(\lambda+\mu+1)_n}{n!}s^{\mu+n}e^{-s}\tfrac{ds}{s}\\
&\qquad\Bigl(s = Cs_2,\ \int_0^\infty s^{\mu+n}e^{-s}\tfrac{ds}{s} = \Gamma(\mu+n)\Bigr)\\
&\sim C^{-\mu}x^{\lambda+2\mu}e^{-\frac{c}{x}}\Bigl(\sum_{n=0}^\infty\frac{(\mu)_n(\lambda+\mu+1)_n}{n!}(-\tfrac{x}{C})^n\Bigr)\qquad (x\to+0)\\
&= C^{-\mu}x^{\lambda+2\mu}e^{-\frac{c}{x}}{}_2F_0(\mu,\lambda+\mu+1;-\tfrac{x}{C}).\end{aligned}$$

Example 3.6. We give an example related the singular point ∞.

$$\begin{aligned}\tilde I_\infty^\mu(x^\lambda) &= x^{\mu-1}\bigl(I_0^\mu x^{-\mu-1-\lambda}\,|_{x\mapsto\frac{1}{x}}\bigr) = x^{\mu-1}\bigl(\tfrac{\Gamma(-\lambda-\mu)}{\Gamma(-\lambda)}x^{-1-\lambda}\,|_{x\mapsto\frac{1}{x}}\bigr)\\ &= \frac{\Gamma(-\lambda-\mu)}{\Gamma(-\lambda)}x^{\lambda+\mu}\qquad (x\to+\infty,\ \lambda+\mu\notin\mathbb{Z}_{\geq 0}).\end{aligned}\tag{3.6}$$

We give other examples on the asymptotic when $x \to \infty$. Owing to (2.25), we have

$$\begin{aligned}
\tilde I^\mu_\infty(x^\lambda e^{-x}) &= \frac{1}{\Gamma(\mu)}\int_x^\infty t^{\lambda+1}e^{-t}(t-x)^{\mu-1}\tfrac{dt}{t}\\
&= \frac{1}{\Gamma(\mu)}\int_1^\infty x^{\lambda+\mu}t^\lambda e^{-xt}(t-1)^{\mu-1}dt\\
&= \frac{x^{\lambda+\mu}e^{-x}}{\Gamma(\mu)}\int_0^\infty (t+1)^\lambda e^{-xt}t^{\mu-1}dt\\
&= \frac{x^\lambda e^{-x}}{\Gamma(\mu)}\int_0^\infty (1+\tfrac{t}{x})^\lambda e^{-t}t^{\mu-1}dt\\
&= \frac{x^\lambda e^{-x}}{\Gamma(\mu)}\int_0^\infty \sum_{n=0}^\infty \frac{(-\lambda)_n}{n!}(-\tfrac{t}{x})^n e^{-t}t^{\mu-1}dt,
\end{aligned}$$

$$\begin{aligned}
\tilde I^\mu_\infty(x^\lambda e^{-x}) &\sim x^\lambda e^{-x}\sum_{n=0}^\infty \frac{(-\lambda)_n(\mu)_n}{n!}(-x)^{-n}\\
&= x^\lambda e^{-x}{}_2F_0(-\lambda,\mu;-\tfrac{1}{x}) \quad (x\to+\infty).
\end{aligned} \tag{3.7}$$

This is also obtained from Example 3.5 and the relation (2.28).

If $\operatorname{Re} C_1 > 0$, then

$$\begin{aligned}
&\tilde I^\mu_{+\infty}(x^\lambda \exp(-C_1x^{m_1}-\cdots)) = x^{\mu-1}\bigl(I^\mu_{+0}(x^{-\mu-1}x^{-\lambda}\exp(-\tfrac{C_1}{x^{m_1}}-\cdots)\,|_{x\to\frac{1}{x}}\bigr)\\
&= x^{\mu-1}\bigl((m_1C_1)^{-\mu}\bigl(1+o(1)\bigr)x^{-\lambda-\mu-1+(m_1+1)\mu}\exp(-\tfrac{C_1}{x^{m_1}}-\cdots)\ \ (x\to+0)\,|_{x\to\frac{1}{x}}\bigr)\\
&= \bigl(1+o(1)\bigr)(m_1C_1)^{-\mu}x^{\lambda-(m_1-1)\mu}\exp(-C_1x^{m_1}-\cdots)\ \ (x\to+\infty).
\end{aligned}$$

In general we have the following theorem by Theorem 3.1 and the equalities (2.25) and (2.28).

Theorem 3.7. *Assume $u(x)$ has an asymptotic*

$$u(x) \sim \sum_{j=0}^k \phi_j(\tfrac{1}{x})x^{-\lambda}\log^j\tfrac{1}{x}\cdot\exp(-C_1x^{m_1}-C_2x^{m_2}-\cdots-C_Kx^{m_K}) \quad \text{for } x\to e^{i\theta_0}\infty \tag{3.8}$$

with $\phi_j\in\mathbb{C}[\![x]\!]$, $\lambda, C_j\in\mathbb{C}$, $\theta_0\in\mathbb{R}$, $m_j\in\mathbb{Z}$ and $m_1>m_2>\cdots>m_K>0$.

i) To define $v(x) = (\tilde I^\mu_{e^{i\theta_0}\infty}u)(x)$ we assume (2.26) or (2.27). Then $v(x)$ has the asymptotic

$$v(x) \sim \sum_{j=0}^k \psi_j(\tfrac{1}{x})x^{-\lambda_0}\log^j\tfrac{1}{x}\cdot\exp\Bigl(-C_1x^{m_1}-C_2x^{m_2}-\cdots-C_Kx^{m_K}\Bigr) \quad \text{for } x\to e^{i\theta_0}\infty. \tag{3.9}$$

Here $\psi_j\in\mathbb{C}[\![x]\!]$ and $\lambda_0\in\mathbb{C}$. Fix $k_0\in\mathbb{Z}$ satisfying $\phi_j(0)=0$ for $k_0<j\le k$ and $-1\le k_0\le k$. Then $\psi_j(0)=0$ for $k_0<j\le k$.

(Case I) Assume (2.22). Then

$$\lambda_0 = \lambda-\mu,\quad \psi_{k_0}(0) = \frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\phi_{k_0}(0). \tag{3.10}$$

Moreover if $k=0$ and $\phi_0(x)=a_0+a_1x+a_2x+\cdots$, then

$$\psi_0(\tfrac{1}{x})=\frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\Bigl(a_0+\frac{(\lambda-\mu)_1a_1}{(\lambda)_1}x^{-1}+\frac{(\lambda-\mu)_2a_2}{(\lambda)_2}x^{-2}+\cdots\Bigr). \tag{3.11}$$

If ϕ_j are convergent power series and $u(x)$ equals the right hand side of (3.8)*, then ψ_j are convergent power series and $v(x)$ equals the right hand side of* (3.9)*.*

(Case II). Assume (2.23)*. Then*

$$\lambda_0=\lambda-(m_1-1)\mu,\quad \psi_{k_0}(0)=(m_1C_1)^{-m_1\mu}\phi_{k_0}(0). \tag{3.12}$$

ii) Let $c\in\mathbb{C}$ and L be a piecewise C^1-path which satisfies

$$L:[\alpha,\infty]\ni t\to\mathbb{C} \text{ with } L(\alpha)=c \text{ and } \lim_{t\to\infty}\operatorname{Arg}L'(t)=\theta_0.$$

For $x=L(t)$ with $t\in(\alpha,\infty)$, we assume that $w(x)=(I_c^\mu u)(x)$ is defined by the path $L|_{[\alpha,t]}$.

(Case III) Assume $K=0$ and $\operatorname{Re}\lambda<1$. Then

$$w(x)=\frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\mu)}\bigl(a+o(1)\bigr)x^{-\lambda+\mu} \text{ for } x\to e^{i\theta_0}\infty.$$

(Case IV) Assume $\operatorname{Re}C_1e^{-im_1\theta_0}<0$. Then

$$w(x)=(-m_1C_1)^{-m_1\mu}\bigl(a+o(1)\bigr)x^{-\lambda-(m_1-1)\mu}\exp\bigl(-C_1x^{m_1}-\cdots-C_Kx^{m_K}\bigr)$$
$$\text{for } x\to e^{i\theta_0}\infty.$$

Remark 3.8. Suppose $k=0$ and $m_1=K=1$ in Theorem 3.7. Owing to Example 3.6, we have

$$\phi_0(x)=a_0+a_1x+\cdots,\quad \psi_0(x)=b_0+b_1x+\cdots$$
$$b_j=\sum_{\nu=0}^{j}a_{j-\nu}\frac{(\mu)_\nu(\lambda+\mu+1+j-\nu)_\nu}{(-C)^\nu\nu!}=\sum_{\nu=0}^{j}a_{j-\nu}\frac{(\mu)_\nu(-\lambda-\mu-j-1)_\nu}{C^\nu\nu!}.$$

4. Middle convolution of a differential operator

In this section we study the differential equation satisfied by the Riemann-Liouville transform of a solution to a linear differential equation on $P^1_{\mathbb{C}}$. The resulting equation is the middle convolution of the original equation.

For $P\in W[x]$ we put

$$\partial^KP=\partial^K\sum a_{i,j}x^i\partial^j=\sum_{i,j}c_{i,j}\partial^i\vartheta^j \tag{4.1}$$

Here we choose $K\in\mathbb{Z}_{\geq0}$ so that $K\geq i-j$ for any (i,j) with $a_{i,j}\neq0$. Then for $\mu\in\mathbb{C}$ we define the middle convolution $\mathrm{mc}_\mu(P)$ by

$$\mathrm{mc}_\mu(P)=\partial^{-N}\sum_{i,j}c_{i,j}\partial^i(\vartheta-\mu)^j\in W[x]. \tag{4.2}$$

Here $N\in\mathbb{Z}_{\geq0}$ is taken to be maximal so that $\sum_{i,j}c_{i,j}\partial^i(\vartheta-\mu)^j\in\partial^NW[x]$ for $P\neq0$. It is clear that $\mathrm{mc}_\mu(P)$ does not depend on the choice of K. Note that

$$\mathrm{mc}_{\mu'}\circ\mathrm{mc}_\mu(P)=\mathrm{mc}_{\mu+\mu'}(P) \text{ and } \mathrm{mc}_{-\mu}\circ\mathrm{mc}_\mu(P)=P \tag{4.3}$$

if $P\notin\partial W[x]$.

Remark 4.1. For $P \in W(x)$ we define $\mathrm{mc}_\mu(P) = \mathrm{mc}_\mu(\phi P)$ with $\phi \in \mathbb{C}(x)$ so that $\phi \neq 0$, $\phi P \in W[x]$ and $\deg_x \phi P$ is minimal. In [**O1**, 1.1] we define the map R of $W(x)$ to $W[x]$ by $\mathrm{R}P = \phi P$.

Theorem 4.2. *Let $P \in W[x]$ and $u(x)$ be a solution to $Pu = 0$ with the asymptotic expansion* (2.20). *If $m_j \in \mathbb{Z}_{>0}$ and*

$$\lambda \notin \mathbb{Z} \text{ and } \lambda + \mu \notin \mathbb{Z} \tag{4.4}$$

or

$$\operatorname{Re} C_1 e^{-im_1\theta_0} > 0 \tag{4.5}$$

in (2.20), *then*

$$\mathrm{mc}_\mu(P)(I_0^\mu u) = 0. \tag{4.6}$$

Proof. Let $Q \in W[x]$. Then Qu has an asymptotic expansion of the form (2.20) by changing λ and $\hat\phi(x)$ by λ_0 and $\hat\phi_0(x)$. Here $\lambda_0 = \lambda + \mu \notin \mathbb{Z}$ if (4.4) is not valid. Under the notation (4.1) and (4.2)

$$0 = I_0^\mu(\partial^K Pu) = I_0^\mu\Bigl(\sum c_{i,j}\partial^i\vartheta^j u\Bigr) = \sum c_{i,j}\partial^i(\vartheta-\mu)I_0^\mu(u) = \partial^N \mathrm{mc}_\mu(P)I_0^\mu(u).$$

Put $u_0 = \mathrm{mc}_\mu(P)I_0^\mu(u)$. The equation $\partial^N u_0 = 0$ means $u_0 \in \mathbb{C}[x]$. On the other hand, Theorem 3.1 implies

$$u_0 \sim e^{-\frac{C_1}{x^{m_1}}-\cdots-\frac{C_K}{x^{m_K}}}\sum x^{\lambda_0}\phi_j(x)\log^j x \quad (x \to e^{i\theta_0}0)$$

and therefore $u_0 = 0$. □

Remark 4.3. i) In the above theorem the condition $Pu = 0$ implies that $\mathrm{mc}_\mu(P)I_0^\mu(u)$ is a polynomial or 0 even without the condition $\lambda + \mu \notin \mathbb{Z}$, which follows from the above proof.

ii) Note that $\partial(x^2\partial + 1)u = 0$ with $u(x) = 1 + e^{\frac{1}{x}}$. Then $(x^2\partial + 1)u \neq 0$ and $u(x) \sim e^{\frac{1}{x}}$ for $x \to +0$ and $u(x) \sim 1$ for $x \to -0$.

We define

$$\begin{gathered} W_0[x]\partial^\mu := W[x] \otimes \mathbb{C}[\partial^{-1}] \otimes \partial^\mu, \\ \partial^{\mu'}\partial^{\mu'} = \partial^{\mu+\mu'} \text{ and } \partial^\mu x = x\partial^\mu + \mu\partial^{\mu-1}. \end{gathered}$$

Then

$$W_0[x]\partial^{\mu'} \times W_0[x]\partial^\mu \ni (P', P) \mapsto P'P \in W[x]_0\partial^{\mu+\mu'}$$

and the map

$$\mathrm{Ad}(\partial^\mu) : W_0[x] \ni P \mapsto \partial^\mu P \partial^{-\mu} \in W_0[x]$$

defines an endomorphism of the ring $W_0[x]$. Then $\mathrm{mc}_\mu(P) = \partial^{N'-\mu}P\partial^\mu$. Here N' is a minimal integer so that $\partial^{N'-\mu}P\partial^\mu \in W[x]$.

Remark 4.4. Replacing $\partial^{-\mu}$ by I_a^μ, $W_0[x]\partial^{-\mu}$ acts on a local singular solution $u(x)$ of a differential equation. Then the following calculation is valid.

$$Pu = 0 \;\Rightarrow\; P\partial^\mu\partial^{-\mu}u = 0 \;\Rightarrow\; (\partial^{N'-\mu}P\partial^\mu)(\partial^{-\mu}u) = 0 \;\Rightarrow \mathrm{mc}_\mu(P)I_a^\mu u = 0. \tag{4.7}$$

EXAMPLE 4.5. We calculate the equation $Pv = 0$ satisfied by $v(x) = (I_0^\mu u)(x)$ with $u(x) = x^\lambda e^{-\frac{C}{x}}$, $\operatorname{Re} C > 0$ and $x > 0$, which is the function in Example 2.2 with $\lambda' = C' = 0$. Namely, $v(x) = \frac{1}{\Gamma(\mu)} \int_0^x t^\lambda e^{-\frac{C}{t}} (x-t)^{\mu-1} dt$ satisfies $Pv = 0$. Since

$$\frac{u'(x)}{u(x)} = \frac{\lambda}{x} + \frac{C}{x^2} = \frac{\lambda x + C}{x^2},$$

we have

$$\begin{aligned}
P &= \mathrm{mc}_\mu \operatorname{Ad}\bigl(x^\lambda e^{-\frac{C}{x}}\bigr)\partial \\
&= \mathrm{mc}_\mu \bigl(x^2\partial - (\lambda x + C)\bigr) \\
&= \mathrm{mc}_\mu \bigl(\partial x^2 \partial - \partial(\lambda x + C)\bigr) \\
&= \mathrm{mc}_\mu \bigl(\partial x(x\partial - \lambda) - C\partial\bigr) \\
&= \mathrm{mc}_\mu \bigl((\vartheta + 1)(\vartheta - \lambda) - C\partial\bigr) \\
&= (\vartheta + 1 - \mu)(\vartheta - \lambda - \mu) - C\partial \\
&= \vartheta^2 - (\lambda + 2\mu - 1)\vartheta + (\mu + \lambda)(\mu - 1) - C\partial \\
&= x^2\partial^2 + x\partial - (\lambda - 2\mu - 1)x\partial + (\mu + \lambda)(\mu - 1) - C\partial \\
&= x^2\partial^2 - \bigl(C + (2\mu - \lambda - 2)x\bigr)\partial + (\mu + \lambda)(\mu - 1).
\end{aligned}$$

REMARK 4.6. To calculate mc_μ the following formula is useful

$$\begin{aligned}
x^n\partial^n &= \vartheta(\vartheta - 1)\cdots(\vartheta - n + 1), \\
\partial^n x^n &= (\vartheta + 1)(\vartheta + 2)\cdots(\vartheta + n), \\
\partial^m \vartheta^n &= (\vartheta + m)^n \partial^m, \\
\vartheta^2 &= x^2\partial^2 + x\partial,\ \vartheta^3 = x^3\partial^3 + 3x^2\partial^2 + x\partial, \\
\vartheta^4 &= x^4\partial^4 + 6x^3\partial^3 + 7x^2\partial^2 + x\partial.
\end{aligned} \tag{4.8}$$

This is easily checked by the action to the function x^λ which is an eigenvector to the operator ϑ. Calculations of $\mathrm{mc}_\mu(P)$, $\operatorname{Ad}(\phi)(P)$ and other related operations in $W(x)$ are realized by a computer program in [**O7**].

5. Differential equation on $P^1_{\mathbb{C}}$

In this section we review the Riemann scheme and the spectral type and the middle convolution mc_μ of a linear differential equation

$$Pu = 0 \tag{5.1}$$

on $P^1_{\mathbb{C}}$. See [**O6**] for the details.

Hereafter in this paper we assume that any singularity of the equation (5.1) is a regular singularity or an unramified irregular singularity. We will define a generalized Riemann scheme, which we denote by GRS in this paper, and a spectral type as in the case of Fuchsian differential equations defined by [**O1**].

Put

$$\lambda(x) = \lambda_0 + \lambda_1 x^{r_1} + \cdots + \lambda_m x^{r_m} \quad (0 < r_1 < \cdots < r_m).$$

Here $\lambda_j \in \mathbb{C}$. We define the *characteristic function* $e_\lambda(x)$ with the *exponent* $\lambda(x)$ by

$$e_\lambda(x) := x^{-\lambda_0} \exp\bigl(-\lambda_1 \tfrac{x^{r_1}}{r_1} - \cdots - \lambda_m \tfrac{x^{r_m}}{r_m}\bigr)$$

and put $\check e_\lambda(x) := e_\lambda(\frac{1}{x}) = x^{\lambda_0}\exp\bigl(-\frac{\lambda_1}{r_1x^{r_1}}-\cdots-\frac{\lambda_m}{r_mx^{r_m}}\bigr)$. Then

$$\begin{aligned}&\bigl(\vartheta+\lambda(x)\bigr)e_\lambda(x)=0,\quad \mathrm{Ad}\bigl(e_\lambda(x)\bigr)\vartheta=\vartheta+\lambda(x),\quad e_\lambda(x)e_\lambda(x)=e_{\lambda+\lambda}(x),\\&\bigl(\vartheta-\lambda(\tfrac{1}{x})\bigr)\check e_\lambda(x)=0,\quad \mathrm{Ad}\bigl(\check e_\lambda(x)\bigr)\vartheta=\vartheta-\lambda(\tfrac{1}{x}).\end{aligned}$$

Definition 5.1. Let $x=c$ be a singularity of the equation (5.1). For a polynomial $\lambda\in\mathbb{C}[x]$ and a positive integer m the equation has a *generalized characteristic exponent* $[\lambda]_{(m)}$ if the equation has formal solutions

$$u_\nu(y)=\check e_{\lambda+\nu}(y)+\psi_\nu(y)\check e_{\lambda+m}(y)\quad(\nu=0,\dots,m-1)$$

with $\psi_\nu\in\mathbb{C}[\![x]\!]_{(\infty)}$ and

$$\mathbb{C}[\![x]\!]_{(\infty)}=\bigoplus_{j=0}^{\infty}\mathbb{C}[\![x]\!]\log^j x.$$

When $m=1$, $[\lambda]_{(1)}$ is called a characteristic exponent of the equation and may be simply denoted by λ.

Here y is given by

$$y=\begin{cases}x-c & (c\neq\infty),\\ \frac{1}{x} & (c=\infty)\end{cases}\tag{5.2}$$

with $c=c_j$.

A *generalized Riemann scheme*, GRS in short, is the table

$$\begin{Bmatrix}x=c_0=\infty & \cdots & x=c_p\\ [\lambda_{0,1}]_{(m_{0,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})}\\ \vdots & \vdots & \vdots\\ [\lambda_{0,n_1}]_{(m_{0,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})}\end{Bmatrix}.\tag{5.3}$$

Here

$$n=m_{j,1}+\cdots+m_{j,n_j}\qquad(j=0,\dots,p)$$

are $(p+1)$ tuples of partitions of n. The Riemann scheme corresponding to (5.3) is given by putting

$$[\lambda]_{(m)}:=\begin{pmatrix}\lambda\\ \lambda+1\\ \vdots\\ \lambda+m-1\end{pmatrix}\ \text{ and }\ [\lambda]_m:=\begin{pmatrix}\lambda\\ \lambda\\ \vdots\\ \lambda\end{pmatrix}\in\mathbb{C}[x]^m.\tag{5.4}$$

Suppose

$$\lambda_{j,\nu'}-\lambda_{j,\nu}\notin\bigl\{0,1,\dots,m_{j,\nu}-1\bigr\}\quad(1\le\nu<\nu'\le n_j,\ j=0,\dots,p).\tag{5.5}$$

Then we define that P has GRS (5.3) if $[\lambda_{j,\nu}]_{(m_{j,\nu})}$ $(\nu=1,\dots,n_j)$ are generalized characteristic exponents at $x=c_j$ for $j=0,\dots,p$. (See the definition of GRS in [**O1**] when (5.5) is not valid.)

Remark 5.2. Suppose

$$\deg(\lambda_{j,\nu}-\lambda_{j,\nu'})>0\ \text{ or }\ \lambda_{j,\nu}-\lambda_{j,\nu'}\notin\mathbb{Z}\quad(1\le\nu<\nu'\le n_j,\ j=0,\dots,p).\tag{5.6}$$

Then P has GRS (5.3) if and only if P has the Riemann scheme corresponding to (5.3) and (5.1) has linearly independent solutions of the form $\psi(y)\check e_\lambda(y)$ with $\psi(x)\in\mathbb{C}[\![x]\!]$, namely, they have not any $\log y$ term.

Remark 5.3. In [**O1**] we use the notation

$$\begin{Bmatrix} x=c & (r_1) & \cdots & (r_m) \\ \lambda_0 & \lambda_1 & \cdots & \lambda_m \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} x=c \\ \lambda_0+\lambda_1x^{r_1}+\cdots+\lambda_m x^{r_m} \end{Bmatrix},$$

which implies the existence of a solution $u(x) \sim x^{-\lambda_0}\exp(-\frac{\lambda_1}{r_1}x^{r_1}-\cdots-\frac{\lambda_m}{r_m}x^{r_m})$ for $x\to\infty$ when $c=\infty$.

Let $\bigl\{[\lambda_{j,1}]_{(m_{j,1})},\ldots,[\lambda_{j,n_j}]_{(m_{j,n_j})}\bigr\}$ be the set of generalized exponents of P at the singular point c_j. Then $n=m_{j,1}+\cdots+m_{j,n_j}$ be a partition of $n=\operatorname{ord}P$. For $r\in\mathbb{Z}_{\ge 0}$ we define equivalence relations $\underset{j,r}{\sim}$ between the elements of $I_n:=\{1,\ldots,n\}$ as follows. For $i\in I_n$, we put $\nu_{j,i}\in\{1,\ldots,n_j\}$ by

$$m_{j,1}+\cdots+m_{j,\nu_{j,i}-1}<i\le m_{j,1}+\cdots+m_{j,\nu_{j,i}}$$

and define

$$i\underset{j,r}{\sim}i' \iff \begin{cases}\nu_{j,i}=\nu_{j,i'} & (r=0),\\ \deg\bigl(\lambda_{j,\nu_{j,i}}(x)-\lambda_{j,\nu_{j,i'}}(x)\bigr)<r & (r\ge 1).\end{cases} \tag{5.7}$$

Let $n_{j,r}$ be the number of equivalence classes under $\underset{j,r}{\sim}$ and let R_j be the Poincaré rank of P at the singular point c_j. Then

$$n_{j,0}=n_j\ge n_{j,1}\ge\cdots\ge n_{j,R_j}\ge n_{j,R_j+1}=1.$$

By a suitable permutation of the indices $\nu\in\{1,\ldots,n_j\}$ of $m_{j,\nu}$ we may assume

$$i\le i''\le i',\ i\underset{j,r}{\sim}i' \ \Rightarrow\ i\underset{j,r}{\sim}i''.$$

Let

$$n=m_{j,1}^{(r)}+\cdots+m_{j,n_{j,r}}^{(r)} \tag{5.8}$$

be the corresponding partition of n such that $m_{j,\nu}^{(0)}=m_{j,\nu}$ and for $\nu=1,\ldots,n_{j,r}$

$$I_{j,\nu}^{(r)}:=\{\nu\in\mathbb{Z}_{\ge 0}\mid m_{j,1}^{(r)}+\cdots+m_{j,\nu-1}^{(r)}<\nu\le m_{j,1}^{(r)}+\cdots+m_{j,\nu}^{(r)}\} \tag{5.9}$$

give the equivalence classes under $\underset{j,r}{\sim}$. Note that $\{I_{j,\nu}^{(r)}\mid\nu=1,\ldots,n_{j,r}\}$ is a refinement of $\{I_{j,\nu}^{(r+1)}\mid\nu=1,\ldots,n_{j,r+1}\}$. Then we define that the $(R_0+\cdots+R_p+p+1)$ tuples of partitions $\mathbf{m}=\bigl(\mathbf{m}_j^{(r)}\bigr)_{\substack{r=0,\ldots,R_j\\ j=0,\ldots,p}}=\bigl(m_{j,\nu}^{(r)}\bigr)_{\substack{\nu=1,\ldots,n_{j,r}\\ r=0,\ldots,R_j\\ j=0,\ldots,p}}$ of n is the *spectral type* of P and that of GRS (5.3). Then the number of full parameters of GRS (5.3) with the spectral type $\bigl(m_{j,\nu}^{(r)}\bigr)_{\substack{\nu=1,\ldots,n_{j,r}\\ r=0,\ldots,R_j\\ j=0,\ldots,p}}$ equals $R=\displaystyle\sum_{j=0}^{p}\sum_{r=0}^{R_j}n_{j,r}-1$. Here we note that we always impose the Fuchs-Hukuhara relation on GRS.

As in the case of Fuchsian differential equation, this spectral type is expressed by writing the numbers $m_{j,\nu}^{(r)}$. The numbers are separated by "," indicating different singular points and by "$|$" indicating different levels of the equivalence relations:

$$m_{0,1}^{(0)}m_{0,2}^{(0)}\cdots m_{0,n_{0,0}}^{(0)}|\cdots|m_{0,1}^{(R_1)}\cdots m_{0,n_{0,R_1}}^{(R_1)},m_{1,1}^{(0)}\cdots,\cdots m_{p,n_{p,R_p}}^{(R_p)}$$

The *index of the rigidity* of GRS (5.3) is defined by that of the tuples of the partitions $\big(m_{j,\nu}^{(r)}\big)$ (cf. [**O3, O1**]):

$$\begin{aligned}
\operatorname{idx}\{\lambda_{\mathbf m}\} &:= \operatorname{idx}\mathbf m = \operatorname{idx}\big(m_{j,\nu}^{(r)}\big) = 2n^2 - \sum_{j=0}^{p}\sum_{r=0}^{R_j}\Big(n^2 - \sum_{\nu=1}^{n_{j,r}}\big(m_{j,\nu}^{(r)}\big)^2\Big)\\
&= 2n^2 - \sum_{j=0}^{p}\Big(n^2 - \sum_{\nu=1}^{n_j} m_{j,\nu}^2\Big)\\
&\quad - \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\sum_{\nu'=1}^{n_j} m_{j,\nu}m_{j,\nu'}\deg\big(\lambda_{j,\nu}(x)-\lambda_{j,\nu'}(x)\big).
\end{aligned}\tag{5.10}$$

Put $\operatorname{ord}\mathbf m = m_{j,1}+\cdots+m_{j,n_j} = n$. As in the Fuchsian case (cf. [**O1**, Definition 4.17]), *the Fuchs-Hukuhara relation* is given by

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu}(0) = \operatorname{ord}\mathbf m - \tfrac12\operatorname{idx}\mathbf m. \tag{5.11}$$

The generalized Riemann scheme of $\mathrm{mc}_\mu(P)$ is given by the following theorem.

Theorem 5.4 ([**O1**, Theorem 5.2], [**Hi**, Theorem 3.2]). *Suppose $P\in W[x]$ has the generalized Riemann scheme $\big\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\big\}_{\substack{\nu=1,\dots,n_j\\ j=0,\dots,p}}$ given in (5.3) and it is irreducible in $W(x)$. We may assume $\lambda_{j,1}=0$ for $j=1,\dots,p$ and $\mu=\lambda_{0,1}-1$. Here some $m_{j,1}$ are allowed to be zero. If $\{\lambda_{j,\nu}\}$ and μ are generic (cf.* [**O1, Hi**]*) under this assumption, $\mathrm{mc}_\mu(P)$ has GRS $\big\{[\lambda'_{j,\nu}]_{(m'_{j,\nu})}\big\}_{\substack{\nu=1,\dots,n_j\\ j=0,\dots,p}}$ given by*

$$\begin{aligned}
&d(\mathbf m) := 2n - \sum_{j=0}^{p}\sum_{r=0}^{R_j}\big(n-m_{j,1}^{(r)}\big),\\
&m'_{j,\nu} = m_{j,\nu} - \delta_{\nu,1}\cdot d(\mathbf m) && (1\le\nu\le n_j,\ 0\le j\le p),\\
&\lambda'_{j,0} = \delta_{j,0}\cdot(1-\mu) && (j=0,\dots,p),\\
&\lambda'_{j,\nu} = \lambda_{j,\nu} + \big((-1)^{\delta_{j,0}} + \deg\lambda_{j,\nu}\big)\cdot\mu && (1\le\nu\le n_j,\ 0\le j\le p)
\end{aligned}$$

and the index of rigidity and the irreducibility of P are kept under mc_μ.

Remark 5.5. The above equality $\lambda'_{j,\nu} = \lambda_{j,\nu} + \big((-1)^{\delta_{j,0}} + \deg\lambda_{j,\nu}\big)\cdot\mu$ was wrongly written as $\lambda'_{j,\nu} = \lambda_{j,\nu} + (-1)^{\delta_{j,0}}\big(1+\deg\lambda_{j,\nu}\big)\cdot\mu$ in [**O6**, Theorem 5.3], which should be corrected as above. Similarly the equalities in [**O6**, Remark 5.2] should be

$$I_\mu x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}\quad (x>0),$$

$$\frac{1}{\Gamma(\mu)}\int_x^{\infty e^{i\theta_0}} u(t)(t-x)^{\mu-1}dt = \Big(\sum_{\nu=0}^{k} c'_\nu x^{-\nu} + o(x^{-k})\Big)x^{-\lambda-(m_0-1)\mu}\exp\Big(-\sum_{j=1}^{K} C_j x^{m_j}\Big)$$
$$(V'_{\theta_0,\theta_1,L}\ni x\to\infty).$$

Remark 5.6. We can construct a differential equation with GRS (5.3) in the following way.

We may assume $m_{j,1}\ge m_{j,\nu}$ for $\nu=1,\dots,n_j$ and $j=0,\dots,p$. Suppose the equation $Pu=0$ has this GRS. We apply suitable additions to this equation so that the resulting GRS satisfies $\lambda_{j,1}=0$ for $j=1,\dots,p$ and $\deg\lambda_{0,1}=0$. Then we apply $\mathrm{mc}_{\lambda_{1,0}-1}$ to this resulting equation. If the GRS is *rigid*, namely, the index of rigidity equals 2, then the order of the equation decreases by the application of this middle convolution (cf. [**O6**, Remark 5.4 ii)]). Repeating this procedure, we finally get the trivial equation $u'=0$. Inverting this procedure, we can construct the differential equation $Pu=0$ with a given rigid GRS together with an integral representation of solutions to the equation.

There exist 345 Fuchsian spectral types with order up to 8, which are listed in [**O1**, 13.2.3]. Hence the rigid spectral types allowing unramified singularities are easily obtained up to order 8. For example, the spectral type $421,43,43,52$ in the list corresponds to the following 5 spectral types with irregular singularities

$$421|43,43,52 \quad 421,43|43,52 \quad 412|52,43,43 \quad 412|52,43|43 \quad 421|43|43,52.$$

The following is an example of the reduction of spectral types.

$$\begin{aligned}\underline{4}12|\underline{5}2,\underline{4}3,\underline{4}3 &\xrightarrow{-3} 112|22,13,13 \rightsquigarrow \underline{2}11|\underline{2}2,\underline{3}1,\underline{3}1 \xrightarrow{-2} 11|2,11,11\\ &\rightsquigarrow \underline{1}1,\underline{1}1,\underline{1}1\,(\text{Gauss}) \xrightarrow{-1} 1,1,1\,(\text{trivial})\end{aligned}$$

The arrow $\to$ represents a middle convolution and $\rightsquigarrow$ represents an addition. This reduction gives a representation of a solution by a triple integral by means of the inverse of the above procedure. The number -3 above the first arrow indicates $-d(\mathbf{m}) = 3+2+3+3-2\cdot 7$. In general there are several way of the reduction of a given GRS. Then we have different integral representations of the local solution. See the last example in this paper.

6. Examples

In this last section we give simple examples to explain our results. The arguments in this section will work in the general case.

6.1. Gauss hypergeometric family. Gauss hypergeometric equation and its confluence are most fundamental and instructive. We examine them according to their spectral types.

Case 1: $\underline{11,11,11}$

We may assume that the singular points of the equation with the spectral type $11,11,11$ are 0, 1 and ∞ by a fractional linear transformation. Then the corresponding GRS is

$$H_2:\begin{Bmatrix} x=0 & 1 & \infty\\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix} \quad \text{(the Fuchs-Hukuhara relation: } \sum\lambda_{i,j}=1)$$

and the equation with this GRS is uniquely determined since the spectral type is rigid. Here $\lambda_{j,\nu}$ are complex parameters. Note that there are symmetries given by $\lambda_{i,\nu} \leftrightarrow \lambda_{i,3-\nu}$ for $i\in\{0,1,2\}$ and $\nu\in\{1,2\}$. Another symmetry $(\lambda_{0,1},\lambda_{0,2})\leftrightarrow(\lambda_{1,1},\lambda_{1,2})$ is induced from a fractional linear transformation. There are six symmetries of this type.

Let $u_{j,\lambda_{j,\nu}}(x)$ be the local solution characterized by

$$u_{j,\lambda_{j,\nu}}(x) = f_j(x)^{\lambda_{j,\nu}}\phi_{j,\nu}(x), \quad f_j(x)=\begin{cases} x & (j=0)\\ 1-x & (j=1)\end{cases},$$

$\phi_{j,\nu}(x)$ are holomorphic at $x=j$ and $\phi_{j,\nu}(j)=1$ for $j=0,1$ and $\nu=1,2$.

Then we have

$$u_{0,\lambda_{0,2}}(x)=\sum_{\nu=1}^{2}c(0:\lambda_{0,2}\rightsquigarrow 1:\lambda_{1,\nu})u_{1,\lambda_{1,\nu}}(x)\quad (x\in(0,1))$$

Here the *connection coefficients* $c(0:\lambda_{0,2}\rightsquigarrow 1:\lambda_{1,\nu})$ holomorphically depend on $\lambda_{j,\nu}$ when $\lambda_{j,1}-\lambda_{j,2}\notin\mathbb{Z}$ for $j=0,1$. They are determined by Theorem 3.1 as follows.

A local solution corresponding to the exponent $\lambda_{0,2}$ is given by

$$v(x)=x^{\tilde\lambda}(1-x)^{\tilde\lambda'}I_0^{\mu}\bigl(x^{\lambda}(1-x)^{\lambda'}\bigr).$$

Corresponding transformation of the GRS

$$\begin{aligned}&\begin{Bmatrix}x=0 & 1 & \infty\\ \underline{\lambda} & \lambda' & -\lambda-\lambda'\end{Bmatrix}\xrightarrow{\partial^{-\mu}}\begin{Bmatrix}x=0 & 1 & \infty\\ 0 & 0 & 1-\mu\\ \underline{\lambda+\mu} & \lambda'+\mu & -\lambda-\lambda'-\mu\end{Bmatrix}\xrightarrow{\times x^{\tilde\lambda}(1-x)^{\tilde\lambda'}}\\ &\begin{Bmatrix}x=0 & 1 & \infty\\ \tilde\lambda=\lambda_{0,1} & \tilde\lambda'=\lambda_{1,1} & 1-\mu-\tilde\lambda-\tilde\lambda'=\lambda_{2,1}\\ \underline{\lambda+\mu+\tilde\lambda}=\lambda_{0,2} & \lambda'+\mu+\tilde\lambda'=\lambda_{1,2} & -\lambda-\lambda'-\mu-\tilde\lambda-\tilde\lambda'=\lambda_{2,2}\end{Bmatrix}\end{aligned}\tag{6.1}$$

follows from Theorem 5.4 (cf. [**O1**, (1.8)]).

Theorem 3.1 (Case I) and (Case III) imply

$$v(x)=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}u_{0,\lambda_{0,2}}(x)=Cu_{1,\lambda_{1,1}}(x)+\frac{\Gamma(-\lambda'-\mu)}{\Gamma(-\lambda')}u_{1,\lambda_{1,2}}(x)\tag{6.2}$$

with $C\in\mathbb{C}$ and therefore we have

$$\begin{aligned}c(0:\lambda_{0,2}\rightsquigarrow 1:\lambda_{1,2})&=\frac{\Gamma(\lambda+\mu+1)\cdot\Gamma(-\lambda'-\mu)}{\Gamma(\lambda+1)\cdot\Gamma(-\lambda')}\\&=\frac{\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)\cdot\Gamma(\lambda_{1,1}-\lambda_{1,2})}{\Gamma(\lambda_{0,2}+\lambda_{1,1}+\lambda_{2,1})\cdot\Gamma(\lambda_{0,2}+\lambda_{1,1}+\lambda_{2,2})}.\end{aligned}$$

By the symmetry $\lambda_{1,1}\leftrightarrow\lambda_{1,2}$ we have

$$u_{0,\lambda_{0,2}}=\sum_{\nu=1}^{2}\frac{\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)\cdot\Gamma(\lambda_{1,3-\nu}-\lambda_{1,\nu})}{\Gamma(\lambda_{0,2}+\lambda_{1,3-\nu}+\lambda_{2,1})\cdot\Gamma(\lambda_{0,2}+\lambda_{1,3-\nu}+\lambda_{2,1})}u_{1,\lambda_{1,\nu}}.\tag{6.3}$$

Here we note that $C\in\mathbb{C}$ is given by Remark 3.3 ii), which corresponds to Theorem 3.1 (Case V). Applying symmetries given by fractional linear transformation, we have other connection relations, which are given in [**O2**, §7 and (9.15)] in a unified form.

Remark 6.1. i) Here the poles of $\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)$ correspond to the existence of logarithmic term of the local solution $u_{0,\lambda_{0,2}}$ and the poles of $\Gamma(\lambda_{1,1}-\lambda_{1,2})$ correspond to the existence of logarithmic term of the local solution at $x=1$ with the exponent $\lambda_{1,2}$. The poles of the numerator correspond to the fact that the local solution $u_{0,\lambda_{0,2}}$ satisfies a differential equation of the first order whose characteristic exponent at $x=1$ equals $\lambda_{1,1}$. This fact holds in general for the connection coefficients of rigid Fuchsian differential equations (cf. [**O1**, Chapter 12]).

ii) The symmetries induced by $\lambda_{j,1}\leftrightarrow\lambda_{j,2}$ for $j=1,2$ and $x\mapsto\frac{x}{x-1}$ give 8 different Riemann-Liouville integral expression of the solution corresponding to the exponent $\lambda_{0,2}$.

iii) The condition for the irreducibility is give by

$$\lambda_{0,1}+\lambda_{1,i}+\lambda_{2,j}\notin\mathbb{Z}\quad (i=1,\,2 \text{ and } j=1,\,2).$$

Equation (3.4) in Theorem 3.1 gives the series expansion of $I_0^\mu\big(x^\lambda(1-x)^{\lambda'}\big)$ in Example 3.4, which shows

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma(\beta)}{\Gamma(\gamma)}x^{\gamma-1}I_0^{\gamma-\beta}\big(x^{1-\beta}(1-x)^{-\alpha}\big).$$

Then the GRS of $F(\alpha,\beta,\gamma;x)$ equals

$$\begin{Bmatrix} x=0 & 1 & \infty & \\ \underline{0} & 0 & \alpha & ;x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix}.$$

Owing to Theorem 4.2, we get Gauss hypergeometric equation. See [**O1**, Example 1.8 i)] for the calculation and related results. Moreover [**WW**, Chapter XIV] gives fundamental results related to Gauss hypergeometric function and [**O2**] gives an elementary approach to Gauss hypergeometric function without an integration.

Case 2: $\underline{11|11,11}$

We may assume that the singular points of the equation with the spectral type $11|11,11$ are ∞ and 0 and moreover ∞ is an irregular singular point. Then the corresponding GRS equals

$$\begin{Bmatrix} x=0 & \infty \\ \lambda_{0,1} & \lambda_{1,1}+\alpha x \\ \lambda_{0,2} & \lambda_{1,2}+\beta x \end{Bmatrix}\quad (\alpha\neq\beta,\ \sum_{j,\nu}\lambda_{j,\nu}=1).$$

We may moreover assume GRS equals

$$\begin{Bmatrix} x=0 & \infty \\ \frac12-m & k-\frac{x}{2} \\ \frac12+m & \underline{-k+\frac{x}{2}} \end{Bmatrix} \tag{6.4}$$

by the transformations $x\mapsto cx$ and $\mathrm{Ad}(x^\lambda e^{\lambda' x})$ with suitable c, λ, λ', m, $k\in\mathbb{C}$. We can construct the equation with this GRS and its solution by the operations, namely, additions and a middle convolution by the transformation of GRS

$$\begin{Bmatrix} x=0 & \infty \\ 0 & 0 \end{Bmatrix} \xrightarrow{x^\lambda e^{-x}} \begin{Bmatrix} x=0 & \infty \\ \lambda & x-\lambda \end{Bmatrix} \xrightarrow{\partial^{-\mu}} \begin{Bmatrix} x=0 & \infty \\ 0 & 1-\mu \\ \lambda+\mu & \underline{-\lambda+x} \end{Bmatrix}$$

$$\xrightarrow{\times(x^{-\frac{\lambda+\mu-1}{2}}e^{\frac{x}{2}})} \begin{Bmatrix} x=0 & \infty \\ \frac12-\frac{\lambda+\mu}{2} & \frac{\lambda-\mu+1}{2}-\frac{x}{2} \\ \frac12+\frac{\lambda+\mu}{2} & \underline{-\frac{\lambda-\mu+1}{2}+\frac{x}{2}} \end{Bmatrix},$$

$$\lambda=m+k-\tfrac12,\ \mu=m-k-\tfrac12,\ m=\tfrac{\lambda+\mu+1}{2},\ k=\tfrac{\lambda-\mu}{2}.$$

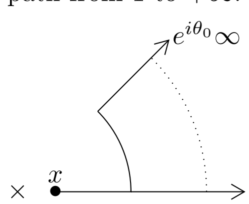
Then the local solution $v(x)$ at ∞ corresponding to the exponent $\frac{x}{2}-k$ has the asymptotic $v(x)\sim x^k e^{-\frac{x}{2}}$ for $x\to+\infty$ is the Whittaker function $W_{k,m}(x)$. In fact (cf. [**WW**, CHAPTER XVI]),

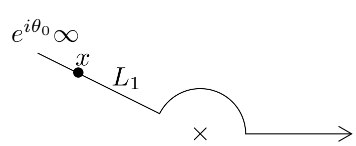
$$
\begin{aligned}
W_{k,m}(x) &:= \frac{x^k e^{-\frac{x}{2}}}{\Gamma(m+\frac12-k)}\int_0^\infty t^{m-k-\frac12}e^{-t}(1+\tfrac{t}{x})^{m+k-\frac12}\,dt\\
&= \frac{x^{-m+\frac12}e^{-\frac{x}{2}}}{\Gamma(m+\frac12-k)}\int_0^\infty t^{m-k-\frac12}e^{-t}(x+t)^{m+k-\frac12}\,dt\\
&= \frac{x^{-m+\frac12}e^{\frac{x}{2}}}{\Gamma(m+\frac12-k)}\int_x^\infty (s-x)^{m-k-\frac12}e^{-s}s^{m+k-\frac12}\,ds\\
&= x^{-m+\frac12}e^{\frac{x}{2}}\tilde I_{+\infty}^{m-k+\frac12}(x^{m+k-\frac12}e^{-x})\\
&\approx x^k e^{-\frac{x}{2}} \quad (x\to+\infty).
\end{aligned}
$$

Here $\approx$ means the ratio of the both sides of the equation takes the value 1 at the limit and this asymptotic follows from Theorem 3.7 (Case I). More precisely, (3.7) shows

$$
W_{k,m}(x) \sim x^k e^{-\frac{x}{2}}\,{}_2F_0(\tfrac12-m-k,\tfrac12+m-k;-\tfrac1x) \qquad \text{for } x\to+\infty. \tag{6.5}
$$

It is clear that $\int_x^\infty$ in the above last integration may be replaced by $\int_x^{e^{i\theta_0}\infty}$ if $|\theta_0|<\frac{\pi}{2}$. Put $\theta_0=\operatorname{Arg}x$. Suppose $|x|>1$ and $\frac{\pi}{2}<|\theta_0|<\frac{3\pi}{2}$. Put $L=L_1+C+L_2$. Here L_1 is the linear path from $e^{i\theta_0}\infty$ to $e^{i\theta_0}$, C is the path $[0,1]\ni t\mapsto e^{i(1-t)\theta_0}$ and L_2 is the path from 1 to $+\infty$. We define $\tilde I^{\mu}_{+\infty}$ by a part of the path L.





Then Theorem 3.7 (Case IV) implies the asymptotic $W_{k,m}(x)\approx x^k e^{-\frac{x}{2}}$ for $x\to e^{i\theta_0}\infty$. Thus we have

$$
W_{k,m}(x)\approx x^k e^{-\frac{x}{2}} \quad \text{for } x\to e^{i\theta_0}\infty \text{ with } |\theta_0|<\tfrac{3\pi}{2} \text{ and } |\theta_0|\neq\tfrac{\pi}{2}. \tag{6.6}
$$

Here it is known that the condition $|\theta_0|\neq\frac{\pi}{2}$ is not necessary (cf. [**Hu2**]).

Moreover Theorem 3.1 Case III implies

$$
W_{k,m}(x)\approx\frac{\Gamma(-\lambda-\mu)}{\Gamma(-\lambda)}x^{\frac12+m}=\frac{\Gamma(-2m)}{\Gamma(\frac12-m-k)}x^{\frac12+m} \quad (\operatorname{Re}(\lambda+\mu)<0,\ x\to+0).
$$

Since

$$
\begin{aligned}
&\operatorname{Ad}(x^{\frac12-m}e^{\frac{x}{2}})\operatorname{mc}_{m+k+\frac12}\operatorname{Ad}(x^{m+k-\frac12}e^{-x})\partial\\
&\quad=\operatorname{Ad}(x^{\frac12-m}e^{\frac{x}{2}})\operatorname{mc}_{m+k+\frac12}\big(\partial-\tfrac{m+k-\frac12}{x}+1\big)\\
&\quad=\operatorname{Ad}(x^{\frac12-m}e^{\frac{x}{2}})\operatorname{mc}_{m-k+\frac12}\big(\partial(x\partial-(m+k-\tfrac12)+x)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Ad}(x^{\frac12 - m} e^{\frac{x}{2}}) \mathrm{mc}_{m-k+\frac12}\big(\partial x(\partial+1) - (m+k-\tfrac12)\partial)\big)\\
&= \mathrm{Ad}(x^{\frac12 - m} e^{\frac{x}{2}})\big((\partial x - (m-k+\tfrac12))(\partial+1) - (m+k-\tfrac12)\partial\big)\\
&= \mathrm{Ad}(x^{\frac12 - m} e^{\frac{x}{2}})\big(x\partial^2 + (x-2m+1)\partial + k - m + \tfrac12\big)\\
&= x(\partial - \tfrac{\frac12 - m}{x} - \tfrac12)^2 + (x-2m+1)(\partial - \tfrac{\frac12-m}{x} - \tfrac12) + k - m + \tfrac12\\
&= x\partial^2 - \tfrac{x}{4} + k + \tfrac{\frac14 - m^2}{x},
\end{aligned}
$$

$W_{k,m}(x)$ is a solution to the Whittaker equation

$$
u'' = \Big(\frac14 - \frac{k}{x} - \frac{\frac14 - m^2}{x^2}\Big)u. \tag{6.7}
$$

Another local solution $M_{k,m}(\mathrm{x})$ given in [**WW**, CHAPTER XVI] corresponding to the exponent $m+\frac12$ at 0 is obtained by the integral

$$
\begin{aligned}
\int_0^1 t^{m-k+\frac12}(1-t)^{m+k-\frac12} e^{xt} dt &= \int_0^1 (1-s)^{m-k-\frac12} s^{m+k-\frac12} e^{x(1-s)} ds\\
&= e^x \int_0^1 (1-s)^{m-k-\frac12} s^{m+k+\frac12} e^{-xs} \tfrac{ds}{s}\\
&= e^x \int_0^x (1-\tfrac{s}{x})^{m-k-\frac12} (\tfrac{s}{x})^{m+k+\frac12} e^{-s} \tfrac{ds}{s}\\
&= x^{-2m} e^x \int_0^x (x-s)^{m-k-\frac12} s^{m+k-\frac12} e^{-s} ds\\
&= x^{-2m} e^x \Gamma(m-k+\tfrac12) I_0^{m-k+\frac12}(x^{m+k-\frac12} e^{-x})
\end{aligned}
$$

and then

$$
\begin{aligned}
M_{k,m}(x) &:= \frac{x^{m+\frac12} e^{-\frac{x}{2}} \Gamma(2m+1)}{\Gamma(\frac12+m-k)\Gamma(\frac12+m+k)} \int_0^1 t^{m-k+\frac12}(1-t)^{m+k-\frac12} e^{xt} dt\\
&= \frac{x^{-m+\frac12} e^{\frac{x}{2}} \Gamma(2m+1)}{\Gamma(m+k+\frac12)} I_0^{m-k+\frac12}(x^{m+k-\frac12} e^{-x})\\
&= \frac{x^{-m+\frac12} e^{\frac{x}{2}} \Gamma(2m+1)}{\Gamma(m+k+\frac12)} I_0^{m-k+\frac12} \sum_{n=0}^\infty x^{m+k-\frac12+n} \frac{(-1)^n x^n}{n!}\\
&= \frac{x^{-m+\frac12} e^{\frac{x}{2}} \Gamma(2m+1)}{\Gamma(m+k+\frac12)} \sum_{n=0}^\infty \frac{\Gamma(m+k+\frac12+n)(-1)^n}{\Gamma(2m+n+1)n!} x^{2m+n}\\
&= x^{m+\frac12} e^{\frac{x}{2}} \sum_{n=0}^\infty \frac{(m+k+\frac12)_n}{(2m+1)_n n!}(-x)^n\\
&= x^{m+\frac12} e^{\frac{x}{2}} {}_1F_1(m+k+\tfrac12; 2m+1; -x).
\end{aligned}
$$

By the symmetry $m \leftrightarrow -m$ of GRS (6.4), $M_{k,-m}(x)$ is a solution to (6.7) and

$$
W_{k,m}(x) = W_{k,-m}(x) = \frac{\Gamma(-2m)}{\Gamma(\frac12 - m - k)} M_{k,m}(x) + \frac{\Gamma(2m)}{\Gamma(\frac12 + m - k)} M_{k,-m}(x). \tag{6.8}
$$

Hence we have

(6.9)

$$\begin{aligned}\tilde{I}_{+\infty}^{m-k+\frac{1}{2}}(x^{m+k-\frac{1}{2}}e^{-x}) &= \frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m+k)} I_0^{m-k+\frac{1}{2}}(x^{m+k-\frac{1}{2}}e^{x}) \\ &\quad + \frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m+k)} I_0^{-m-k+\frac{1}{2}}(x^{-m+k-\frac{1}{2}}e^{-x}) \\ &= \frac{\sin(m+k-\frac{1}{2})\pi}{\sin 2m\pi} I_0^{m-k+\frac{1}{2}}(x^{m+k-\frac{1}{2}}e^{-x}) \\ &\quad + \frac{\sin(m-k+\frac{1}{2})\pi}{\sin 2m\pi} I_0^{-m-k+\frac{1}{2}}(x^{-m+k-\frac{1}{2}}e^{-x}).\end{aligned}$$

Note that GRS (6.4) has the coordinate symmetry

$$\left\{\begin{matrix} x=0 & \infty & \\ \frac{1}{2}-m & k-\frac{x}{2} & ;x \\ \underline{\frac{1}{2}+m} & \underline{-k+\frac{x}{2}} & \end{matrix}\right\} = \left\{\begin{matrix} x=0 & \infty & \\ \frac{1}{2}-m & k+\frac{x}{2} & ;-x \\ \underline{\frac{1}{2}+m} & \underline{-k-\frac{x}{2}} & \end{matrix}\right\} \tag{6.10}$$

corresponding to $(x,k)\mapsto(-x,-k)$. Hence

$$M_{-k,m}(e^{\pm\pi i}x) = e^{\pm(m+\frac{1}{2})\pi i}M_{k,m}(x) \tag{6.11}$$

and $W_{-k,m}(e^{\pm\pi i}x)$ are solutions to (6.7). Note that

$$W_{-k,m}(x) \approx x^{-k}e^{-\frac{x}{2}} \quad \text{for } x\to e^{i\theta_0}\infty \text{ with } |\theta_0|<\tfrac{3}{2}\pi.$$

Then we have

(6.12)

$$\begin{aligned}W_{-k,m}(e^{\pm\pi i}x) &= \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m+k)}M_{-k,m}(e^{\pm\pi i}x) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m+k)}M_{-k,-m}(e^{\pm\pi i}x) \\ &= \frac{\Gamma(-2m)e^{\pm(\frac{1}{2}+m)\pi i}}{\Gamma(\frac{1}{2}-m+k)}M_{k,m}(x) + \frac{\Gamma(2m)e^{\pm(\frac{1}{2}-m)\pi i}}{\Gamma(\frac{1}{2}+m+k)}M_{k,-m}(x), \\ &\approx e^{\mp k\pi i}x^{-k}e^{\frac{x}{2}} \text{ for } x\to e^{i(\theta_0\mp\pi)}\infty \text{ with } |\theta_0|<\tfrac{3}{2}\pi.\end{aligned}$$

Theorem 3.7 (Case IV) shows

$$\begin{aligned}M_{k,m}(x) &= \frac{x^{-m+\frac{1}{2}}e^{\frac{x}{2}}\Gamma(2m+1)}{\Gamma(m+k+\frac{1}{2})} I_0^{m-k+\frac{1}{2}}(x^{m+k-\frac{1}{2}}e^{-x}) \\ &\approx \frac{\Gamma(2m+1)}{\Gamma(m+k+\frac{1}{2})} e^{\mp(m-k+\frac{1}{2})\pi i}x^k e^{-\frac{x}{2}} \quad \text{for } x\to e^{\mp\pi i}\infty\end{aligned} \tag{6.13}$$

and

$$M_{k,m}(x) = \frac{\Gamma(2m+1)e^{\mp(m-k+\frac{1}{2})\pi i}}{\Gamma(m+k+\frac{1}{2})}W_{k,m}(x) + C_\pm W_{-k,m}(e^{\pm\pi i}x) \tag{6.14}$$

with $C_\pm\in\mathbb{C}$ and $|\operatorname{Arg} x\pm\frac{\pi}{2}|<\pi$, which is determined by the asymptotic of $M_{k,m}(x)$ for $x\to+\infty$. Namely, (6.11), (6.13) and (6.12) imply

$$M_{k,m}(x) = e^{\mp(m+\frac{1}{2})\pi i}M_{-k,m}(e^{\pm\pi i}x) \approx \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{1}{2})}x^{-k}e^{\frac{x}{2}} \quad (x\to+\infty),$$

$$C_\pm = \frac{\Gamma(2m+1)e^{\pm k\pi i}}{\Gamma(m-k+\frac{1}{2})}. \tag{6.15}$$

On the other hand, we can also get (6.14) by (6.8) and (6.12).

Equation (6.7) is irreducible if and only if

$$m \pm k + \tfrac{1}{2} \notin \mathbb{Z}. \tag{6.16}$$

This is proved as follows. First we note that the reducibility is a closed condition. If $m \pm k + \frac{1}{2} \in \mathbb{Z}$, the relations (6.8) and (6.12) imply the reducibility because of the vanishing of one of the connection coefficients.

If the equation is reducible, there exists a solution $u(x)$ satisfies an equation of the first order. Then $u(x) \sim Cx^{\frac{1}{2}-m}$ or $Cx^{\frac{1}{2}+m}$ with $C \neq 0$ for $x \to +0$ and $u(x) \sim C'x^k e^{-\frac{x}{2}}$ or $C'x^{-k}e^{\frac{x}{2}}$ with $C' \neq 0$ for $x \to +\infty$. For an example, suppose $u(x) \sim Cx^{\frac{1}{2}-m}$ for $x \to +0$ and $u(x) \sim C'x^k e^{-\frac{x}{2}}$ for $x \to +\infty$. Since $u(x)$ has no singularity in $\mathbb{C} \setminus \{0\}$, the Fuchs-Hukuhara relation for the differential equation of the first order implies $d := m + k - \frac{1}{2} \in \mathbb{Z}_{\geq 0}$ and $u(x) = C'W_{k,m}(x)$. Note that the equation satisfied by $u(x)$ may have apparent singularities. Then d is a number of zeros of $u(x)$ on $\mathbb{C} \setminus \{0\}$ and $x^{m-\frac{1}{2}}e^{\frac{x}{2}}u(x)$ corresponds a Laguerre polynomial of degree d under a certain normalization.

Case 3: $\underline{11|11|11}$. There is only one singular point and we may assume that it is ∞.

$$\begin{Bmatrix} x = \infty \\ ax^2 + bx \end{Bmatrix} \xrightarrow{\mathrm{mc}_{-\mu}} \begin{Bmatrix} x = \infty \\ 1 + \mu \\ ax^2 + bx - \mu \end{Bmatrix} \to \begin{Bmatrix} x = \infty \\ -\frac{x^2}{2} + m + \frac{1}{2} \\ \underline{\frac{x^2}{2} - m + \frac{1}{2}} \end{Bmatrix}, \ \mu = m - \tfrac{1}{2}. \tag{6.17}$$

The above last transformation is obtained by a transformation $x \mapsto C_1 x + C_0$ and $\mathrm{Ad}(e^{\frac{x^2}{4}})$ with $a = 1$, $b = 0$ and $\mu = m - \frac{1}{2}$. Note that the normalized solution $u(x)$ corresponding to the exponent $\frac{x^2}{2} - m + \frac{1}{2}$ has the asymptotic $u(x) \sim x^{-m-\frac{1}{2}}e^{-\frac{x^2}{2}}$ for $x \to +\infty$. Then

$$\begin{aligned} u(x) &= e^{\frac{x^2}{4}} \tilde{I}_{+\infty}^{-\mu}(e^{-\frac{x^2}{2}}) = \frac{e^{\frac{x^2}{4}}}{\Gamma(\mu)} \int_x^\infty e^{-\frac{t^2}{2}}(t-x)^{-\mu-1}dt \\ &= \frac{e^{\frac{x^2}{4}}}{\Gamma(\mu)} \int_0^\infty e^{-\frac{(s+x)^2}{2}} s^{-\mu-1}dt \quad (s = t - x) \\ &= \frac{e^{-\frac{x^2}{4}}}{\Gamma(\mu)} \int_0^\infty e^{-\frac{t^2}{2}-tx} t^{-\mu-1}dt \quad (s \mapsto t) \\ &= D_\mu(x), \end{aligned}$$

$$D_\mu(x) \approx x^\mu e^{-\frac{x^2}{4}} \text{ for } \ x \to +\infty \quad \text{(cf. Theorem 3.7 (Case IV))}.$$

Here the parabolic cylinder function $D_\mu(x)$ is a solution to the Weber equation $Pu = 0$, where

$$\begin{aligned} P &= \mathrm{Ad}(x^{\frac{x^2}{4}})\mathrm{mc}_{-\mu}\,\mathrm{Ad}(e^{-\frac{x^2}{2}})\partial \\ &= \mathrm{Ad}(x^{\frac{x^2}{4}})\mathrm{mc}_{-\mu}(\partial + x) \\ &= \mathrm{Ad}(x^{\frac{x^2}{4}})\mathrm{mc}_{-\mu}(\partial^2 + x\partial + 1) \\ &= \mathrm{Ad}(x^{\frac{x^2}{4}})(\partial^2 + x\partial + \mu + 1) \end{aligned}$$

$$\begin{aligned}&= (\partial - \tfrac{x}{2})^2 + x(\partial - \tfrac{x}{2}) + \mu + 1\\ &= \partial^2 + \tfrac12 + \mu - \tfrac{x^2}{4}.\end{aligned}$$

Note that $D_\mu(x) = D_\mu(-x)$ and $D_{-\mu-1}(ix)$ is another solution to the equation.

It follows from GRS (6.17) that $\sqrt{x}D_\mu(\sqrt{2x})$ satisfies the Whittaker equation and the asymptotic for $x \to +\infty$ shows

$$D_\mu(x) = 2^{\frac{\mu}{2}+\frac14}x^{-\frac12}W_{\frac{\mu}{2},-\frac14}(\tfrac{x^2}{2}).$$

The condition for the irreducibility is

$$\mu = m - \tfrac12 \notin \mathbb{Z}.$$

Note that $e^{-\frac{x^2}{4}}D_\mu(x)$ is the Hermite polynomial of degree μ for $\mu \in \mathbb{Z}_{\ge 0}$.

Versal unfolding (Gauss hypergeometric family)

The versal equation $Pu = 0$ of Gauss hypergeometric family is given in [**O1**, Example 2.5], which is

$$\begin{aligned}P &= \mathrm{mc}_\mu \circ \mathrm{Ad}\Bigl(\exp(-\int \frac{\lambda_1 dx}{1-c_1x} - \int \frac{\lambda_2 x dx}{(1-c_1x)(1-c_2x)})\Bigr)\partial\\ &= \mathrm{mc}_\mu\bigl(\partial + \frac{\lambda_1}{1-c_1x} + \frac{\lambda_2 x}{(1-c_1x)(1-c_2x)}\bigr)\\ &= \mathrm{mc}_\mu\bigl(\partial(1-c_1x)(1-c_2x)\partial + \partial(\lambda_1(1-c_2x)+\lambda_2x)\bigr)\\ &= \bigl((1-c_1x)\partial + c_1(\mu-1)\bigr)\bigl((1-c_2x)\partial + c_2\mu\bigr) + \lambda_1\partial + (\lambda_2-\lambda_1c_2)(x\partial+1-\mu)\\ &= (1-c_1x)(1-c_2x)\partial^2\\ &\quad + \bigl((c_1+c_2)(\mu-1) + \lambda_1 + \bigl(2c_1c_2(1-\mu) + \lambda_2 - \lambda_1c_2\bigr)x\bigr)\partial\\ &\quad + (\mu-1)(c_1c_2\mu + \lambda_1c_2 - \lambda_2)\end{aligned}$$

and a solution is given by

$$\begin{aligned}&I_c^\mu \exp\Bigl(-\int \frac{\lambda_1 dx}{1-c_1x} - \int \frac{\lambda_2 x dx}{(1-c_1x)(1-c_2x)}\Bigr)\\ &= I_c^\mu\bigl((1-c_1x)^{\frac{\lambda_1}{c_1}+\frac{\lambda_2}{c_1(c_1-c_2)}}(1-c_2x)^{\frac{\lambda_2}{c_2(c_2-c_1)}}\bigr).\end{aligned}$$

Here $c = \frac{1}{c_1}$ or $\frac{1}{c_2}$ or ∞. If $c_1c_2(c_1-c_2) \neq 0$, the corresponding GRS is

$$\begin{Bmatrix} x = \frac{1}{c_1} & \frac{1}{c_2} & \infty\\ 0 & 0 & 1-\mu\\ \frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_1(c_1-c_2)} + \mu & \frac{\lambda_2}{c_2(c_2-c_1)} + \mu & -\frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_1c_2} - \mu\end{Bmatrix}.$$

Since

$$\bigl(\tfrac{\lambda_2}{c_2(c_2-c_1)} + \mu\bigr) + \bigl(-\tfrac{\lambda_1}{c_1} + \tfrac{\lambda_2}{c_1c_2} - \mu\bigr) = -\tfrac{\lambda_1}{c_1} + \tfrac{\lambda_2}{c_1(c_2-c_1)}$$

and

$$\lim_{c_2\to 0} c_2\bigl(\tfrac{\lambda_2}{c_2(c_2-c_1)} + \mu\bigr) = -\tfrac{\lambda_2}{c_1} \text{ and } \lim_{c_2\to 0}(1-c_2x)^{-\frac{K}{c_2}} = e^{Kx},$$

we have

$$\begin{Bmatrix} x = \frac{1}{c_1} & \infty\\ 0 & 1-\mu\\ \frac{\lambda_1}{c_1} + \frac{\lambda_2}{c_1^2} + \mu & -\frac{\lambda_2}{c_1}x - \frac{\lambda_1}{c_1} - \frac{\lambda_2}{c_1^2}\end{Bmatrix} \quad (c_1 \neq c_2 = 0),$$

which is also obtained by

$$\tfrac{\lambda_1}{1-c_1x}+\tfrac{\lambda_2x}{1-c_1x}=\tfrac{\lambda_1}{1-c_1x}+\tfrac{\lambda_2}{c_1(1-c_1x)}-\tfrac{\lambda_2}{c_1}.$$

6.2. Hypergeometric family. The spectral type of the generalized hypergeometric function ${}_nF_{n-1}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_{n-1};x)$ is $\overbrace{1\cdots1}^{n},(n-1)1,\overbrace{1\cdots1}^{n}$ and it is explained in [**O1**, 13.4].

We examine a confluence of the generalized hypergeometric function ${}_3F_2$ with the following spectral type:

$\underline{111|21,111}$

The corresponding GRS is realized by

$$\begin{Bmatrix}x=\infty & 0\\ \underline{x-\lambda} & \underline{\lambda}\end{Bmatrix}\xrightarrow{\mathrm{mc}_\mu}\begin{Bmatrix}x=\infty & 0\\ 1-\mu & 0\\ \underline{x-\lambda} & \underline{\lambda+\mu}\end{Bmatrix}\xrightarrow{\mathrm{Ad}(x^{\lambda'})}\begin{Bmatrix}x=\infty & 0\\ 1-\mu-\lambda' & \lambda'\\ \underline{x-\lambda-\lambda'} & \underline{\lambda+\mu+\lambda'}\end{Bmatrix}$$

$$\xrightarrow{\mathrm{mc}_{\mu'}}\begin{Bmatrix}x=\infty & x=0\\ 1-\mu'=1-\lambda_{1,1}+\lambda_{0,1} & 0\\ 1-\mu-\lambda'-\mu'=1-\lambda_{1,2}+\lambda_{0,1} & \lambda'+\mu'=\lambda_{0,2}-\lambda_{0,1}\\ \underline{x-\lambda-\lambda'}=x-\lambda_{1,3}+\lambda_{0,1} & \underline{\lambda+\mu+\lambda'+\mu'}=\lambda_{0,3}-\lambda_{0,1}\end{Bmatrix}$$

with $\mu=\lambda_{1,2}-\lambda_{0,2},\quad \mu'=\lambda_{1,1}-\lambda_{0,1},\quad \lambda=\lambda_{0,3}-\lambda_{1,2},\quad \lambda'=\lambda_{0,2}-\lambda_{1,1},$
$\lambda_{0,1}+\lambda_{0,2}+\lambda_{0,3}=\lambda_{1,1}+\lambda_{1,2}+\lambda_{1.3}.$

Solutions to the corresponding equation are given by

$$\begin{aligned}u_\infty(x)&=\tilde I^{\mu'}_{+\infty}x^{\lambda'}\tilde I^{\mu}_{+\infty}(x^\lambda e^{-x})\\
&\sim \tilde I^{\mu'}_{\infty}x^{\lambda+\lambda'}\sum_{n=1}^\infty\frac{(-\lambda)_n(\mu)_n(-1)^n}{n!}x^{-n}e^{-x}\\
&\sim\Bigl(\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(-\lambda-\lambda'+n)_m(-\lambda)_n(\mu')_m(\mu)_n}{m!n!}(-x)^{-m-n}\Bigr)x^{\lambda+\lambda'}e^{-x}\\
&=\Bigl(\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(\lambda_{0,1}-\lambda_{1,3})_{m+n}(\lambda_{1,2}-\lambda_{0,3})_n(\lambda_{1,1}-\lambda_{0,1})_m(\lambda_{1,2}-\lambda_{0,2})_n}{(\lambda_{0,1}-\lambda_{1,3})_n m!n!}\\
&\qquad\times(-\tfrac1x)^{m+n}\Bigr)x^{\lambda_{1,3}-\lambda_{0,1}}e^{-x}\approx x^{\lambda_{1,3}-\lambda_{0,1}}e^{-x}\quad(x\to+\infty),\\
u_\infty(x)&\approx\frac{\Gamma(-\lambda-\mu)\cdot\Gamma(-\lambda-\lambda'-\mu-\mu')}{\Gamma(-\lambda)\cdot\Gamma(-\lambda-\lambda'-\mu)}x^{\lambda+\lambda'+\mu+\mu'}\\
&=\frac{\Gamma(\lambda_{0,2}-\lambda_{0,3})\cdot\Gamma(\lambda_{0,1}-\lambda_{0,3})}{\Gamma(\lambda_{1,2}-\lambda_{0,3})\cdot\Gamma(\lambda_{1,1}-\lambda_{0,3})}x^{\lambda_{0,3}-\lambda_{0,1}}\quad(x\to+0\\
&\qquad\text{if }\mathrm{Re}\,(\lambda_{0,3}-\lambda_{0,2})<0\text{ and }\mathrm{Re}\,(\lambda_{0,3}-\lambda_{0,1})<0),\\
u_0(x)&:=I^{\mu'}_0x^{\lambda'}I^{\mu}_0(x^\lambda e^{-x})=\frac{1}{\Gamma(\mu)\cdot\Gamma(\mu')}\int_0^x t^{\lambda'}(x-t)^{\mu'}\int_0^t s^\lambda e^{-s}(t-s)^\mu\tfrac{ds}{s}\tfrac{dt}{t}\\
&=I^{\mu'}_0x^{\lambda'}\sum_{n=0}^\infty\frac{(-1)^n\Gamma(\lambda+1+n)}{\Gamma(\lambda+\mu+1+n)n!}x^{\lambda+\mu+n}\\
&=\sum_{n=0}^\infty\frac{(-1)^n\Gamma(\lambda+1+n)\cdot\Gamma(\lambda+\lambda'+\mu+1+n)}{\Gamma(\lambda+\mu+1+n)\cdot\Gamma(\lambda+\lambda'+\mu+\mu'+1+n)}x^{\lambda+\lambda'+\mu+\mu'+n}\end{aligned}$$

$$= \frac{\Gamma(\lambda_{0,3}-\lambda_{1,2}+1)\cdot\Gamma(\lambda_{0,3}-\lambda_{1,1}+1)}{\Gamma(\lambda_{0,3}-\lambda_{0,2}+1)\cdot\Gamma(\lambda_{0,3}-\lambda_{0,1}+1)}x^{\lambda_{0,3}-\lambda_{0,1}} \times {}_2F_2(\lambda_{0,3}-\lambda_{1,2}+1,\lambda_{0,3}-\lambda_{1,1}+1;\lambda_{0,3}-\lambda_{0,2}+1,\lambda_{0,3}-\lambda_{0,1}+1;-x).$$

Here we remark that the connection coefficient of the solution $u_\infty(x)$ to the local solution $u_0(x)$ at the origin equals

$$\frac{\sin(\lambda_{0,2}-\lambda_{0,3})\pi\cdot\sin(\lambda_{0,1}-\lambda_{0,3})\pi}{\sin(\lambda_{1,2}-\lambda_{0,3})\pi\cdot\sin(\lambda_{1,1}-\lambda_{0,3})\pi}. \tag{6.18}$$

Applying $\mathrm{Ad}(x^{\lambda_{0,1}})$ to our GRS, it changes into

$$\begin{Bmatrix} x=\infty & x=0 \\ 1-\lambda_{1,1} & \lambda_{0,1} \\ 1-\lambda_{1,2} & \lambda_{0,2} \\ x-\lambda_{1,3} & \lambda_{0,3} \end{Bmatrix} \quad \text{with} \quad \lambda_{0,1}+\lambda_{0,2}+\lambda_{0,3}=\lambda_{1,1}+\lambda_{1,2}+\lambda_{1,3}. \tag{6.19}$$

Then we have obtained the normalized connection coefficients

$$c(\infty : x-\lambda_{1,3} \rightsquigarrow 0 : \lambda_{0,3}) = \frac{\Gamma(\lambda_{0,2}-\lambda_{0,3})\cdot\Gamma(\lambda_{0,1}-\lambda_{0,3})}{\Gamma(\lambda_{1,2}-\lambda_{0,3})\cdot\Gamma(\lambda_{1,1}-\lambda_{0,3})}. \tag{6.20}$$

The poles of the denominator of the connection coefficient implies that $u_{\infty,x-\lambda_{1,3}}$ satisfies a differential equation of the second order.

To get a solution corresponding to another exponent at ∞, we consider the procedure

$$\begin{Bmatrix} x=\infty & 0 \\ x-\lambda & \lambda \end{Bmatrix} \xrightarrow{\mathrm{mc}_\mu} \begin{Bmatrix} x=\infty & 0 \\ 1-\mu & 0 \\ \underline{x-\lambda} & \underline{\lambda+\mu} \end{Bmatrix} \xrightarrow{\mathrm{Ad}(x^{\lambda'}e^x)} \begin{Bmatrix} x=\infty & x=0 \\ -x+1-\mu-\lambda' & \lambda' \\ \underline{-\lambda-\lambda'} & \underline{\lambda+\mu+\lambda'} \end{Bmatrix}$$

$$\xrightarrow{\mathrm{mc}_{\mu'}} \begin{Bmatrix} x=\infty & x=0 \\ 1-\mu' & 0 \\ -x+1-\mu-\lambda'-\mu' & \lambda'+\mu' \\ \underline{-\lambda-\lambda'-\mu'} & \underline{\lambda+\mu+\lambda'+\mu'} \end{Bmatrix} \xrightarrow{\mathrm{Ad}(x^{\lambda_{0,1}})}$$

$$\xrightarrow{x\mapsto -x} \begin{Bmatrix} x=\infty & x=0 \\ 1-\mu'-\lambda_{0,1}=1-\lambda_{1,1} & \lambda_{0,1} \\ x+1-\mu-\lambda'-\mu'-\lambda_{0,1}=x-\lambda_{1,3} & \lambda'+\mu'+\lambda_{0,1}=\lambda_{0,2} \\ \underline{-\lambda-\lambda'-\mu'-\lambda_{0,1}}=1-\lambda_{1,2} & \underline{\lambda+\mu+\lambda'+\mu'+\lambda_{0,1}}=\lambda_{0,3} \end{Bmatrix}$$

and put

$$\begin{aligned}
\tilde u_\infty(x) &:= \tilde I^{\mu'}_{+\infty} x^{\lambda'} e^x \tilde I^{\mu}_{+\infty}(x^\lambda e^{-x}) \\
&\sim \tilde I^{\mu'}_{+\infty}\bigl(x^{\lambda+\lambda'}{}_2F_0(-\lambda,\mu;-\tfrac{1}{x})\bigr) \quad (x\to+\infty) \\
&\sim \frac{\Gamma(-\lambda-\lambda'-\mu')}{\Gamma(-\lambda-\lambda')}x^{\lambda+\lambda'+\mu'}{}_3F_1(-\lambda,\mu,-\lambda-\lambda'-\mu';-\lambda-\lambda';-\tfrac{1}{x}) \\
&\qquad (x\to+\infty), \\
\tilde u_\infty(x) &\approx \frac{\Gamma(-\lambda-\mu)\cdot\Gamma(-\lambda-\lambda'-\mu-\mu')}{\Gamma(-\lambda)\cdot\Gamma(-\lambda-\lambda'-\mu)}x^{\lambda+\lambda'+\mu+\mu'} \quad (x\to+0,\ \mathrm{Re}\,(\lambda+\mu)<0, \\
&\qquad \mathrm{Re}\,(\lambda+\lambda'+\mu+\mu')<0).
\end{aligned}$$

Thus we have

$$\begin{aligned}&c(-\infty:1-\lambda_{1,2}\rightsquigarrow 0:\lambda_{0,3})\\ &\quad=e^{\lambda_{0,3}\pi i}\frac{\Gamma(\lambda_{1,1}-\lambda_{1,2}+1)\cdot\Gamma(\lambda_{0,2}-\lambda_{0,3})\cdot\Gamma(\lambda_{0,1}-\lambda_{0,3})}{\Gamma(\lambda_{0,1}-\lambda_{1,2}+1)\cdot\Gamma(\lambda_{0,2}-\lambda_{1,2}+1)\cdot\Gamma(\lambda_{1,1}-\lambda_{0,3})}\end{aligned}\tag{6.21}$$

and

$$\begin{aligned}&{}_3F_1(-\lambda,-\lambda-\lambda'-\mu',\mu;-\lambda-\lambda';-\tfrac{1}{x})\\ &\quad={}_3F_1(\lambda_{0,1}-\lambda_{1,2}+1,\lambda_{0,2}-\lambda_{1,2}+1,\lambda_{0,3}-\lambda_{1,2}+1;\lambda_{1,2}-\lambda_{1,1}+1;-\tfrac{1}{x}).\end{aligned}$$

Here the local solution $u_{-\infty,1-\lambda_{1,2}}$ at $-\infty$ is given by

$$\begin{aligned}u_{-\infty,1-\lambda_{1,2}}(x)&:=\frac{\Gamma(-\lambda-\lambda')}{\Gamma(-\lambda-\lambda'-\mu)}(e^{\pi i}x)^{\lambda_{0,1}}\tilde u_\infty(e^{\pi i}x)\\ &\approx(e^{\pi i}x)^{\lambda_{1,2}-1}\quad\text{for}\quad x\to e^{-\pi i}\infty\end{aligned}\tag{6.22}$$

which is uniquely characterized by this asymptotic and then

$$u_{-\infty,1-\lambda_{1,2}}=\sum_{\nu=1}^{3}c(-\infty:1-\lambda_{1,2}\rightsquigarrow 0:\lambda_{0,\nu})u_{0,\lambda_{0,\nu}}.\tag{6.23}$$

If $\lambda_{0,\nu}-\lambda_{1,2}+1\in\mathbb{Z}_{\le 0}$, $u_{-\infty,1-\lambda_{1,2}}$ satisfies a differential equation of the first order for $\nu=1,\,2,\,3$. If $\lambda_{1,1}-\lambda_{0,\nu}\in\mathbb{Z}_{\le 0}$, $u_{-\infty,1-\lambda_{1,2}}$ satisfies a differential equation of the second order for $\nu=1,\,2,\,3$.

By the symmetry of GRS (6.19) with respect to the exponents $\lambda_{0,\nu}$ for $\nu=1,\,2,\,3$ and the exponents $1-\lambda_{1,\nu}$ for $\nu=1,\,2$, we have normalized 3 solutions corresponding to the 3 exponents at each singular point and the connection relation between the singular points 0 and ∞.

The above procedure is valid for the spectral type

$\underline{\overbrace{1\cdots1}^{n}|(n-1)1,\overbrace{1\cdots1}^{n}}.$

Namely, applying $\mathrm{Ad}(x^{\lambda''})\mathrm{mc}_{\mu''}$ to (6.19) several times with various (λ'',μ''), we get

$$\begin{Bmatrix}x=\infty & x=0\\ 1-\lambda_{1,\nu}\ (1\le\nu<n) & \lambda_{0,\nu}\ (1\le\nu\le n)\\ x-\lambda_{1,n} & \end{Bmatrix}\qquad\Bigl(\sum_{\nu=1}^{n}\lambda_{0,\nu}=\sum_{\nu=1}^{n}\lambda_{1,\nu}\Bigr),$$

which is symmetric with respect to the parameters in $\{\lambda_{0,\nu}\mid 1\le\nu\le n\}$ and the parameters in $\{\lambda_{1,\nu}\mid 1\le\nu<n\}$. The local solutions are given by

$$\begin{aligned}u_{0,\lambda_{0,n}}&=x^{\lambda_{0,\nu}}{}_{n-1}F_{n-1}(\lambda_{0,n}-\lambda_{1,\nu}\ (1\le\nu<n)\ ;\ \lambda_{0,n}-\lambda_{0,\nu}\ (1\le\nu<n)\ ;\ x),\\ u_{\infty,1-\lambda_{1,1}}&=(e^{\pi i}x)^{\lambda_{1,1}-1}{}_nF_{n-2}(\lambda_{0,\nu}-\lambda_{1,1}+1\ \ (1\le\nu\le n)\ ;\\ &\qquad\lambda_{1,1}-\lambda_{1,\nu}\ (1<\nu<n)\ ;\ -\tfrac{1}{e^{\pi i}x})\qquad(x\to e^{-\pi i}\infty),\\ u_{\infty,x-\lambda_{1,n}}&=x^{\lambda_{0,1}}\tilde I_\infty^{\lambda_{1,1}-\lambda_{0,1}}x^{\lambda_{0,2}-\lambda_{1,1}}\tilde I_\infty^{\lambda_{1,2}-\lambda_{0,2}}x^{\lambda_{0,3}-\lambda_{1,2}}\cdots\\ &\qquad\cdots\tilde I_\infty^{\lambda_{1,n-1}-\lambda_{0,n-1}}(x^{\lambda_{0,n}-\lambda_{1,n-1}}e^{-x})\\ &\approx x^{\lambda_{1,n}}e^{-x}\qquad(x\to+\infty)\end{aligned}$$

and the connection coefficients are

$$\begin{aligned} c(-\infty : 1-\lambda_{1,1} \rightsquigarrow 0 : \lambda_{0,n}) &= e^{\lambda_{0,n}\pi i} \\ &\quad \times \frac{\prod_{1<\nu<n}\Gamma(\lambda_{1,\nu}-\lambda_{1,1}+1)\cdot\prod_{1\le\nu<n}\Gamma(\lambda_{0,\nu}-\lambda_{0,n})}{\prod_{1<\nu<n}\Gamma(\lambda_{1,\nu}-\lambda_{0,n})\cdot\prod_{1\le\nu<n}\Gamma(\lambda_{0,\nu}-\lambda_{1,1}+1)}, \\ c(\infty : x-\lambda_{1,n} \rightsquigarrow 0 : \lambda_{0,n}) &= \frac{\prod_{1\le\nu<n}\Gamma(\lambda_{0,\nu}-\lambda_{0,n})}{\prod_{1\le\nu<n}\Gamma(\lambda_{1,\nu}-\lambda_{0,n})}. \end{aligned}$$

6.3. Jordan Pochhammer family. The spectral type of Jordan-Pochhammer equation is

$$\underbrace{(p-1)1,(p-1)1,\cdots,(p-1)1}_{p+1\text{blocks}} \quad (p=2 \ \Rightarrow \text{Gauss})$$

Integral representation of the solution $u(x)$ of the corresponding versal equation is given by

$$u(x) = I_c^\mu(\phi), \tag{6.24}$$

$$\begin{aligned} \phi(x) &= \exp\Bigl(-\int\sum_{k=1}^p \frac{\lambda_k x^{k-1}}{(1-c_1x)(1-c_2x)\cdots(1-c_kx)}\Bigr) \\ &= \prod_{j=1}^p (1-c_jx)^{\sum_{k=j}^p \frac{\lambda_j}{c_j\prod_{j\le\nu\le k,\,\nu\ne j}(c_j-c_\nu)}} \qquad (c_i\ne c_j\ne 0 \text{ for } i\ne j). \end{aligned} \tag{6.25}$$

The GRS of the equation is

$$\left\{\begin{matrix} x=\infty & x=\frac{1}{c_j}\ (j=1,\dots,p) \\ [1-\mu]_{(p-1)} & [0]_{(p-1)} \\ \displaystyle\sum_{k=1}^p\frac{(-1)^k\lambda_k}{\prod_{\nu=1}^k c_\nu}-\mu & \displaystyle\sum_{k=j}^p\frac{\lambda_k}{c_j\prod_{1\le\nu\le k,\nu\ne j}(c_j-c_\nu)}+\mu \end{matrix}\right\} \qquad (c_i\ne c_j\ne 0 \text{ for } \ i\ne j),$$

$$\left\{\begin{matrix} x=\infty \\ [1-\mu]_{(p-1)} \\ \displaystyle\sum_{k=1}^p \lambda_k x^k+(p-1)\mu \end{matrix}\right\} \qquad (c_1=\cdots=c_p=0).$$

Suppose $\lambda_p = 1$ and $c_1=\cdots=c_p=0$. In view of Remark 4.6 the corresponding equation $Pu=0$ is given by

$$\begin{aligned} P &= \mathrm{mc}_\mu\bigl(\partial+\sum_{k=1}^p\lambda_kx^{k-1}\bigr) \\ &= \mathrm{mc}_\mu\bigl(\partial^p+\sum_{k=1}^p\partial^{p-1}\lambda_kx^{k-1}\bigr) \\ &= \mathrm{mc}_\mu\bigl(\partial^p+\sum_{k=1}^p\lambda_k(\vartheta+p-1)(\vartheta+p-2)\cdots(\vartheta+p-k+1)\partial^{p-k}\bigr) \\ &= \partial^p+\sum_{k=1}^p\lambda_k(\vartheta-\mu+p-1)(\vartheta-\mu+p-2)\cdots(\vartheta-\mu+p-k+1)\partial^{p-k} \end{aligned}$$

and we have its solutions

$$\begin{aligned} u_m(x) &= \frac{1}{\Gamma(\mu)} \int_x^{e^{\frac{2m\pi}{p}i}\infty} e^{-\sum_{k=1}^p \lambda_k \frac{t^k}{k}} (t-x)^{\mu-1} dt \\ &\approx x^{-(p-1)\mu} e^{-\frac{x^p}{p}} \quad (x \to e^{\frac{2m\pi}{p}i}\infty) \end{aligned}$$

for $m = 0, \dots, p-1$.

We consider the spectral type $21|21,21,21$.

$\underline{2}1|\underline{2}1,\underline{2}1,\underline{2}1 \xrightarrow{-2} 1|1,1,1$

$$\begin{Bmatrix} x=\infty & 0 & y \\ [1-\lambda_3]_{(2)} & [0]_{(2)} & [0]_{(2)} \\ \alpha x + \lambda_2 & \lambda_0 & \lambda_1 \end{Bmatrix} \qquad (\alpha \neq 0,\ \lambda_0+\lambda_1+\lambda_2 = 2\lambda_3)$$

$$\xrightarrow{\mathrm{mc}_{-\lambda_3}} \begin{Bmatrix} x=\infty & 0 & 1 \\ \alpha x + \lambda_2 & \lambda_0 - \lambda_3 & \lambda_1 - \lambda_3 \end{Bmatrix}.$$

We can consider another reduction.

$21|21,21,21 \rightsquigarrow \underline{2}1|\underline{2}1,\underline{2}1,\underline{1}2 \xrightarrow{-1} 11|11,11,2 \rightsquigarrow \underline{1}1|\underline{1}1,\underline{1}1 \xrightarrow{-1} 1,1,1$

$$\begin{Bmatrix} x=\infty & 0 & y \\ [1-\lambda_3]_{(2)} & [0]_{(2)} & [0]_{(2)} \\ \alpha x + \lambda_2 & \lambda_0 & \lambda_1 \end{Bmatrix} \qquad (\lambda_0+\lambda_1+\lambda_2 = 2\lambda_3)$$

$$\xrightarrow{\mathrm{Ad}\left((y-x)^{-\lambda_1}\right)} \begin{Bmatrix} x=\infty & 0 & y \\ [1+\lambda_1-\lambda_3]_{(2)} & [0]_{(2)} & [-\lambda_1]_{(2)} \\ \alpha x + \lambda_1 + \lambda_2 & \lambda_0 & 0 \end{Bmatrix}$$

$$\xrightarrow{mc_{\lambda_1-\lambda_3}} \begin{Bmatrix} x=\infty & 0 & y \\ 1-\lambda_1+\lambda_3 & 0 & [-\lambda_3]_{(2)} \\ \alpha x + \lambda_1 + \lambda_2 & \lambda_0+\lambda_1-\lambda_3 & \end{Bmatrix}$$

$$\xrightarrow{\mathrm{Ad}(y-x)^{\lambda_3}} \begin{Bmatrix} x=\infty & 0 \\ 1-\lambda_1 & 0 \\ \alpha x + \lambda_1 + \lambda_2 - \lambda_3 & \lambda_0+\lambda_1-\lambda_3 \end{Bmatrix}$$

$$\xrightarrow{\mathrm{mc}_{-\lambda_1}} \begin{Bmatrix} x=\infty & 0 \\ \alpha x + \lambda_1 + \lambda_2 - \lambda_3 & \lambda_0-\lambda_3 \end{Bmatrix}.$$

Then the normalized local solution u_{0,λ_0} corresponding to the exponent λ_0 is given by

$$\begin{aligned} u_{0,\lambda_0} &= \frac{\Gamma(\lambda_0+1)}{\Gamma(\lambda_0-\lambda_3+1)} I_0^{\lambda_3}\left(x^{\lambda_0-\lambda_3}(1-\tfrac{x}{y})^{\lambda_1-\lambda_3} e^{-\alpha x}\right) \\ &= \frac{\Gamma(\lambda_0+1)}{\Gamma(\lambda_0-\lambda_3+1)} I_0^{\lambda_3-\lambda_1}\Big((1-\tfrac{x}{y})^{-\lambda_3} I_0^{\lambda_1}\left(x^{\lambda_0-\lambda_3}e^{-\alpha x}\right)\Big). \end{aligned}$$

Here $|y| > |x| > 0$.

Remark 6.2. i) $I_0^{\lambda_1}\left(x^{\lambda_0-\lambda_3}e^{-\alpha x}\right)$ is expressed by using the Whittaker function $M_{k,m}(x)$.

ii) If $\alpha = 0$, the equation is reducible.

iii) We remark that the function $u_{0,\lambda_0}(x,y)$ satisfies a confluent KZ equation with the variables x and y which is given in [**O5**, Example 7.6] with $(x_0,x_1,x_2) = (x,y,0)$ and $a = 0$. We will discuss such equations in another paper. The equation is a confluence of Appell's F_1. Note that the coordinate transformation $(x,y)\mapsto(x,\frac{x}{y})$ is an automorphism of confluent KZ equations of this type (cf. [**O3**, §6]).

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